PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 21 – French, Chapter 8

Spring 2016 Semester

Matthew Jones

Midterm Exam:

Date: Thursday, March 10th

Time: 8:00 – 10:00 pm

Room: MSEE B012

Material: French, chapters 1-8

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- The solution, $y(x,t)$, describes the shape of a string as a function of x and t.
- This is a transverse wave: the displacement is perpendicular to the direction of propagation.
- This would confuse the following discussion…
- Instead, let’s now consider longitudinal waves, like the pressure waves due to the propagation of sound in a gas.

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- The solution, $p(x,t)$, describes the excess pressure in the gas as a function of x and t.
- What if the wave was propagating in the y-direction?

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- What if the wave was propagating in the z-direction?

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

Waves in Three Dimensions

- The excess pressure is now a function of $\vec{x}$ and t.
- Wave equation in three dimensions:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- But we like to write it this way:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- Where ∇^2 is called the "Laplacian operator", but you just need to think of it as a bunch of derivatives:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Waves in Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- How do we solve this? Here's how…

$$p(\vec{x},t) = p_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

- One partial derivatives:

Chain rule…

$$\frac{\partial p}{\partial x} = i p_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \frac{\partial}{\partial x}\left(\vec{k}\cdot\vec{x}-\omega t\right)$$

$$= i p_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \frac{\partial}{\partial x}\left(k_x x + k_y y + k_z z - \omega t\right)$$

$$= i k_x p(\vec{x},t)$$

- Second derivative:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2\, p(\vec{x},t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- Second derivatives:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2\, p(\vec{x},t)$$
$$\frac{\partial^2 p}{\partial y^2} = -k_y^2\, p(\vec{x},t)$$
$$\frac{\partial^2 p}{\partial z^2} = -k_z^2\, p(\vec{x},t)$$
$$\frac{\partial^2 p}{\partial t^2} = -\omega^2\, p(\vec{x},t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$
$$-\left(k_x^2 + k_y^2 + k_z^2\right)p(\vec{x},t) = -\frac{\omega^2}{v^2}p(\vec{x},t)$$

- Any values of k_x, k_y, k_z satisfy the equation, provided that

$$\omega = v\sqrt{k_x^2 + k_y^2 + k_z^2} = v\left|\vec{k}\right|$$

- If $k_y = k_z = 0$ then $p(\vec{x},t) = p_0 e^{i(k_x x - \omega t)}$ but this described a wave propagating in the $+x$ direction.

Waves in Three Dimensions

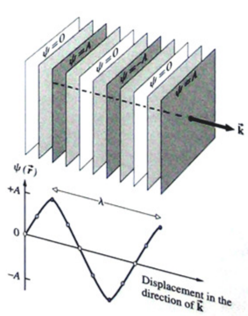
- As usual, we are mainly interested in the real component:

$$\psi(\vec{r},t) = A\cos\left(\vec{k}\cdot\vec{x} - \omega t\right)$$

- A wave propagating in the opposite direction would be described by

$$\psi'(\vec{r},t) = A'\cos\left(\vec{k}\cdot\vec{x} + \omega t\right)$$

- The points in a plane with a common phase is called the "wavefront".



Waves in Three Dimensions

$$\psi(\vec{r},t) = A\cos(\vec{k}\cdot\vec{x} \mp \omega t)$$

- Sometimes we are free to pick a coordinate system in which to describe the wave motion.
- If we choose the x-axis to be in the direction of propagation, we get back the one-dimensional solution we are familiar with:

$$\psi(\vec{r},t) = A\cos(kx \mp \omega t)$$

- But in one-dimension we saw that any function that satisfied $f(x \pm vt)$ was a solution to the wave equation.
- What is the corresponding function in three dimensions?

Waves in Three Dimensions

$$\omega = v\sqrt{k_x^2 + k_y^2 + k_z^2} = v|\vec{k}|$$

- General solution to the wave equation are functions that are twice-differentiable of the form:

$$\psi(\vec{r},t) = C_1 f(\hat{k}\cdot\vec{r} - vt) + C_2 g(\hat{k}\cdot\vec{r} + vt)$$

where $\hat{k} = \vec{k}/|\vec{k}|$

- Just like in the one-dimensional case, these do not have to be harmonic functions.

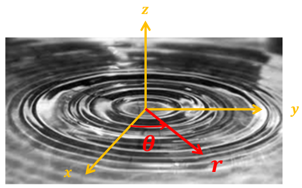
Waves in Two Dimensions

- Plane waves frequently provide a good description of physical phenomena, but this is usually an approximation:

- This looks like a wave… can the wave equation describe this?

Waves in Two Dimensions

- Rotational symmetry:
  - Cartesian coordinates are not well suited for describing this problem.
  - Use polar coordinates instead.
  - Motion should depend on r but should be independent of θ



Waves in Two Dimensions

- Wave equation: $\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$
- How do we write ∇^2 in polar coordinates?

$$r = \sqrt{x^2 + y^2}$$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

- Derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

Waves in Two Dimensions

$$= \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\right]$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}$$

Similarly,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}.$$

Taking one more derivative, we see

$$\frac{\partial^2 u}{\partial \theta^2} = r\frac{\partial}{\partial \theta}\left[-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}\right]$$

$$= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r\left[-\sin\theta\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial \theta}\right) + \cos\theta\left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}\right)\right]$$

$$= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r\left[-\sin\theta\left(-r\sin\theta\frac{\partial^2 u}{\partial x^2} + r\cos\theta\frac{\partial^2 u}{\partial x \partial y}\right) + \cos\theta\left(-r\sin\theta\frac{\partial^2 u}{\partial x \partial y} + r\cos\theta\frac{\partial^2 u}{\partial y^2}\right)\right]$$

$$= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r^2\left[\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\right]$$

$$= -r\frac{\partial u}{\partial r} + r^2\left[\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\right]$$

Now we're ready to put everything together:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}$$

$$= (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial x^2} + (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}$$

Waves in Two Dimensions

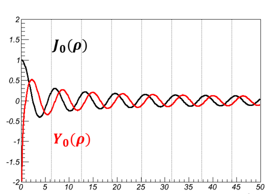
- Laplacian in polar coordinates:
$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$
- When the geometry does not depend on θ or z:
$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)$$
- Wave equation:
$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Waves in Two Dimensions

- Wave equation:
$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$
- If we assume that $\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi$ then the equation is:
$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\omega^2}{v^2}\psi = 0$$
- Change of variables: Let $\rho = r\omega/v$
$$\frac{\omega^2}{v^2}\frac{\partial^2\psi}{\partial\rho^2} + \frac{\omega^2}{v^2}\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\omega^2}{v^2}\psi = 0$$
$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \psi(\rho) = 0$$

Waves in Two Dimensions

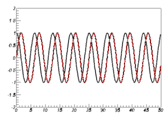
- Bessel's Equation:
$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \psi(\rho) = 0$$
- Solutions are "Bessel functions": $J_0(\rho), Y_0(\rho)$



Bessel Functions?

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega^2}{v^2}\psi = 0$$

- Solutions: $\sin kx, \cos kx$
- Graphs:

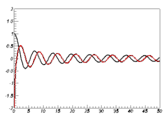


- Series representation:

$$\cos kx = \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^{2n}}{(2n)!}$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\omega^2}{v^2}\psi = 0$$

- Solutions: $J_0(kr)$, $Y_0(kr)$
- Graphs:

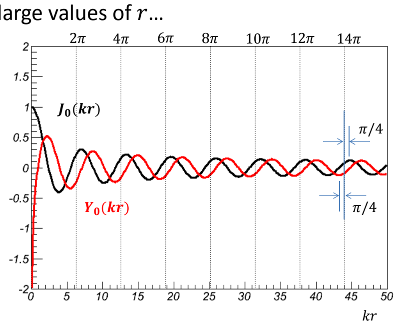


- Series representation:

$$J_0(kr) = \sum_{n=0}^{\infty} \frac{(-1)^n (kr)^{2n}}{2^{2n}(n!)^2}$$

Asymptotic Properties

- At large values of r…

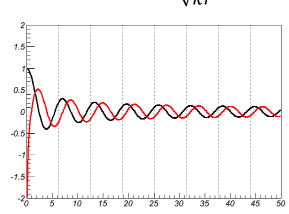


Asymptotic Properties

- When r is large, for example, $kr \gg 1$

$$J_0(kr) \approx \sqrt{2/\pi}\,\frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$

$$Y_0(kr) \approx \sqrt{2/\pi}\,\frac{\sin(kr - \pi/4)}{\sqrt{kr}}$$



Example

- What are the frequencies of the rotationally symmetric normal modes of oscillation for the surface of a circular drum of radius R for which the speed of wave propagation is v?
  - The speed would depend on things like the surface tension and mass per unit area, but the solution only depends on the value of v.

Example

- How is this similar to the string with fixed ends?
  - Look for solutions to the wave equation that satisfy the boundary conditions.
  - When $y(x,t) = 0$ for $x = 0$ and $x = L$, these were
$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t$$
  - We substitute this back into the wave equation to find ω_n:
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$
$$\left[\left(\frac{n\pi}{L}\right)^2 - \frac{\omega_n^2}{v^2}\right] y_n(x,t) = 0$$
$$\omega_n = \frac{n\pi v}{L}$$

Example

- In the case of a circular drum we have to pick the form of the solution that we expect:
  - It can't be $Y_0(kr)$ because this one diverges at $r = 0$
  - It must be $J_0(kr)$ but only when k makes it satisfy the boundary condition $J_0(kR) = 0$.
  - Asymptotic form of the solution:
$$J_0(kr) \approx \sqrt{2/\pi}\,\frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$
  - The argument of the cosine function must be:
$$kR - \pi/4 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, etc\ldots$$

Example

$$kR - \pi/4 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

- In general, $k_n R - \pi/4 = (2n-1)\pi/2$

$$k_n = \frac{1}{R}\left(\frac{\pi}{2}(2n-1) + \frac{\pi}{4}\right)$$
$$= \frac{n\pi}{R} - \frac{\pi}{4R}$$

- Frequencies are

$$\omega_n = v k_n = \frac{v}{R}\left(n\pi - \frac{\pi}{4}\right)$$

Example

- Various numerical methods are available to evaluate $J_0(kr)$ and to find its roots
  - Just like there are numerical methods at your disposal to evaluate $\sin(kx)$ and $\cos(kx)$.

n	Rk_n (approx)	z_n (exact)
1	$3\pi/4 = 2.3562$	2.4048
2	$7\pi/4 = 5.4978$	5.5201
3	$11\pi/4 = 8.6394$	8.6537
4	$15\pi/4 = 11.7810$	11.7915
5	$19\pi/4 = 14.9226$	14.9309

Energy

- The energy carried by a wave is proportional to the square of the amplitude.
- When $\psi(r,t) \sim A\frac{\cos kr}{\sqrt{r}}$ the energy density decreases as $1/r$
- But the wave is spread out on a circle of circumference $2\pi r$
- The total energy is constant, independent of r
- At large r they look like plane waves: