

Physics 42200

Waves & Oscillations

Lecture 20 – French, Chapter 8

Spring 2016 Semester

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Midterm Exam:

Date: Thursday, March 10th

Time: 8:00 – 10:00 pm

Room: MSEE B012

Material: French, chapters 1-8

Impedance

- Mechanical impedance:

$$Z = T/v = \sqrt{T\mu}$$

- Electrical impedance:

$$Z = \sqrt{L'/C'}$$

- Capacitance per unit length: $C' = \frac{2\pi\epsilon_0}{\log\left(\frac{R_2}{R_1}\right)}$
- Inductance per unit length: $L' = \frac{\mu_0}{2\pi} \log\left(\frac{R_2}{R_1}\right)$
- Speed in cable: $v = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\epsilon_r\epsilon_0\mu_0}} = \frac{c}{\sqrt{\epsilon_r}}$
- Impedance: $Z = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_r\epsilon_0}} \log\left(\frac{R_2}{R_1}\right)$

Coaxial
cable

Electrical Impedance

$$Z = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}} \log \left(\frac{R_2}{R_1} \right)$$

- This depends on the geometry of the cable
- The dimensions are ohms

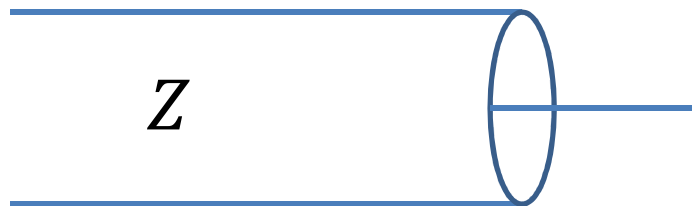
$$Z = \frac{60 \, \Omega}{\sqrt{\epsilon_r}} \log \left(\frac{R_2}{R_1} \right)$$

- As far as pulses are concerned, the cable looks like a resistance and satisfies Ohm's law:

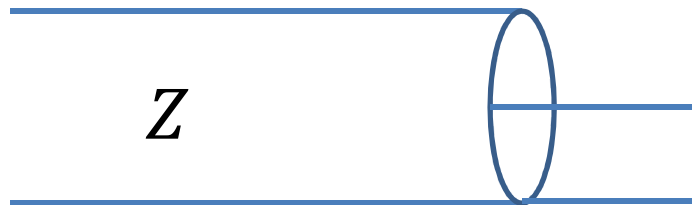
$$I = V/Z$$

Electrical Impedance

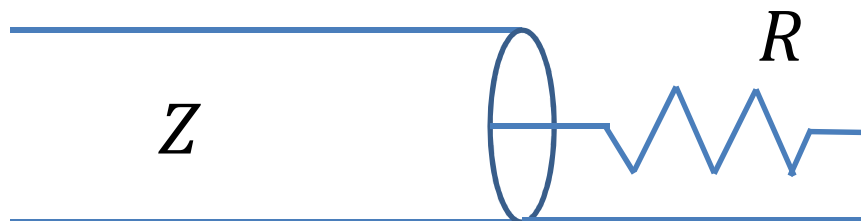
- Impedance at the end of the cable:



Open circuit, $Z' = \infty$



Short circuit, $Z' = 0$



Resistor, $Z' = R$

Electrical Impedance

- Reflection coefficient:

$$\rho = \frac{Z' - Z}{Z' + Z}$$

- Transmission coefficient:

$$\tau = \frac{2Z'}{Z' + Z}$$

- Limiting cases to remember:

- Open circuit: $\rho = 1, \tau = 2$
- Short circuit: $\rho = -1, \tau = 0$
- Matched, $Z' = Z$: $\rho = 0, \tau = 1$.

Power Transmission

- How much power is reflected?

- Open circuit or short circuit:

$$P_r = \frac{V_r^2}{Z} = \frac{V_i^2}{Z} \left(\frac{Z' - Z}{Z' + Z} \right)^2 = \frac{V_i^2}{Z} = P_i$$

- Reflected power is zero when $Z' = Z$
 - In this case, all power must be transmitted
 - Maximal power is transferred to a load of resistance R when $R = Z$.
 - This is called impedance matching.

Source Impedance

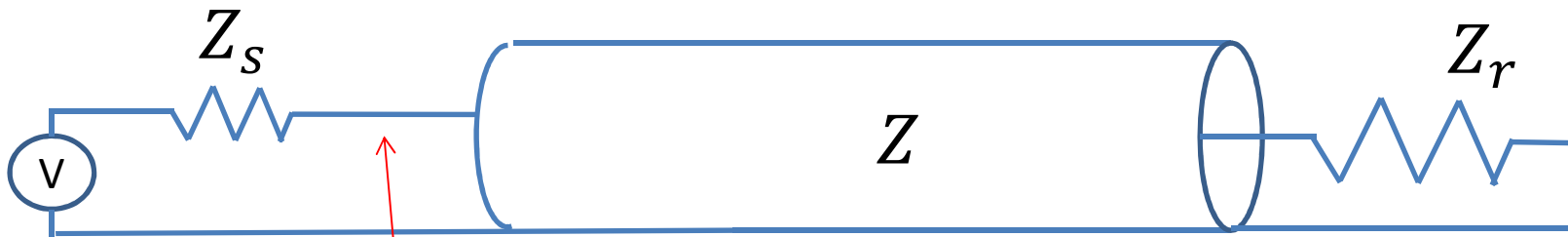
- An ideal voltage source provides a given voltage, independent of the current.
 - But real voltage sources can't deliver arbitrarily large currents
- Voltage sources are modelled by an ideal voltage source and a resistor:



This resistance is called the source impedance. Sometimes we want the source impedance to be finite...

Drivers/Receivers

- Now we can model the entire cable:



- Current from the source:

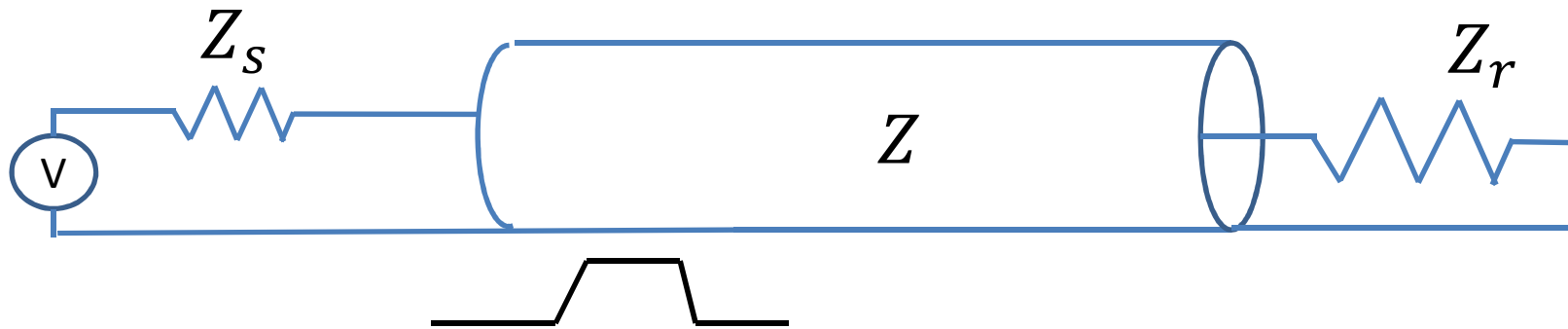
$$I = \frac{V}{Z_s + Z}$$

- Voltage at the left end of the cable:

$$V_i = V - I Z_s = V \frac{Z}{Z_s + Z}$$

Drivers/Receivers

- Voltage pulse propagates to the receiver



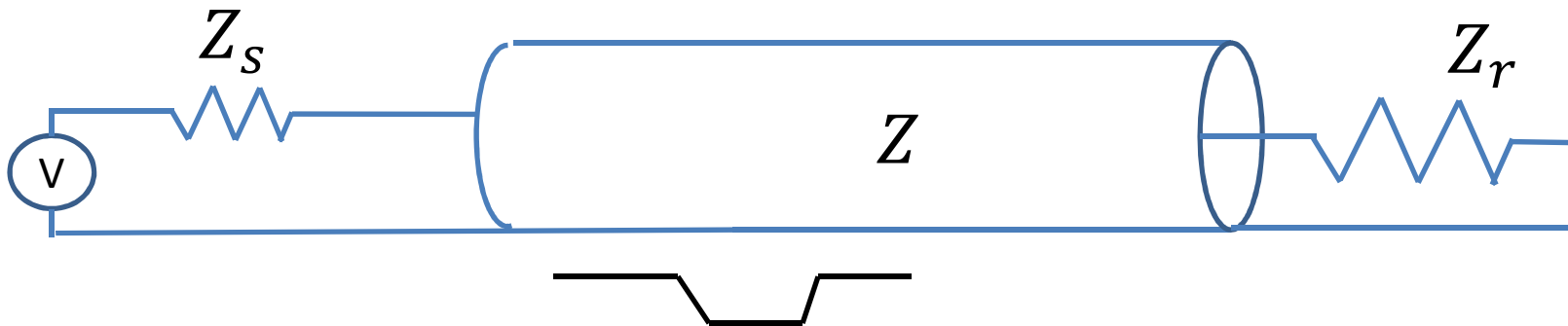
- The pulse might be reflected at the receiver

$$\rho = \frac{Z_r - Z}{Z_r + Z}$$

- When $Z_r < Z$ the reflected pulse is inverted.

Drivers/Receivers

- Voltage pulse propagates back to the source



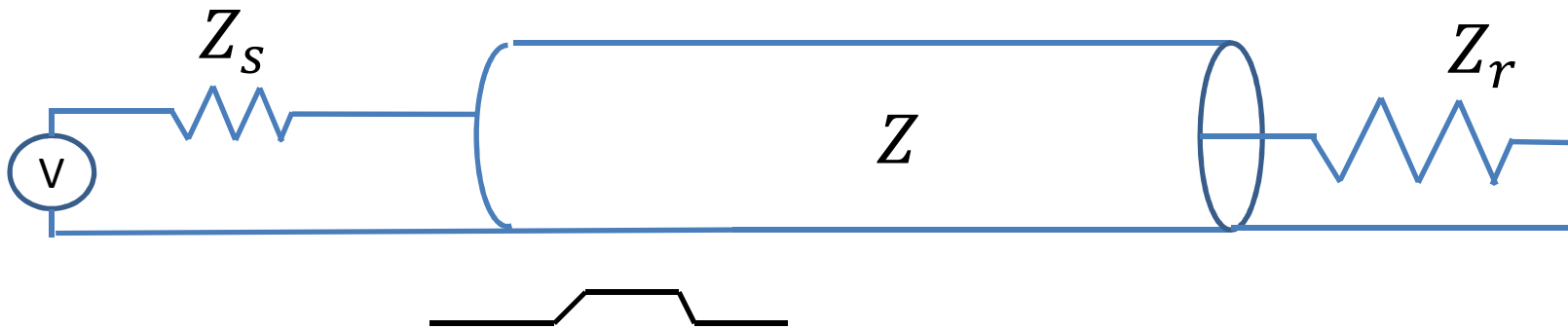
- The pulse can also be reflected from an impedance mismatch at the source:

$$\rho = \frac{Z_s - Z}{Z_s + Z}$$

- When $Z_s > Z$ the reflected pulse is not inverted.

Drivers/Receivers

- Reflected pulse propagates to the receiver



- The system is linear, so the observed signal at any point is the sum of all incident and reflected waves.

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

- The solution, $y(x, t)$, describes the shape of a string as a function of x and t .
- This is a transverse wave: the displacement is perpendicular to the direction of propagation.
- This would confuse the following discussion...
- Instead, let's now consider longitudinal waves, like the pressure waves due to the propagation of sound in a gas.

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- The solution, $p(x, t)$, describes the excess pressure in the gas as a function of x and t .
- What if the wave was propagating in the y -direction?

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- What if the wave was propagating in the z -direction?

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

Waves in Three Dimensions

- The excess pressure is now a function of $\vec{x}$ and t .
- Wave equation in three dimensions:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- But we like to write it this way:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- Where ∇^2 is called the “Laplacian operator”, but you just need to think of it as a bunch of derivatives:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Waves in Three Dimensions

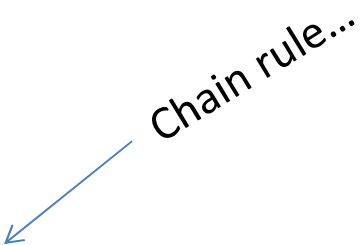
- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- How do we solve this? Here's how...

$$p(\vec{x}, t) = p_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

- One partial derivatives:

$$\begin{aligned} \frac{\partial p}{\partial x} &= ip_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \frac{\partial}{\partial x} (\vec{k} \cdot \vec{x} - \omega t) \\ &= ip_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \frac{\partial}{\partial x} (k_x x + k_y y + k_z z - \omega t) \\ &= ik_x p(\vec{x}, t) \end{aligned}$$


- Second derivative:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2 p(\vec{x}, t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- Second derivatives:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2 p(\vec{x}, t)$$

$$\frac{\partial^2 p}{\partial y^2} = -k_y^2 p(\vec{x}, t)$$

$$\frac{\partial^2 p}{\partial z^2} = -k_z^2 p(\vec{x}, t)$$

$$\frac{\partial^2 p}{\partial t^2} = -\omega^2 p(\vec{x}, t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

$$-(k_x^2 + k_y^2 + k_z^2)p(\vec{x}, t) = -\frac{\omega^2}{v^2}p(\vec{x}, t)$$

- Any values of k_x, k_y, k_z satisfy the equation, provided that

$$\omega = v \sqrt{k_x^2 + k_y^2 + k_z^2} = v|\vec{k}|$$

- If $k_y = k_z = 0$ then $p(\vec{x}, t) = p_0 e^{i(k_x x - \omega t)}$ but this describes a wave propagating in the $+x$ direction.

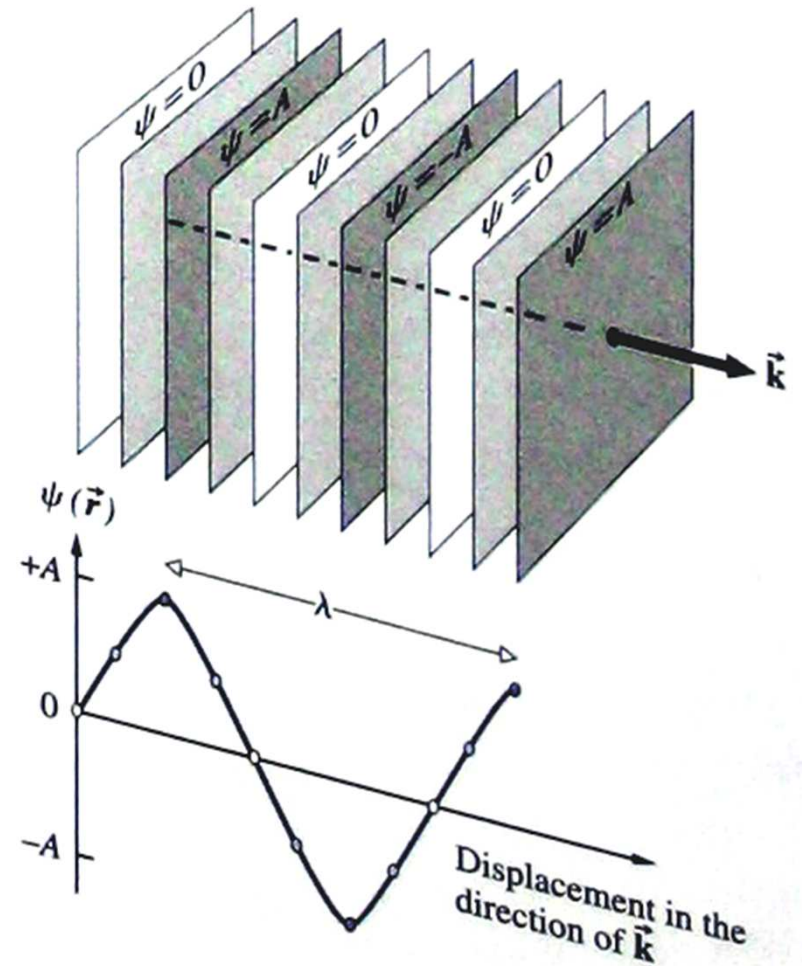
Waves in Three Dimensions

$$p(\vec{x}, t) = p_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

- The vector, $\vec{k}$, points in the direction of propagation
- The wavelength is $\lambda = 2\pi/|\vec{k}|$
- How do we visualize this solution?
 - Pressure is equal at all points $\vec{x}$ such that $\vec{k} \cdot \vec{x} - \omega t = \phi$ where ϕ is some constant phase.
 - Let $\vec{x}'$ be some other point such that $\vec{k} \cdot \vec{x}' - \omega t = \phi$
 - We can write $\vec{x}' = \vec{x} + \vec{u}$ and this tells us that $\vec{k} \cdot \vec{u} = 0$.
 - $\vec{k}$ and $\vec{u}$ are perpendicular.
 - *All points in the plane perpendicular to $\vec{k}$ have the same phase.*

Waves in Three Dimensions

- As usual, we are mainly interested in the real component:
$$\psi(\vec{r}, t) = A \cos(\vec{k} \cdot \vec{x} - \omega t)$$
- A wave propagating in the opposite direction would be described by
$$\psi'(\vec{r}, t) = A' \cos(\vec{k} \cdot \vec{x} + \omega t)$$
- The points in a plane with a common phase is called the “wavefront”.



Waves in Three Dimensions

$$\psi(\vec{r}, t) = A \cos(\vec{k} \cdot \vec{x} \mp \omega t)$$

- Sometimes we are free to pick a coordinate system in which to describe the wave motion.
- If we choose the x -axis to be in the direction of propagation, we get back the one-dimensional solution we are familiar with:

$$\psi(\vec{r}, t) = A \cos(kx \mp \omega t)$$

- But in one-dimension we saw that any function that satisfied $f(x \pm vt)$ was a solution to the wave equation.
- What is the corresponding function in three dimensions?

Waves in Three Dimensions

$$\omega = v \sqrt{k_x^2 + k_y^2 + k_z^2} = v |\vec{k}|$$

- General solution to the wave equation are functions that are twice-differentiable of the form:

$$\psi(\vec{r}, t) = C_1 f(\hat{k} \cdot \vec{r} - vt) + C_2 g(\hat{k} \cdot \vec{r} + vt)$$

$$\text{where } \hat{k} = \vec{k} / |\vec{k}|$$

- Just like in the one-dimensional case, these do not have to be harmonic functions.

Example

- Is the function $\psi(\vec{x}, t) = (ax + bt + c)^2$ a solution to the wave equation?

- It should be because we can write it as

$$\psi(\vec{x}, t) = (a(\mathbf{x} + \mathbf{v}t) + c)^2$$

where $v = b/a$ which is of the form $g(x + vt)$

- We can check explicitly:

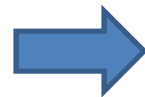
$$\frac{\partial \psi}{\partial x} = 2a(ax + bt + c)$$

$$\frac{\partial \psi}{\partial x} = 2b(ax + bt + c)$$

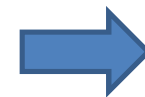
$$\frac{\partial^2 \psi}{\partial x^2} = 2a^2$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2b^2$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$



$$2a^2 = 2b^2/v^2$$



$$v = b/a$$

Example

- Is the function $\psi(\vec{x}, t) = ax^{-2} + bt$, where $a > 0, b > 0$, a solution to the wave equation?
- It is twice differentiable...

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{6a}{x^4} \qquad \frac{\partial^2 \psi}{\partial t^2} = 0$$

- But it is not a solution:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad \Rightarrow \quad \frac{6a}{x^4} = 0$$

- Only true if $a = 0$, which we already said was not the case.
- This is not a solution to the wave equation.

Waves in Two Dimensions

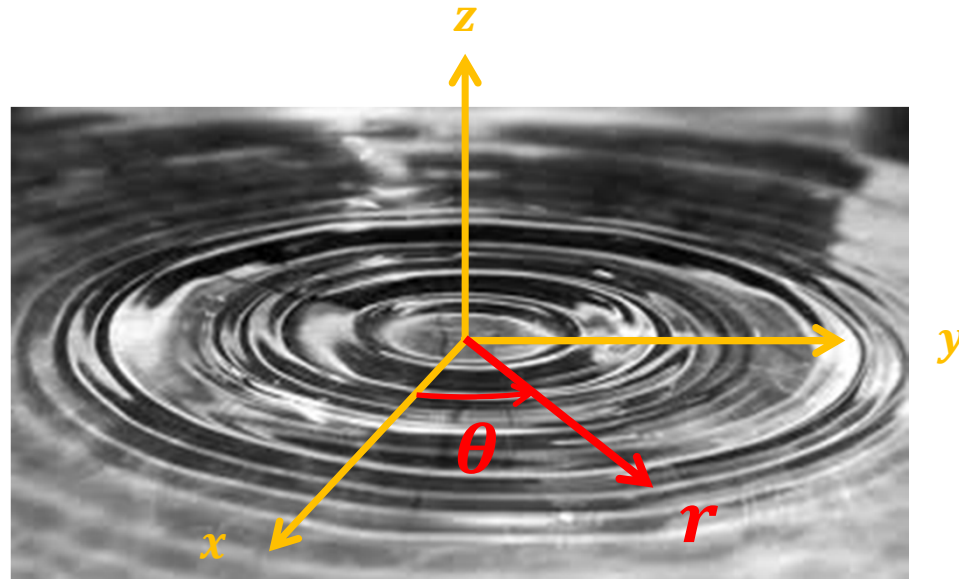
- Plane waves frequently provide a good description of physical phenomena, but this is usually an approximation:



- This looks like a wave... can the wave equation describe this?

Waves in Two Dimensions

- Rotational symmetry:
 - Cartesian coordinates are not well suited for describing this problem.
 - Use polar coordinates instead.
 - Motion should depend on r but should be independent of θ



Waves in Two Dimensions

- Wave equation: $\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$
- How do we write ∇^2 in polar coordinates?

$$\left. \begin{aligned} r &= \sqrt{x^2 + y^2} \\ x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \right\} \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

- Derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

Waves in Two Dimensions

$$\begin{aligned}
 &= \cos \theta \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial r} \right] + \sin \theta \left[\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial r} \right] \\
 &= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2}
 \end{aligned}$$

Similarly,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial u}{\partial x} + r \cos \theta \frac{\partial u}{\partial y}.$$

Taking one more derivative, we see

$$\begin{aligned}
 \frac{\partial^2 u}{\partial \theta^2} &= r \frac{\partial}{\partial \theta} \left[-\sin \theta \frac{\partial u}{\partial x} + \cos \theta \frac{\partial u}{\partial y} \right] \\
 &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r \left[-\sin \theta \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \theta} \right) + \cos \theta \left(\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \theta} \right) \right] \\
 &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r \left[-\sin \theta \left(-r \sin \theta \frac{\partial^2 u}{\partial x^2} + r \cos \theta \frac{\partial^2 u}{\partial x \partial y} \right) + \cos \theta \left(-r \sin \theta \frac{\partial^2 u}{\partial x \partial y} + r \cos \theta \frac{\partial^2 u}{\partial y^2} \right) \right] \\
 &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r^2 \left[\sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \right] \\
 &= -r \frac{\partial u}{\partial r} + r^2 \left[\sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \right]
 \end{aligned}$$

Now we're ready to put everything together:

$$\begin{aligned}
 \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2} - \frac{1}{r} \frac{\partial u}{\partial r} + \sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \\
 &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 u}{\partial x^2} + (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 u}{\partial y^2} - \frac{1}{r} \frac{\partial u}{\partial r}
 \end{aligned}$$

Waves in Two Dimensions

- Laplacian in polar coordinates:

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

- When the geometry does not depend on θ or z :

$$\begin{aligned} \nabla^2 \psi &= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) \end{aligned}$$

- Wave equation:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

Waves in Two Dimensions

- Wave equation:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

- If we assume that $\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$ then the equation is:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\omega^2}{v^2} \psi = 0$$

- Change of variables: Let $\rho = r\omega/v$

$$\frac{\omega^2}{v^2} \frac{\partial^2 \psi}{\partial \rho^2} + \frac{\omega^2}{v^2} \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\omega^2}{v^2} \psi = 0$$

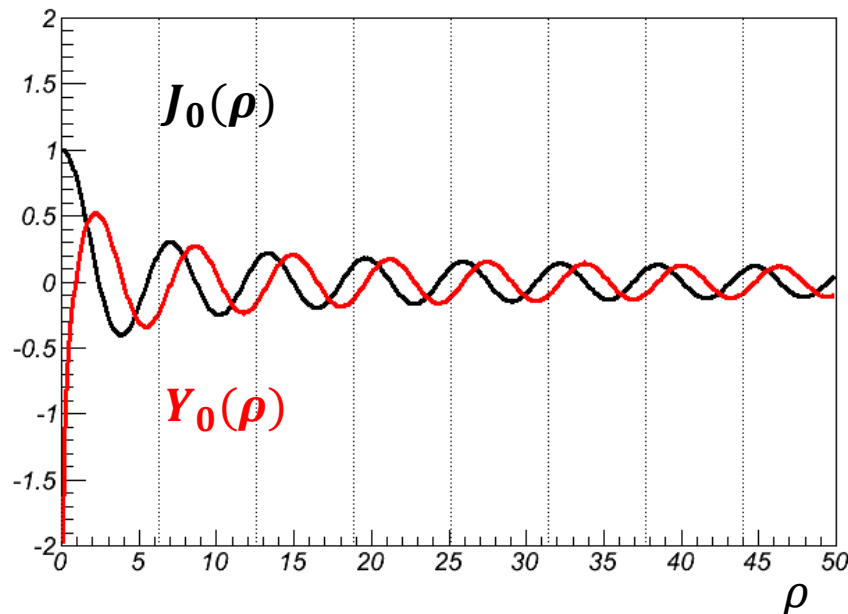
$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \psi(\rho) = 0$$

Waves in Two Dimensions

- Bessel's Equation:

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \psi(\rho) = 0$$

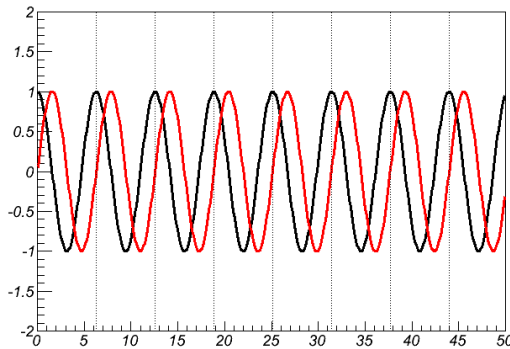
- Solutions are “Bessel functions”: $J_0(\rho)$, $Y_0(\rho)$



Bessel Functions?

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega^2}{v^2} \psi = 0$$

- Solutions: $\sin kx, \cos kx$
- Graphs:

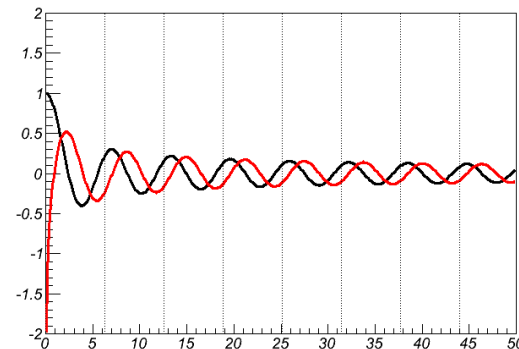


- Series representation:

$$\cos kx = \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^{2n}}{(2n)!}$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\omega^2}{v^2} \psi = 0$$

- Solutions: $J_0(kr), Y_0(kr)$
- Graphs:

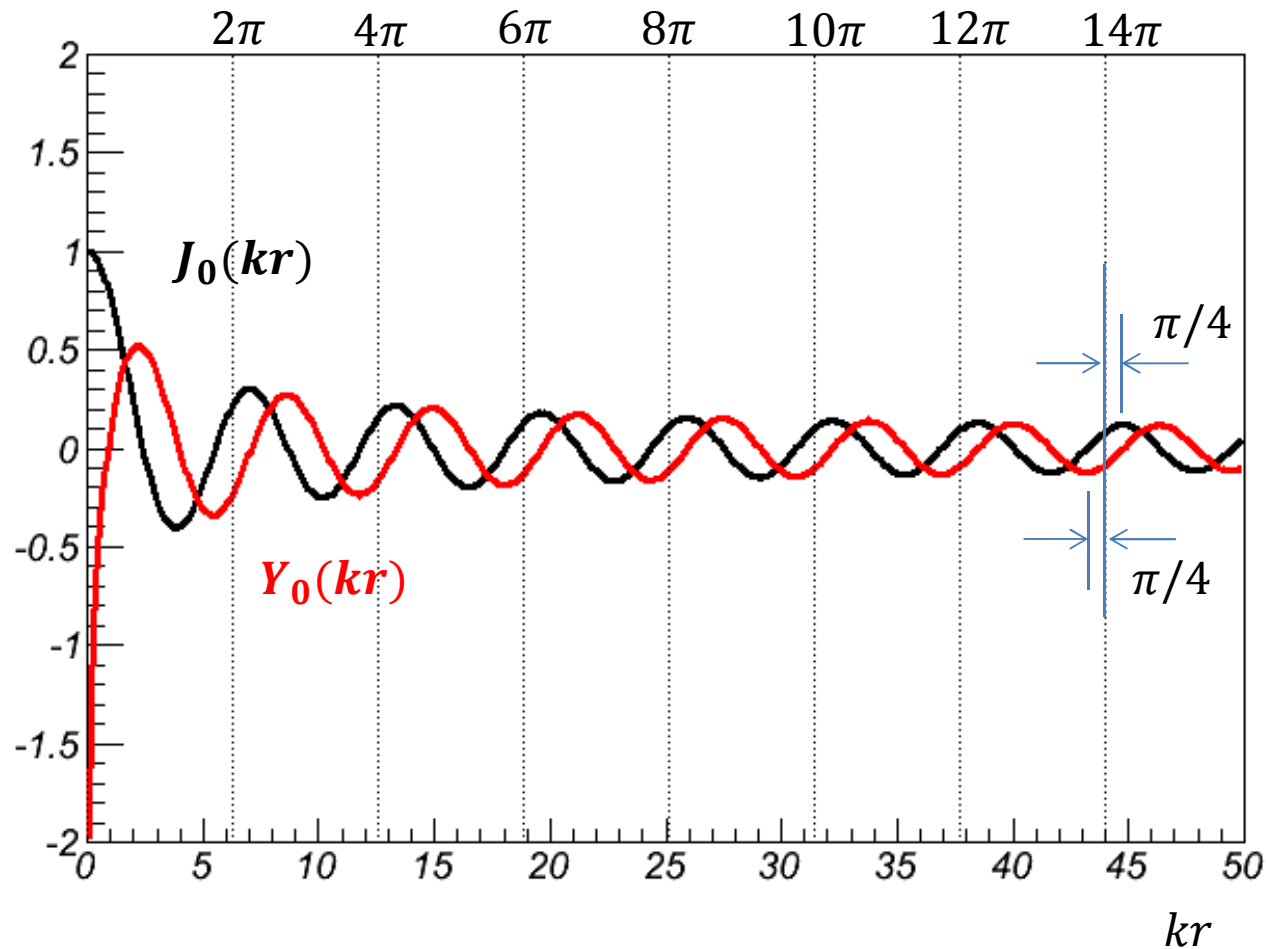


- Series representation:

$$J_0(kr) = \sum_{n=0}^{\infty} \frac{(-1)^n (kr)^{2n}}{2^{2n} (n!)^2}$$

Asymptotic Properties

- At large values of r ...

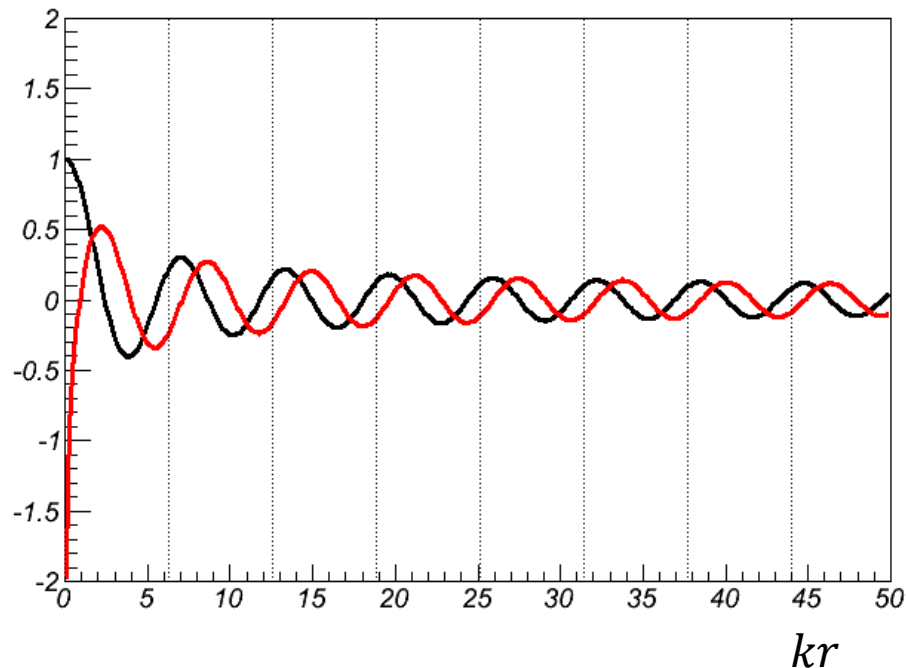


Asymptotic Properties

- When r is large, for example, $kr \gg 1$

$$J_0(kr) \approx \sqrt{2/\pi} \frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$

$$Y_0(kr) \approx \sqrt{2/\pi} \frac{\sin(kr - \pi/4)}{\sqrt{kr}}$$



Energy

- The energy carried by a wave is proportional to the square of the amplitude.
- When $\psi(r, t) \sim A \frac{\cos kr}{\sqrt{r}}$ the energy density decreases as $1/r$
- But the wave is spread out on a circle of circumference $2\pi r$
- The total energy is constant, independent of r
- At large r they look like plane waves:

