

Physics 42200  
**Waves & Oscillations**

Lecture 19 – French, Chapter 7

Spring 2016 Semester

**Announcement:**

There will be no class on Friday March 11<sup>th</sup>.

There is a midterm on Thursday, March 10<sup>th</sup>:

Date: Thursday, March 10<sup>th</sup>  
 Time: 8:00 – 10:00 pm  
 Room: MSEE B012?  
 Material: French, chapters 1-8

**Reflection from Boundaries**

- Consider a pulse propagating on a string, moving to the right, towards a fixed end:



- We learned that this can be represented as a linear combination of normal modes:

$$y(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

- Since the problem is linear, we just need to analyze one normal mode to see what happens next...

### Reflection from a Boundary

- Considering just one normal mode:

$$y_n(x, t) = a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

- Trigonometric identity:

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

- Re-write this as two travelling waves:

$$y_n(x, t) = \frac{a_n}{2} \left[ \sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right) \right]$$

- At the end of the string,  $x = L$ , this is:

$$y_n(L, t) = \pm \frac{A_n}{2} [\sin(\omega_n t) - \sin(\omega_n t)] = 0$$

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### Reflection from a Boundary

$$y_n(L, t) = \pm \frac{a_n}{2} [\sin(\omega_n t) - \sin(\omega_n t)] = 0$$

- The component of the wave that moves to the left has a displacement that is equal and opposite to the displacement of the incident wave.

Another way to see this:

- The function  $y(x, t)$  that describes the shape of the string has two components that move in opposite directions:  $y(x, t) = y_i(x, t) + y_r(x, t)$
- At the end of the string, the two components are equal and opposite, which ensures that the boundary condition  $y(L, t) = 0$  is satisfied.

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### Reflection from a Boundary



- Wave equation for the potential difference between the conductors along a transmission line:

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2}$$

- If the inner and outer conductor are shorted at  $x = L$ , then the potential difference is zero.

$$V(L, t) = 0$$

- A reflected pulse will propagate back towards the source with opposite amplitude.

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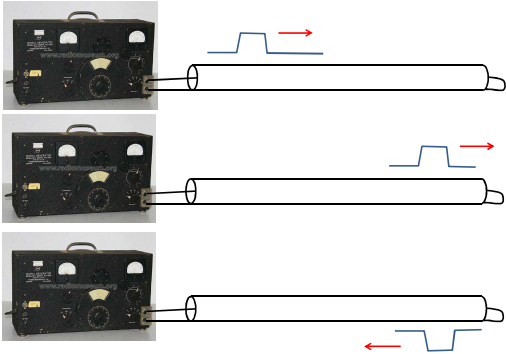
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### Reflection from a Boundary




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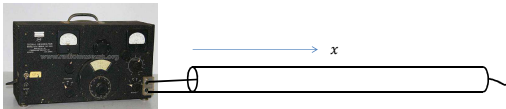
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### Reflection from a Boundary



- If the end of the transmission line is open, then the incident pulse produces a voltage across the end:  

$$V(L, t) = V_L(t)$$
- This acts like a source for a wave propagating to the left.
- The reflected wave is not inverted

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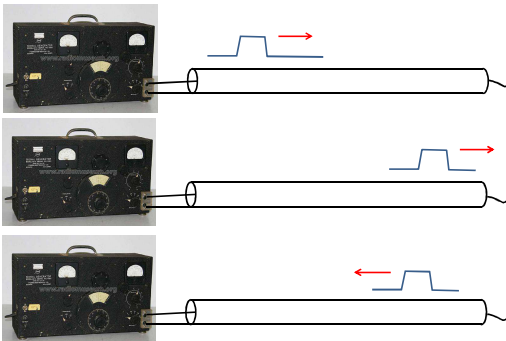
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### Reflection from a Boundary




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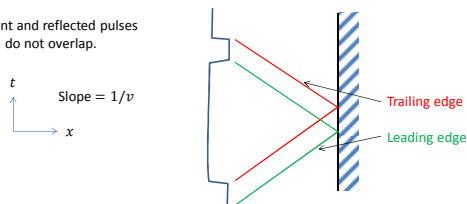
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## Reflection from a Boundary

- So far we have considered just two cases:
  - Reflection with inversion
  - Reflection without inversion
- We can draw a graphical description of the incident and reflected waves:

Incident and reflected pulses do not overlap.




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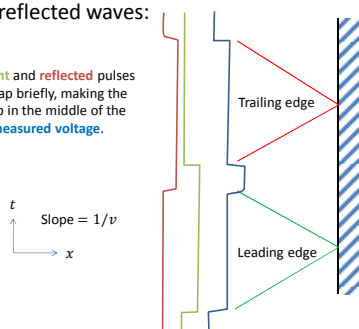
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## Reflection from a Boundary

- We can draw a graphical description of the incident and reflected waves:

Incident and reflected pulses overlap briefly, making the bump in the middle of the measured voltage.




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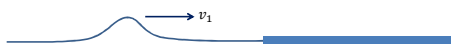
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## Reflection from a Discontinuity

- Suppose a pulse propagates on a string with an abrupt change in mass per unit length:



- Velocity in each section:
 
$$v_1 = \sqrt{T/\mu_1} \quad v_2 = \sqrt{T/\mu_2}$$
- The function  $y(x, t)$  needs three components:
  - Incident pulse:  $y_i(x - v_1 t)$
  - Reflected pulse:  $y_r(x + v_1 t)$
  - Transmitted pulse:  $y_t(x - v_2 t)$
- Boundary conditions:
  - Continuity of the function  $y(x, t)$  and its derivative.

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### Reflection from a Discontinuity

- Function for the wave in the string with  $\mu_1$ :  

$$y_1(x, t) = y_i(x - v_1 t) + y_r(x + v_1 t)$$
- Function for the wave in string with  $\mu_2$ :  

$$y_2(x, t) = y_t(x - v_2 t)$$
- Continuity at the boundary at  $x = 0$ :  

$$y_i(0, t) + y_r(0, t) = y_t(0, t)$$

$$\frac{\partial y_i}{\partial x} + \frac{\partial y_r}{\partial x} = \frac{\partial y_t}{\partial x}$$
- But there is a relation between the derivatives:
  - Let  $u = x - vt$
  - Then  $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial y}{\partial u}$  and  $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial t} = -v \frac{\partial y}{\partial u}$

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### Reflection from a Discontinuity

- $$\frac{\partial y}{\partial x} = \pm \frac{1}{v} \frac{\partial y}{\partial t}$$
- Continuity at the boundary at  $x = 0$ :  

$$y_i(0, t) + y_r(0, t) = y_t(0, t)$$

$$\frac{\partial y_i}{\partial x} + \frac{\partial y_r}{\partial x} = \frac{\partial y_t}{\partial x}$$

$$-\frac{1}{v_1} \frac{\partial y_i}{\partial t} + \frac{1}{v_1} \frac{\partial y_r}{\partial t} = -\frac{1}{v_2} \frac{\partial y_t}{\partial t}$$
  - This can be written:  

$$v_2 y_i'(0, t) - v_2 y_r'(0, t) = v_1 y_t'(t)$$
  - Now we can integrate with respect to  $t$ :  

$$v_2 y_i(0, t) - v_2 y_r(0, t) = v_1 y_t(0, t)$$

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### Reflection from a Discontinuity

- $$y_i(0, t) + y_r(0, t) = y_t(0, t)$$
- $$v_2 y_i(0, t) - v_2 y_r(0, t) = v_1 y_t(0, t)$$
- Since we specified the initial function  $y_i(x, t)$  we just have two equations in two unknowns:  

$$-y_r + y_t = y_i$$

$$v_2 y_r + v_1 y_t = v_2 y_i$$

$$\begin{pmatrix} -1 & 1 \\ v_2 & v_1 \end{pmatrix} \begin{pmatrix} y_r \\ y_t \end{pmatrix} = \begin{pmatrix} y_i \\ v_2 y_i \end{pmatrix}$$

$$y_r = y_i \begin{pmatrix} v_2 - v_1 \\ v_2 + v_1 \end{pmatrix}$$

$$y_t = y_i \begin{pmatrix} 2v_2 \\ v_2 + v_1 \end{pmatrix}$$

Just use Kramer's rule...

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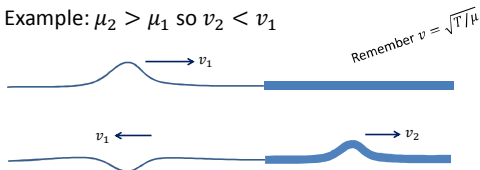
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## Reflection from a Discontinuity

- Reflection coefficient:  $\rho = (v_2 - v_1)/(v_2 + v_1)$
- Transmission coefficient:  $\tau = 2v_2/(v_2 + v_1)$
- Example:  $\mu_2 > \mu_1$  so  $v_2 < v_1$



(This special case applies to the string, but not in general...)

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## Example from Optics

- The index of refraction is defined as the ratio:

$$n = \frac{c}{v}$$

- Speed of light in a dense medium:  $v = c/n$

- Reflection coefficient:

$$\rho = \frac{v_2 - v_1}{v_2 + v_1} = \frac{1/n_2 - 1/n_1}{1/n_2 + 1/n_1} = \frac{n_1 - n_2}{n_1 + n_2}$$

- Phase reversal when  $n_2 > n_1$ , but not when  $n_1 > n_2$

- Transmission coefficient:

$$\tau = \frac{2v_2}{v_2 + v_1} = \frac{2/n_2}{n_1 + n_2}$$

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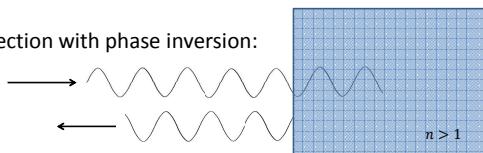
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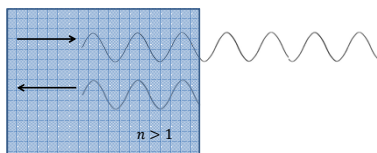
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## Reflection

Reflection with phase inversion:



Reflection without phase inversion:




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## Reflection from a Discontinuity

- But be careful! So far we have assumed that the tension on both sides of the boundary are equal.

$$v_1 = \sqrt{T/\mu_1} \quad v_2 = \sqrt{T/\mu_2}$$

- In other situations this is not always the case:

Solid	Liquid	Gas
$v = \sqrt{Y/\rho}$ $Y$ is Young's modulus	$v = \sqrt{K/\rho}$ $K$ is the bulk modulus	$v = \sqrt{\gamma p/\rho}$ $p$ is the gas pressure $\gamma$ is a property of the gas.

- The restoring force can be produced by different physical effects.
- Next, let's look at how energy propagates in the medium...

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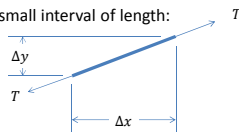
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## Energy Carried by a Pulse

- Potential energy is stored in an elastic string when it is stretched into the shape of a pulse.
- Potential energy in one small interval of length:



- Work needed to stretch the string in the vertical direction:

$$\Delta W = T \int_0^{\Delta y} \frac{y}{\Delta x} dy = \frac{1}{2} \frac{T}{\Delta x} (\Delta y)^2$$

- Work per unit length:

$$\frac{\Delta W}{\Delta x} = \frac{1}{2} T \left( \frac{\Delta y}{\Delta x} \right)^2$$

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## Energy Carried by a Pulse

- Work per unit length:

$$\frac{\Delta W}{\Delta x} = \frac{1}{2} T \left( \frac{\Delta y}{\Delta x} \right)^2$$

- If the pulse maintains this shape but moves with velocity  $v$  then this is the potential energy per unit length.

- Total potential energy is

$$U = \frac{1}{2} T \int \left( \frac{\partial y}{\partial x} \right)^2 dx$$

- Written in terms of linear density and velocity,  $T = \mu v^2$ ,

$$U = \frac{1}{2} \mu v^2 \int \left( \frac{\partial y}{\partial x} \right)^2 dx$$

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## Power Carried by a Wave

- A pulse has a finite amount of energy that moves with speed  $v$
- It is also convenient to describe harmonic waves

$$y(x, t) = A \cos\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

which extend in space over many wavelengths

- First derivative:

$$y'(x) = -\frac{2\pi A}{\lambda} \sin\frac{2\pi x}{\lambda}$$

- Energy in one wavelength (cycle):

$$\begin{aligned} U' &= \frac{1}{2} \mu v^2 \int_0^\lambda \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2} \mu v^2 \frac{\lambda}{2} \left(\frac{2\pi A}{\lambda}\right)^2 \\ &= \frac{1}{2} \lambda \mu \omega^2 A^2 \end{aligned}$$

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## Power Carried by a Wave

- Energy per cycle:

$$U' = \frac{1}{2} \lambda \mu \omega^2 A^2$$

- Cycles passing a point in space per unit time:  $v/\lambda$
- Average power carried by the wave:

$$P = \frac{1}{2} \mu \omega^2 A^2 v = \frac{1}{2} Z \omega^2 A^2$$

- Depends on the characteristic impedance of the medium
  - In this case,  $Z = \mu v = T/v$
- Also depends on the properties of the wave
  - Amplitude and frequency

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## Impedance

- For the string, the impedance is

$$Z = \mu v = T/v = \sqrt{T\mu}$$

$$\begin{aligned} v^2 &= \frac{T}{\mu} \\ v &= \sqrt{T/\mu} \end{aligned}$$

- In general, we can describe the “impedance” as the ratio of the transverse “force”, divided by the “flow” in the x-direction.
- What is the impedance of a transmission line?
  - Recall that  $v = 1/\sqrt{L'C'}$ ,  $k = 2\pi/\lambda = \omega/v$
  - When  $V(x, t) = V_0 e^{-i(kx - \omega t)}$  (this is the “force”)

$$I(x, t) = -\frac{1}{X} \frac{\partial V}{\partial x} \approx -\frac{1}{i\omega L'} \frac{\partial V}{\partial x} = \frac{v}{L'} V(x, t) = \sqrt{\frac{C'}{L'}} V(x, t)$$

- Impedance is  $Z = V/I = \sqrt{L'/C'}$

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## Transmission and Reflection

- For the pulse propagating on the string we had:  

$$\rho = (v_2 - v_1)/(v_2 + v_1)$$

$$\tau = 2v_2/(v_2 + v_1)$$
- We want to write this in terms of the properties of the medium, not just the velocity.
- In this case,  

$$Z_1 = T/v_1 = \sqrt{T\mu_1} \quad Z_2 = T/v_2 = \sqrt{T\mu_2}$$
- General expressions:

$$\rho = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\tau = \frac{2Z_2}{Z_2 + Z_1}$$

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## Reflections in Elastic Media

Solid	Liquid	Gas
$v = \sqrt{Y/\rho}$ $Y$ is Young's modulus $Y \sim 10^{10} \text{ N/m}^2$	$v = \sqrt{K/\rho}$ $K$ is the bulk modulus $K \sim 10^9 \text{ N/m}^2$	$v = \sqrt{\gamma p/\rho}$ $p$ is the gas pressure $\gamma$ is a property of the gas. For air, $\gamma p \sim 1.42 \times 10^5 \text{ N/m}^2$

- Consider reflection coefficients for a typical interface:
  - Air,  $v_{\text{air}} = 340 \text{ m/s}$ ,  $Z_{\text{air}} = 417 \sqrt{\text{kg/s}}$
  - Water,  $v_{\text{water}} = 1500 \text{ m/s}$ ,  $Z_{\text{water}} = 1.47 \times 10^6 \sqrt{\text{kg/s}}$
- Reflection coefficient:  

$$\rho = \frac{Z_1 - Z_2}{Z_1 + Z_2} = -0.9994 \approx -1$$
- Transmission coefficient:  

$$\tau = \frac{2Z_1}{Z_1 + Z_2} = 0.0006$$
- How much power is transferred across the interface?

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## Transmitted Power

- Transmitted amplitude:  $A_t = \tau A$
- Power carried by a wave:  

$$P = \frac{1}{2} Z \omega^2 A^2$$
- Incident power:  

$$P_i = \frac{1}{2} Z_1 \omega^2 A^2$$
- Transmitted power:  

$$P_t = \frac{1}{2} Z_2 \omega^2 (\tau A)^2 = \frac{2Z_2 Z_1^2}{(Z_1 + Z_2)^2} \omega^2 A^2$$

$$\frac{P_t}{P_i} = \frac{4Z_1^2}{(Z_1 + Z_2)^2}$$
- Reflected power:  $P_r = P_i - P_t$  (conserves energy).

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