PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 18 – French, Chapter 6

Spring 2016 Semester

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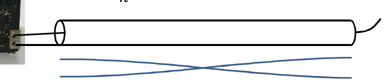
Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
  - **Open circuit: $I(L) = 0$**
  - Short circuit: $V(L) = 0$

$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi\, v \frac{n}{2L} \qquad f_n = \frac{nv}{2L}$$



If $L = 1\ m$ and $v = 20\ cm/ns$ … $\quad n = 1 \quad f = 100\ MHz$

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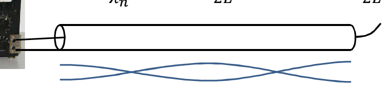

If $L = 1\ m$ and $v = 20\ cm/ns$ … $\quad n = 2 \quad f = 200\ MHz$

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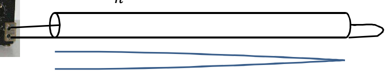

If $L = 1\ m$ and $v = 20\ cm/ns$ ... $\quad n = 1 \quad f = 50\ MHz$

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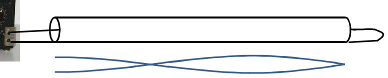
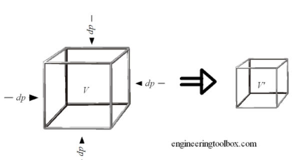


If $L = 1\ m$ and $v = 20\ cm/ns$ ... $\quad n = 2 \quad f = 150\ MHz$

Longitudinal Waves in a Gas



- Increased pressure on a volume of gas decreases its volume
- Bulk modulus of elasticity is defined

$$K = -V\frac{dp}{dV}$$

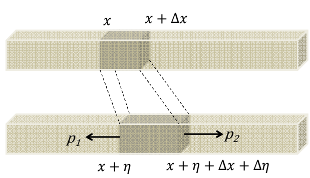
Longitudinal Waves in a Gas

- Equations of state for a gas:
  - Ideal gas law: $pV = NkT$
  - Adiabatic gas law: $pV^{\gamma} = constant$
- In an adiabatic process, no heat is absorbed
  - Absorbing heat would remove mechanical energy from a system
  - Propagation of sound waves through a gas is an example of an adiabatic process
- Bulk modulus calculated from equation of state:

$$V^{\gamma} dp + \gamma p V^{\gamma-1} dV = 0$$
$$\frac{dp}{dV} = -\gamma p / V$$
$$K = -V \frac{dp}{dV} = \gamma p$$

Longitudinal Waves in a Gas

- By analogy with the solid rod, we consider an element of gas at position x of thickness Δx that is displaced by a distance $\eta(x)$:



Longitudinal Waves in a Gas

- Wave equation:

$$\frac{\partial^2 \eta}{dx^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2}$$

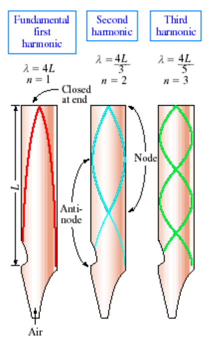
- For a sold rod, $v = \sqrt{Y/\rho}$
- For a gas, $v = \sqrt{K/\rho} = \sqrt{\gamma p/\rho}$
- Changes in pressure and density are very small compared with the average pressure and density.
- At standard temperature and pressure, air has

$$\gamma = 1.40$$
$$p = 101.3 \text{ kPa}$$
$$\rho = 1.2 \text{ kg/m}^3$$
$$v = \sqrt{\frac{(1.40)(101.3\times10^3 N/m^2)}{(1.2\ kg/m^3)}} = 343\ m/s$$

The Physics of Organ Pipes

Resonant Cavities

- Air under pressure enters at the bottom
  - Entering air rapidly oscillates between the pipe and the lip
  - The lower end is a displacement anti-node
- Top end can be open or closed
  - Open end is a pressure node/displacement anti-node
  - Closed end a displacement node/pressure anti-node



Wave Propagation

- The wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- We worked out solutions that satisfied specific boundary conditions.
- A general solution is any function that is of the form

$$y(x,t) = f(x \pm vt)$$

- Are these two pictures compatible?

Wave Propagation

- Solutions for normal modes:

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

$$\omega_n = \frac{\pi n v}{L}$$

- Trigonometric identity:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

- This gives,

$$y_n(x,t) = \frac{1}{2}\left[\sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right]$$

Wave Propagation

$$y_n(x,t) = \frac{1}{2}\left[\sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right]$$

- Write this as

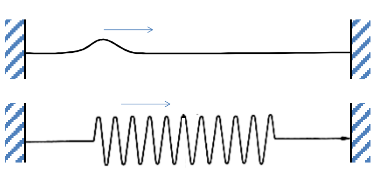
$$y_n(x,t) = \frac{1}{2}[\sin(k(x+vt)) + \sin(k(x-vt))]$$

$$k = \frac{n\pi}{L}$$

- This is the equation for two sine-waves moving in opposite directions.
- The text refers to these as "progressive waves".
- The "standing waves" that satisfy the boundary conditions are the superposition of "progressive waves" that move in opposite directions.

Wave Propagation

- Waves can propagate in either direction.
- Easiest to visualize in terms of a pulse, or wave packet:



- If this disturbance is far from the ends, the effect is the same as letting $L \to \infty$

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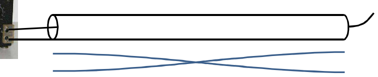

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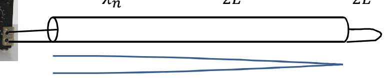

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Wave Propagation

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n)$$

$y(0,t) = y(L,t) = 0$

- In general, we could write

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n)$$

- In the limit where the disturbance is very far from either boundary, the Fourier sine transform is:

$$B(k) = \int_{-\infty}^{\infty} u(x)\sin(kx)\,dx$$

- Similarly, we can define the Fourier cosine transform:

$$A(k) = \int_{-\infty}^{\infty} u(x)\cos(kx)\,dx$$

Wave Propagation

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- Similarly, we can define the Fourier cosine transform:

$$A(k) = \int_{-\infty}^{\infty} u(x)\cos(kx)\,dx$$

- The original function is represented by:

$$u(x) = \frac{1}{\pi}\int_{0}^{\infty} A(k)\cos(kx)\,dk + \frac{1}{\pi}\int_{0}^{\infty} B(k)\sin(kx)\,dk$$

- If $A(k) = A(-k)$ and $B(k) = -B(-k)$ then we can make this more symmetric:

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\cos(kx)\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} B(k)\sin(kx)\,dk$$

Wave Propagation

- To make this even more symmetric we can change slightly the definition of $A(k)$ and $B(k)$:

$$B(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)\sin(kx)\,dx$$

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)\cos(kx)\,dx$$

- Then,

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\cos(kx)\,dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B(k)\sin(kx)\,dk$$

Wave Propagation

- Previously, we interpreted the coefficients a_n as the amplitude of the normal mode with frequency ω_n
  - wavelength $\lambda_n = 2L/n$
  - wavenumber $k_n = 2\pi/\lambda_n = \pi n/L$
- Now, we interpret $A(k)$ and $B(k)$ as the amplitude for harmonic waves with wavenumbers between k and $k + dk$.
- It can be important to decompose a pulse into its frequency components because in real materials, the nature of wave propagation can depend on the frequency.

Example

- Consider a pulse that has a Gaussian shape:

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- Special case:
  - Peak position is at $x = 0$
  - Width of the peak is $\sigma = 1$
- Other Gaussian functions can be transformed into this special case by linear change of variables.
- What is the continuous Fourier transform?

Example

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx)\, dx$$

- The Gaussian function $g(x)$ is an even function:
$$g(x) = g(-x)$$
- The function $\sin(kx)$ is an odd function:
$$\sin(-kx) = -\sin(kx)$$
- This integral must vanish…
$$B(k) = 0$$

Example

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx)\, dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2} \cos(kx)\, dx$$

- From your table of integrals:
$$\int_{-\infty}^{\infty} e^{-ax^2} \cos bx\, dx = \sqrt{\frac{\pi}{a}} e^{-b^2/4a}$$
- In this case, $a = 1/2$ and $b = k$
$$A(k) = \frac{1}{2\pi} \times \sqrt{2\pi} e^{-k^2/2} = \frac{1}{\sqrt{2\pi}} e^{-k^2/2}$$
- This is a Gaussian distribution of wavenumbers $k = \omega/v$.

Notes about Fourier Transforms

- For the Gaussian pulse,

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

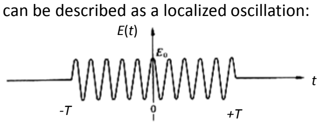
- The amplitudes of the frequency components are:

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma^2/2},\ B(k) = 0$$

- When the pulse is narrow, $\sigma \ll 1$, then the exponent in $A(k)$ is large for a large range of k
  - Since $\omega = v/k$, a narrow pulse has a wide range of frequency components.
- Conversely, a wide pulse has a narrow range of frequencies.

Another Example

- A photon can be described as a localized oscillation:



At $x = 0$, $E(t) = \begin{cases} E_0 \cos(\omega t) & \text{when } |t| < T \\ 0 & \text{otherwise} \end{cases}$

At $t = 0$, $E(x) = \begin{cases} E_0 \cos(kx) & \text{when } |x| < cT \\ 0 & \text{otherwise} \end{cases}$

$$A(k') = \frac{E_0}{\sqrt{2\pi}} \int_{-cT}^{cT} \cos(kx)\cos(k'x)\,dx$$

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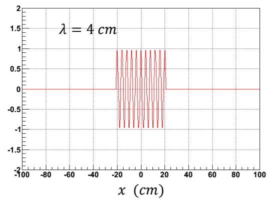
- Trigonometric identity:

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$

$$A(k') = \frac{E_0}{\sqrt{2\pi}}\left[\frac{\sin\big((k-k')cT\big)}{k-k'} + \frac{\sin\big((k+k')cT\big)}{k+k'}\right]$$

Another Example



$k = \frac{2\pi}{\lambda} = 1.571\ cm^{-1}$