Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 17 – French, Chapter 6

Spring 2016 Semester

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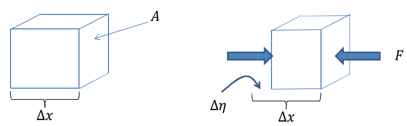
Other Continuous Systems

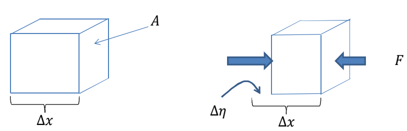


$$\frac{F}{A} = Y\,\frac{\Delta\eta}{\Delta x}$$

Equal and opposite forces squish the cube of elastic material. Net force is zero so there is no acceleration.

$$F(x) = AY\frac{d\eta}{dx}$$

Other Continuous Systems



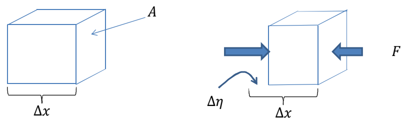
Suppose the force changes over distance Δx…

$$F(x+\Delta x) = F(x) + \frac{dF}{dx}\Delta x$$

Net force is

$$F(x+\Delta x) - F(x) = \frac{dF}{dx}\,\Delta x = AY\frac{d^2\eta}{dx^2}\,\Delta x$$

Other Continuous Systems



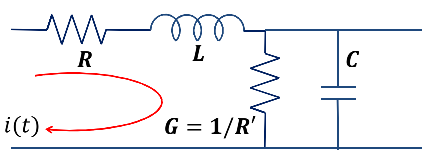
$$AY\frac{d^2\eta}{dx^2}\,\Delta x = \rho A\Delta x\,\frac{d^2\eta}{dt^2}$$

$$\frac{d^2\eta}{dx^2} = \frac{\rho}{Y}\frac{d^2\eta}{dt^2}$$

$$\frac{\partial^2\eta}{dx^2} = \frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2} \qquad v = \sqrt{Y/\rho}$$

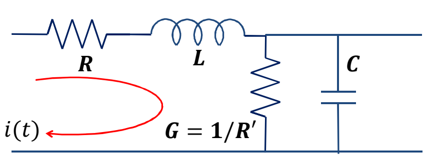
Electrical Circuits

- First, consider one "lump" of a circuit:



- It is convenient to describe the resistor that is in parallel with the capacitor in terms of its conductance, $G = 1/R'$.

Electrical Circuits

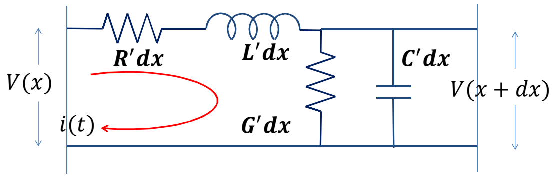


- Calculate the total impedance of the lump:

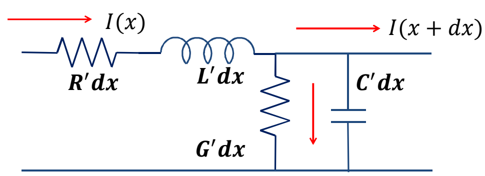
$$\left.\begin{aligned} Z_R &= R \\ Z_L &= i\omega L \\ Z_C &= \frac{1}{i\omega C} \\ Z_G &= 1/G \end{aligned}\right\} \qquad \begin{aligned} X &= R + i\omega L \\ Y &= G + i\omega C \end{aligned} \qquad \begin{aligned} V(t) &= V_0\, e^{i\omega t} \\ \Delta V(t) &= I(t)\, X \\ \Delta I(t) &= V(t)\, Y \end{aligned}$$

Electrical Circuits

- Suppose the resistance, inductance, capacitance and conductance were distributed uniformly with length:
  - Let R' be the resistance per unit length, L' be the inductance per unit length, etc…
- Consider the voltage on either side of the lump:



Electric Circuits



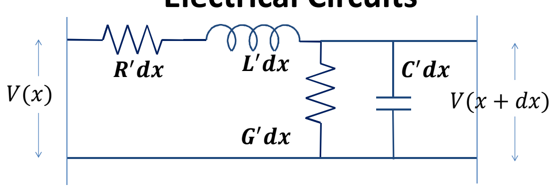
- Current flowing through G' and C' is

$$\Delta I = \frac{V(x)}{Z_{G'+C'}} = V(x)Y$$

$$I(x+dx) = I(x) - V(x)Y$$

$$\frac{\partial I}{\partial x} = \frac{I(x+dx) - I(x)}{dx} = -V(x)Y$$

Electrical Circuits



- Voltage drop across the lump:

$$V(x+dx) = V(x) - I(x)X$$

$$\frac{\partial V}{\partial x} = \frac{V(x+dx) - V(x)}{dx} = -I(x)X$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\partial I}{\partial x}X = XY\,V(x)$$

Electrical Circuits

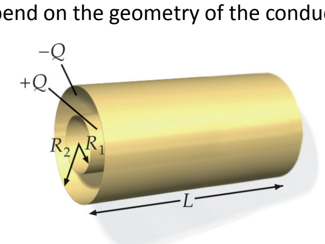
- When we assume that the voltage is of the form
$$V(x,t) = V(x)e^{i\omega t}$$
$$\frac{\partial^2 V}{\partial t^2} = -\omega^2 V(x)$$
- Using the previous result, $\frac{\partial^2 V}{\partial x^2} = XY\,V(x)$ we get:
$$\frac{\partial^2 V}{\partial x^2} + \frac{XY}{\omega^2}\frac{\partial^2 V}{\partial t^2} = 0$$
- Does this resemble the wave equation?
  - Expand out $XY = (R' + i\omega L')(G' + i\omega C')$
  - When R' and G' are small, which is frequently the case then $XY \approx -\omega^2 L'C'$

Electrical Circuits

- Wave equation:
$$\frac{\partial^2 V}{\partial x^2} = L'C'\frac{\partial^2 V}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}$$
- Speed of wave propagation is
$$v = \frac{1}{\sqrt{L'C'}}$$

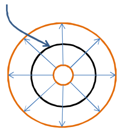
Current in a Transmission Line

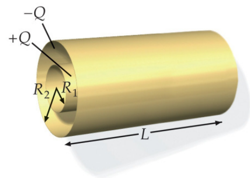
- Speed of wave propagation depends on inductance per unit length and capacitance per unit length
- These depend on the geometry of the conductors
- Example:



Gauss's Law

Radius of Gaussian surface is r





$$\oint_S \hat{n} \cdot \vec{E}\, dA = 2\pi r \ell E = \frac{Q_{inside}}{\epsilon_0}$$

$\vec{E}$ is uniform everywhere on the Gaussian surface
Surface area is $A = 2\pi r \ell$
Linear charge density: $\lambda = Q/\ell$

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

Potential Difference and Capacitance

Work needed to move a charge between the conductors:

$$V = -\int_{R_2}^{R_1} \vec{E} \cdot d\vec{r} = -\frac{\lambda}{2\pi\epsilon_0}\int_{R_2}^{R_1} \frac{dr}{r}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \log\left(\frac{R_2}{R_1}\right)$$

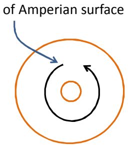
Capacitance is defined by $C = Q/V$

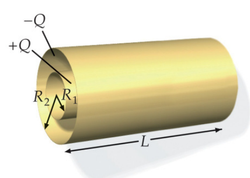
Charge inside is $Q = \lambda\ell$

Capacitance per unit length: $C' = \frac{2\pi\epsilon_0}{\log\left(\frac{R_2}{R_1}\right)}$

Ampere's Law

Radius of Amperian surface is r





Ampere's law: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$

$\vec{B}$ is uniform on the circular path of length $2\pi r$:

$$B = \frac{\mu_0 I}{2\pi r}$$

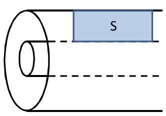
Magnetic Flux and Inductance

Magnetic flux is defined:

$$\phi_m = \int_S \vec{B} \cdot d\vec{a} = \frac{\mu_0 I \ell}{2\pi} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{\mu_0 I \ell}{2\pi} \log\left(\frac{R_2}{R_1}\right)$$

Inductance is defined: $\phi_m = LI$

Inductance per unit length: $L' = \frac{\mu_0}{2\pi} \log\left(\frac{R_2}{R_1}\right)$



Wave Propagation in a Coaxial Cable

- Capacitance per unit length: $C' = \frac{2\pi\epsilon_0}{\log\left(\frac{R_2}{R_1}\right)}$
- Inductance per unit length: $L' = \frac{\mu_0}{2\pi} \log\left(\frac{R_2}{R_1}\right)$
- Speed of wave propagation:

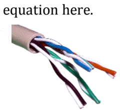
$$v = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$$

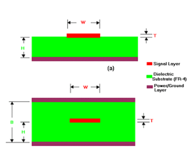
- In practice, the conductors are separated by a dielectric with relative permittivity ϵ_r so the speed of wave propagation is $v = c/\sqrt{\epsilon_r}$

Transmission Lines

- Coaxial:  $\epsilon_r = 2.3,\quad v = 0.66\,c$

Type equation here.

- Twisted pair:  $\epsilon_r = 2.1,\quad v = 0.69\,c$
- Microstrip:
- Stripline:

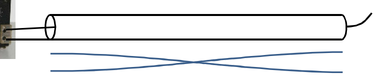


$v \approx 0.6 - 0.8\,c$

Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
  - ***Open circuit:*** $\boldsymbol{I(L) = 0}$
  - Short circuit: $V(L) = 0$

$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi\, v \frac{n}{2L}$ $\quad f_n = \frac{nv}{2L}$



If $L = 1\,m$ and $v = 20\,cm/ns$ … $\quad n = 1 \quad f = 100\,MHz$

Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
  - ***Open circuit:*** $\boldsymbol{I(L) = 0}$
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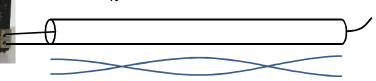
$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi\, v \frac{n}{2L}$ $\quad f_n = \frac{nv}{2L}$

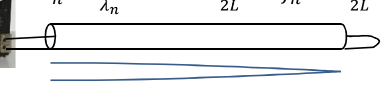


If $L = 1\,m$ and $v = 20\,cm/ns$ … $\quad n = 2 \quad f = 200\,MHz$

Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
  - Open circuit: $I(L) = 0$
  - ***Short circuit:*** $\boldsymbol{V(L) = 0}$

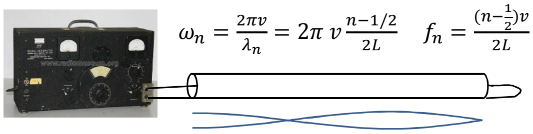
$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi\, v \frac{n-1/2}{2L}$ $\quad f_n = \frac{(n-\frac{1}{2})v}{2L}$



If $L = 1\,m$ and $v = 20\,cm/ns$ … $\quad n = 1 \quad f = 50\,MHz$

Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
  - Open circuit: $I(L) = 0$
  - ***Short circuit: $V(L) = 0$***



$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi\, v \frac{n-1/2}{2L} \qquad f_n = \frac{(n-\frac{1}{2})v}{2L}$$

If $L = 1\ m$ and $v = 20\ cm/ns$ … $\quad n = 2 \quad f = 150\ MHz$

Fourier Analysis

- Wave equation: $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$
- Normal modes:

$$f_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

(when $f_n(0,t) = f_n(L,t) = 0$)

- The general initial value problem specifies the initial displacement and velocity at $t = 0$
- How can we represent the general solution as the sum of normal modes?

Fourier Analysis

- The general solution can be expressed

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

- Initial conditions:

$$y(x,0) = u(x)$$
$$\dot{y}(x,0) = v(x)$$

- How do we determine the constants A_n and δ_n?
- At $t = 0$ the general solution looks like this:

$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ where $B_n = A_n \cos\delta_n$

Fourier Analysis

- Fourier transform:

$$B_k = \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) u(x) dx$$

- Really? Let's prove it by demonstration:
- At $t = 0$, $u(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ so we want to calculate

$$\frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) u(x) dx = \frac{2}{L}\sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

Fourier Analysis

Use the trigonometric identity:

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

$$\int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \cos\left(\frac{(k-n)\pi x}{L}\right) dx + \frac{1}{2}\int_0^L \cos\left(\frac{(k+n)\pi x}{L}\right) dx$$

$$= \frac{L}{2(k-n)\pi}\sin((k-n)\pi) + \frac{L}{2(k+n)\pi}\sin((k+n)\pi)$$

This vanishes unless $k = n$ in which case,

$$\frac{1}{2}\int_0^L \cos\left(\frac{(k-n)\pi x}{L}\right) dx \to \frac{1}{2}\int_0^L dx = \frac{L}{2}$$

So we write:

$$\frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \delta_{kn}$$

$$\delta_{kn} = \begin{cases} 0 \text{ if } k \neq n \\ 1 \text{ if } k = n \end{cases}$$

Fourier Analysis

With this result, we can write

$$\frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) u(x) dx = \frac{2}{L}\sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \sum_{n=1}^{\infty} B_n\,\delta_{kn} = B_k$$

Example

- How to describe a square wave in terms of normal modes:

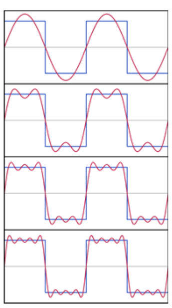
$$u(x) = \begin{cases} +1 \text{ when } 0 < x < \lambda/2 \\ -1 \text{ when } \lambda/2 < x < \lambda \end{cases}$$

$$B_n = \frac{2}{\lambda}\int_0^{\lambda/2} \sin(nkx)dx - \frac{2}{\lambda}\int_{\frac{\lambda}{2}}^{\lambda} \sin(nkx)dx$$

$$= \frac{2}{n\pi}[1 - \cos(n\pi)]$$

$$B_1 = \frac{4}{\pi}, B_3 = \frac{4}{3\pi}, B_5 = \frac{4}{5\pi}, \cdots$$

Example



$$B_n = \frac{2}{n\pi}[1 - \cos(n\pi)]$$

$$B_1 = \frac{4}{\pi}, B_3 = \frac{4}{3\pi}, B_5 = \frac{4}{5\pi}, \cdots$$

$$B_2 = 0, B_4 = 0, B_6 = 0, \cdots$$