

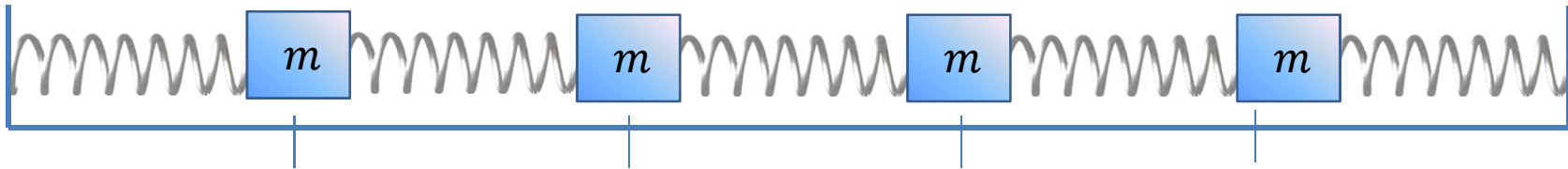
Physics 42200  
**Waves & Oscillations**

Lecture 16 – French, Chapter 6

Spring 2016 Semester

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# Summary



- General solution:

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \theta_k)$$

- Frequencies of normal modes of oscillation:

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- Fourier coefficients:

$$a_k \cos \theta_k = \frac{2}{N} \sum_{n=1}^N x_n(0) \sin\left(\frac{nk\pi}{N+1}\right)$$
$$a_k \omega_k \sin \theta_k = \frac{2}{N} \sum_{n=1}^N \dot{x}_n(0) \sin\left(\frac{nk\pi}{N+1}\right)$$

# Summary



- General solution:

$$y(x, t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) \cos(\omega_k t - \theta_k)$$

- Frequencies of normal modes of oscillation:

$$\omega_k = \frac{k\pi v}{L}$$

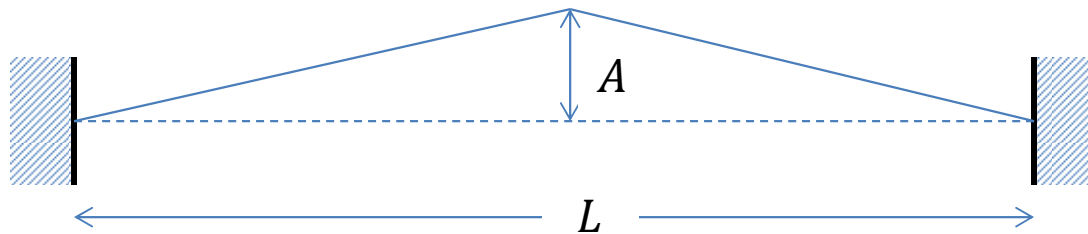
- Fourier coefficients:

$$a_k \cos \theta_k = \frac{2}{L} \int_0^L y(x, 0) \sin\left(\frac{k\pi x}{L}\right) dx$$

$$a_k \omega_k \sin \theta_k = \frac{2}{L} \int_0^L \dot{y}(x, 0) \sin\left(\frac{k\pi x}{L}\right) dx$$

# Example

- When a string is plucked in the middle, what sound will it make?



- This is a question about the amplitudes of the different normal modes of vibration.

$$y(x, t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) \cos(\omega_k t)$$

$$\omega_k = \frac{k\pi v}{L}$$

$$a_k = \frac{2}{L} \int_0^L y(x, 0) \sin\left(\frac{k\pi x}{L}\right) dx$$

# Example

- The initial shape of the string is the function:

$$f(x) = \begin{cases} 2Ax/L & \text{when } x < L/2 \\ 2A - 2Ax/L & \text{when } x > L/2 \end{cases}$$

- Fourier coefficients:

$$\begin{aligned} a_k &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx \\ &= \frac{2}{L} \int_0^{L/2} f(x) \sin\left(\frac{k\pi x}{L}\right) dx \\ &\quad + \frac{2}{L} \int_{L/2}^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx \end{aligned}$$

# Example

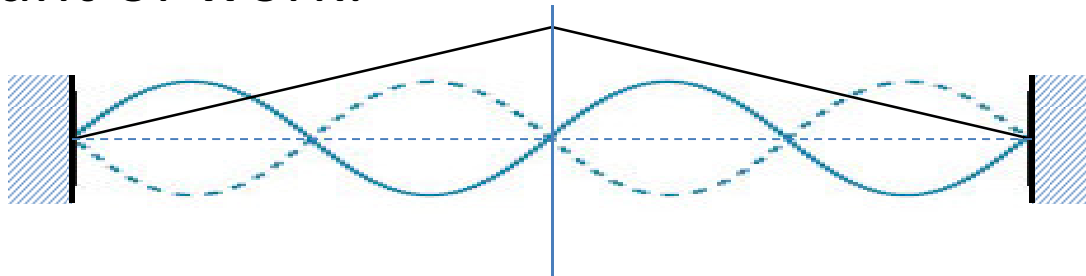
- We have only two kinds of integrals:

$$\int \sin\left(\frac{k\pi x}{L}\right) dx = -\frac{L}{k\pi} \cos\left(\frac{k\pi x}{L}\right)$$

$$\begin{aligned} \int x \sin\left(\frac{k\pi x}{L}\right) dx \\ = -\frac{L}{k\pi} \cos\left(\frac{k\pi x}{L}\right) + \frac{L^2}{k^2\pi^2} \sin\left(\frac{k\pi x}{L}\right) \end{aligned}$$

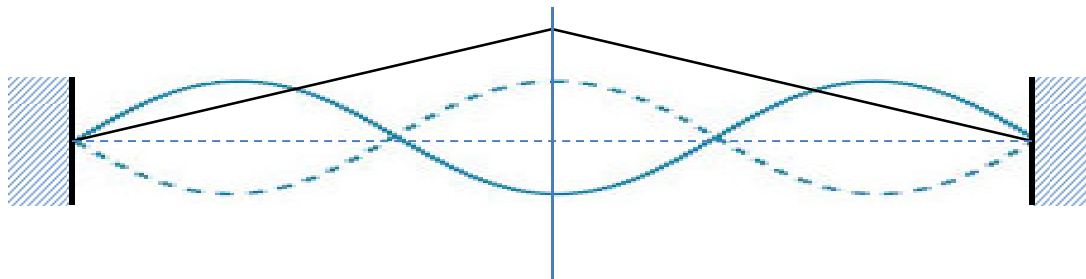
# Example

- It is often useful to use symmetries to simplify the amount of work:



Left and right integrals will cancel.

$$a_2 = a_4 = a_6 = \cdots = 0$$



Left and right integrals are equal.

# Example

$$a_k = \frac{8A}{L^2} \int_0^{L/2} x \sin\left(\frac{k\pi x}{L}\right) dx$$

- Use a table of integrals:

$$(91) \quad \int x \sin(ax) dx = -\frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax$$

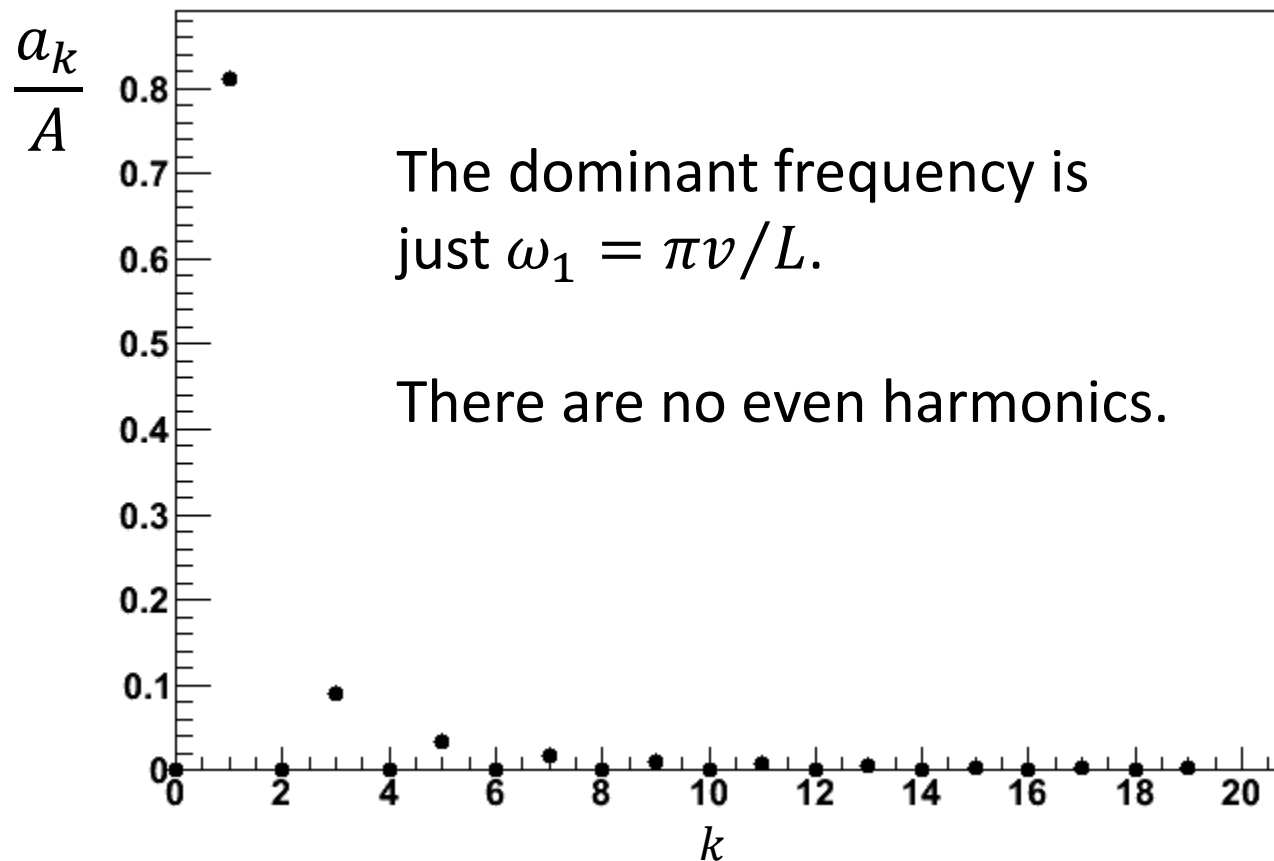
$$a_k = -\frac{4A}{k\pi} \cos\left(\frac{k\pi}{2}\right) + \frac{8A}{k^2\pi^2} \sin\left(\frac{k\pi}{2}\right)$$

- But we only care about  $k = 1, 3, 5, 7 \dots$

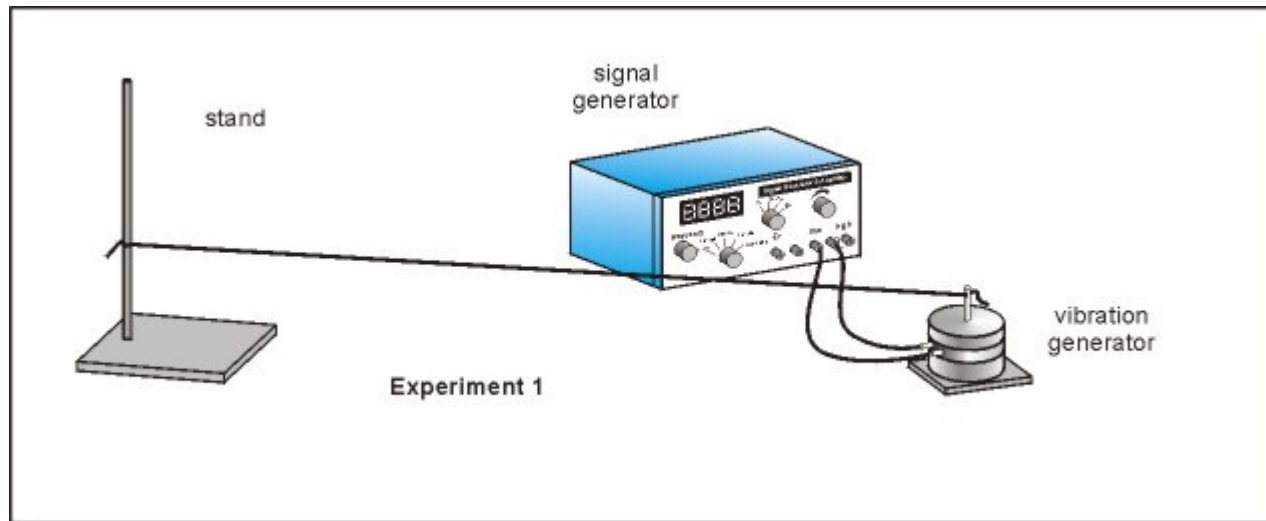
$$a_k = \frac{8A}{\pi^2}, -\frac{8A}{9\pi^2}, \frac{8A}{25\pi^2}, -\frac{8A}{49\pi^2}, \dots$$

# Example

- These are the amplitudes of each frequency component:



# Forced Oscillations



- One end of the string is fixed, the other end is forced with the function  $Y(t) = B \cos \omega t$ .

$$y(0, t) = B \cos \omega t$$

$$y(L, t) = 0$$

- The wave equation still holds so we expect solutions to be of the form

$$y(x, t) = f(x) \cos \omega t$$

# Forced Oscillations

- This time we can't constrain  $f(x)$  to be zero at both ends.
- Now, let  $f(x) = A \sin(kx + \alpha)$ 
  - The constant  $k$  is just  $\omega/v$ .
  - We need to solve for  $A$  and  $\alpha$
- Boundary condition at  $x = L$ :

$$\sin\left(\frac{\omega L}{v} + \alpha\right) = 0 \Rightarrow \frac{\omega L}{v} + \alpha = p\pi$$
$$\alpha_p = p\pi - \frac{\omega L}{v}$$

- Condition at  $x = 0$ :

$$B = A_p \sin \alpha_p$$

# Forced Oscillations

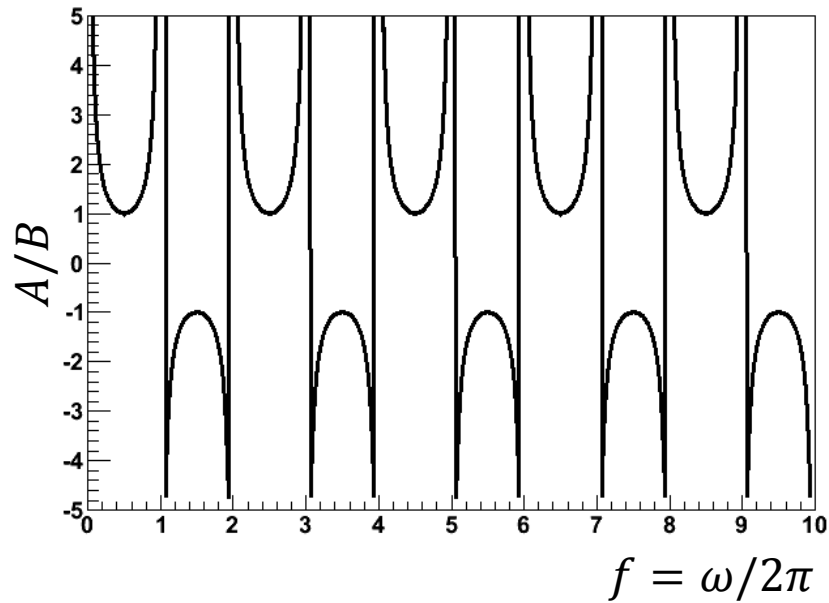
- Amplitude of oscillations:

$$A_p = \frac{B}{\sin(p\pi - \omega L/v)}$$

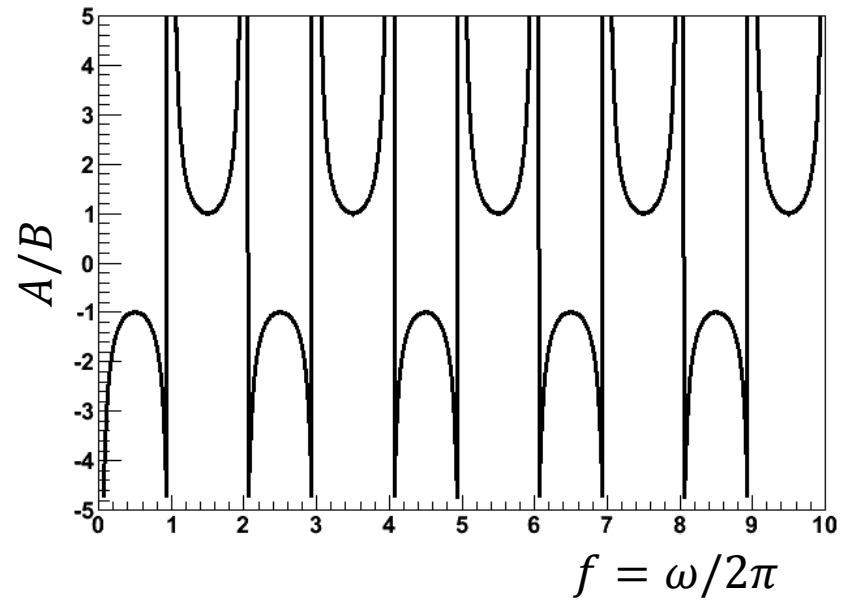
- What does this mean?
  - The driving force can excite many normal modes of oscillation
  - When  $\omega = p\pi v/L$ , the amplitude gets very large

# Forced Oscillations

$$p = 1$$



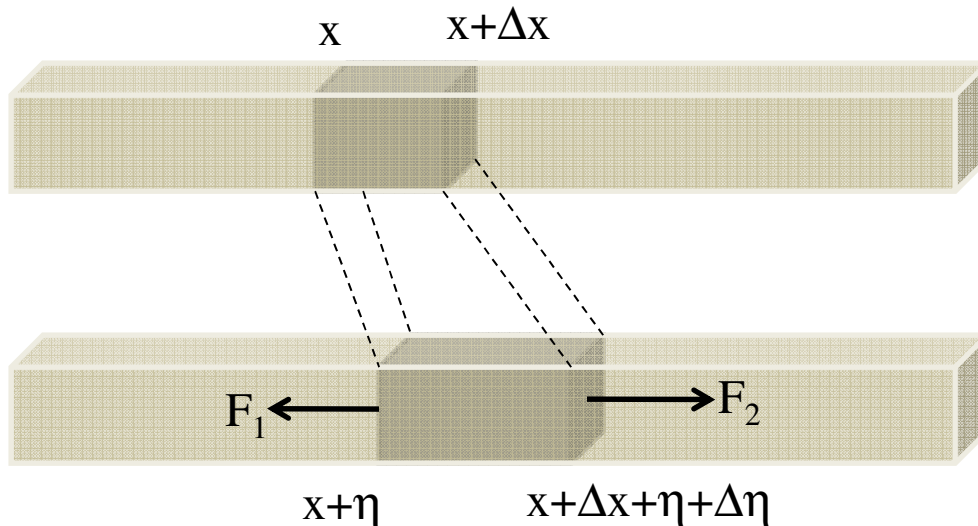
$$p = 2$$



$$L = 5 \text{ m}$$
$$v = 10 \text{ m/s}$$

# Other Continuous Systems

- Longitudinal waves in a solid rod:

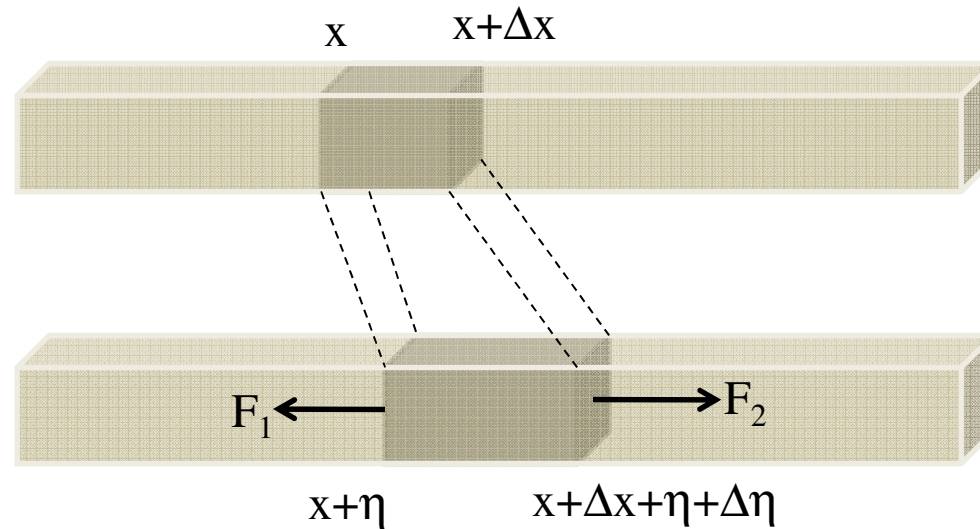


Notation:

- $x$  labels which piece of the rod we are considering, analogous to the index  $n$  when counting discrete masses.
- $\eta$  quantifies how much the element of mass has moved.

- Recall that strain was defined as the fractional increase in length of a small element:  $\Delta \eta / \Delta x$
- Stress was defined as  $\Delta F / A$
- These were related by  $\Delta F / A = Y \Delta \eta / \Delta x$

# Longitudinal Waves in a Solid Rod



$$\Delta F / A = Y \Delta \eta / \Delta x$$

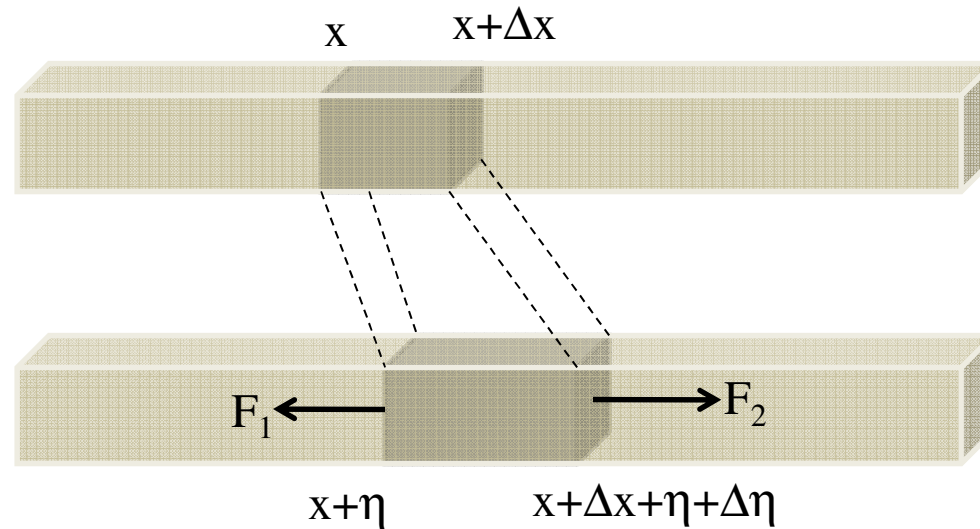
- Force on one side of the element:

$$F_1 = AY \Delta \eta / \Delta x = AY \partial \eta / \partial x$$

- Force on the other side of the element:

$$F_2 = F_1 + AY \frac{\partial^2 \eta}{\partial x^2} \Delta x$$

# Longitudinal Waves in a Solid Rod



- Newton's law:

$$m\ddot{\eta} = F_2 - F_1$$
$$F_2 - F_1 = AY \frac{\partial^2 \eta}{\partial x^2} \Delta x = \rho A \Delta x \frac{\partial^2 \eta}{\partial t^2}$$

- Wave equation:

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\rho}{Y} \frac{\partial^2 \eta}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2}$$

# Longitudinal Normal Modes

- What is the solution for a rod of length  $L$ ?

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2} \qquad v = \sqrt{Y/\rho}$$

- Boundary conditions:

- Suppose one end is fixed

$$\eta(0) = 0$$

- No force at the free end of the rod so the stress is zero there.  
Strain  $\propto$  stress, so the strain is also zero.

$$F = AY \partial \eta / \partial x$$

$$\frac{\partial \eta}{\partial x}_{x=L} = 0$$

- Look for solutions that are of the form

$$\eta(x) = f(x) \cos \omega t$$

# Longitudinal Normal Modes

$$\eta(x) = f(x) \cos \omega t$$

- Inspired by the continuous string problem, we let

$$f(x) = A \sin(kx)$$

- Derivatives:

$$\frac{\partial^2 \eta}{\partial x^2} = -k^2 \eta$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\omega^2 \eta$$

$$\frac{\partial^2 \eta}{dx^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2} \Rightarrow k = \frac{\omega}{v}$$

# Longitudinal Normal Modes

$$f(x) = A \sin\left(\frac{\omega x}{v}\right)$$

- This automatically satisfies the boundary condition at  $x = 0$ .
- At  $x = L$ ,  $\partial\eta/\partial x = 0$ :

$$\frac{\partial\eta}{\partial x}_{x=L} \propto \cos\left(\frac{\omega L}{v}\right) = 0$$

- This means that  $\frac{\omega L}{v} = (n - \frac{1}{2})\pi$
- Angular frequencies of normal modes are

$$\omega_n = \frac{\pi}{L} \left(n - \frac{1}{2}\right) \sqrt{Y/\rho}$$

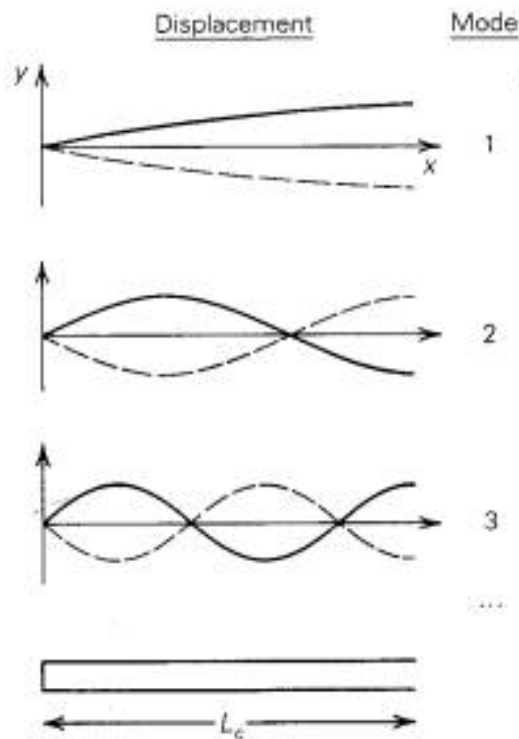
- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L} \sqrt{Y/\rho}$$

# Longitudinal Normal Modes

- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L} \sqrt{Y/\rho}$$



Lowest possible frequency:

$$\nu_1 = \frac{1}{4L} \sqrt{\frac{Y}{\rho}}$$

# Frequencies of Metal Chimes

- Suppose a set of chimes were made of copper rods, with lengths between 30 and 40 cm, rigidly fixed at one end.

- What frequencies should we expect if

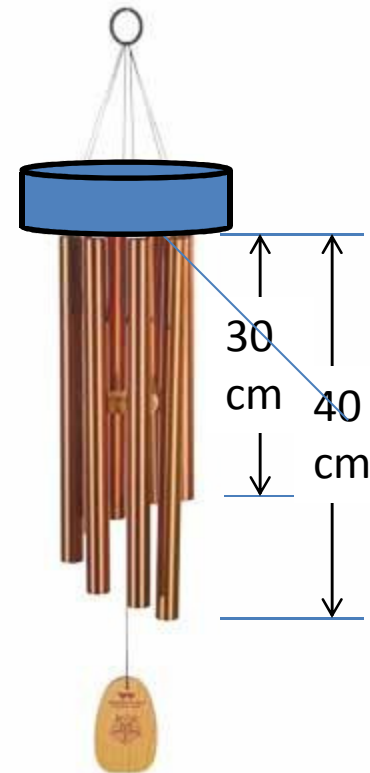
$$Y = 117 \times 10^9 \text{ N} \cdot \text{m}^{-2}$$

$$\rho = 8.96 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$$

$$\nu_1 = \frac{1}{4L} \sqrt{\frac{117 \times 10^9 \text{ N} \cdot \text{m}^{-2}}{8.96 \times 10^3 \text{ kg} \cdot \text{m}^{-3}}}$$

$$= 2260 - 3010 \text{ Hz}$$

(highest octave on a piano)



# Frequencies of Metal Chimes

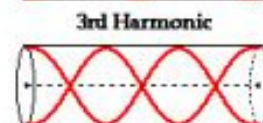
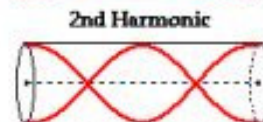
- If the metal rods were not fixed at one end then the boundary conditions at both ends would be:

$$\frac{\partial \eta}{\partial x} = 0$$

- Allowed frequencies of normal modes:

$$v_n = \frac{n}{2L} \sqrt{Y/\rho}$$

Open at Both Ends



Harmonic

Wavelength  $\lambda$

Frequency  $f$

1<sup>st</sup>

$2L$

$f_1$

2<sup>nd</sup>

$L$

$2f_1$

3<sup>rd</sup>

$2L/3$

$3f_1$