Physics 42200

Waves & Oscillations

Lecture 15 – French, Chapter 6

Spring 2016 Semester

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Review of Coupled Oscillators

- General observations:
  - Forces depend on positions x_i of multiple masses
  - Coupled set of differential equations

$$m_i \ddot{x}_i = F(x_1, x_2, \dots, x_N)$$

$$\ddot{x}_i = \frac{1}{m_i} F(x_1, x_2, \dots, x_N)$$

  - In the problems we will consider, $F(x_1, x_2, \dots, x_N)$ is a linear function of x_i

$$\ddot{\vec{x}} + \boldsymbol{A}\,\vec{x} = 0$$

$$(\boldsymbol{A} - \omega^2 \boldsymbol{I})\vec{x} = 0$$

  - If this is true then

$$\det(\boldsymbol{A} - \omega^2 \boldsymbol{I}) = 0$$

  - The eigenvalues of the matrix $\boldsymbol{A}$ are ω^2

Review of Coupled Oscillators

- In general, a system with N masses can have N distinct eigenvalues

$$\left(\boldsymbol{A} - \omega_i^2 \boldsymbol{I}\right)\vec{u}_i = 0$$

- There are N eigenvectors $\vec{u}_i$
- The eigenvectors are orthogonal:

$$\vec{u}_i \cdot \vec{u}_j = 0 \text{ when } i \neq j$$

- If $\vec{u}_i$ is an eigenvector, then so is $\alpha\, \vec{u}_i$ for any real number α.
- The eigenvectors can be normalized so that

$$\vec{u}_i \cdot \vec{u}_i = 1$$

$$\vec{u}_i \cdot \vec{u}_j = \delta_{ij}$$

Review of Coupled Oscillators

- An arbitrary vector $\vec{x}$ can be expressed as a linear combination of eigenvectors:

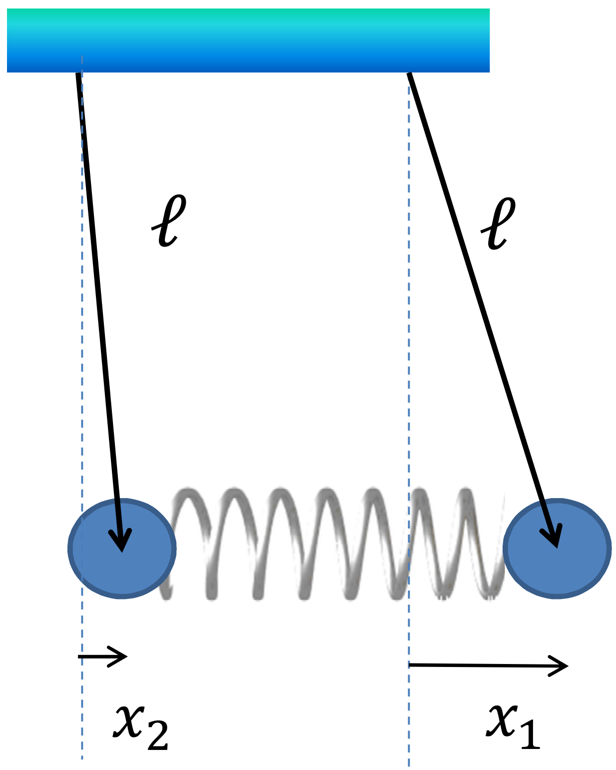
$$\vec{x} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_N\vec{u}_N$$

$$= \sum_{i=1}^{N} a_i\vec{u}_i$$

- How do we solve for the coefficients a_i?

$$\vec{u}_j \cdot \vec{x} = \sum_{i=1}^{N} a_i\vec{u}_j \cdot \vec{u}_i = \sum_{i=1}^{N} a_i\delta_{ij} = a_j$$

Two Coupled Oscillators



- The spring is stretched by the amount $x_1 - x_2$
- Restoring force on pendulum 1:
$$F_1 = -k(x_1 - x_2)$$
- Restoring force on pendulum 2:
$$F_2 = k(x_1 - x_2)$$

$$m\ddot{x}_1 + \frac{mg}{\ell}x_1 + k(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + \frac{mg}{\ell}x_2 - k(x_1 - x_2) = 0$$

Two Coupled Oscillators

$$\ddot{x}_1 + [(\omega_0)^2 + (\omega_c)^2]x_1 - (\omega_c)^2 x_2 = 0$$
$$\ddot{x}_2 + [(\omega_0)^2 + (\omega_c)^2]x_2 - (\omega_c)^2 x_1 = 0$$

- Eigenvalues are

$$\omega_1^2 = \omega_0^2$$
$$\omega_2^2 = (\omega_0)^2 + 2(\omega_c)^2$$

- Eigenvectors are

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\binom{1}{1}$$
$$\vec{u}_2 = \frac{1}{\sqrt{2}}\binom{1}{-1}$$

Two Coupled Oscillators

- Normal modes of oscillation:

$$\vec{q}_1(t) = \vec{u}_1 \cos(\omega_1 t)$$
$$\vec{q}_2(t) = \vec{u}_2 \cos(\omega_2 t)$$

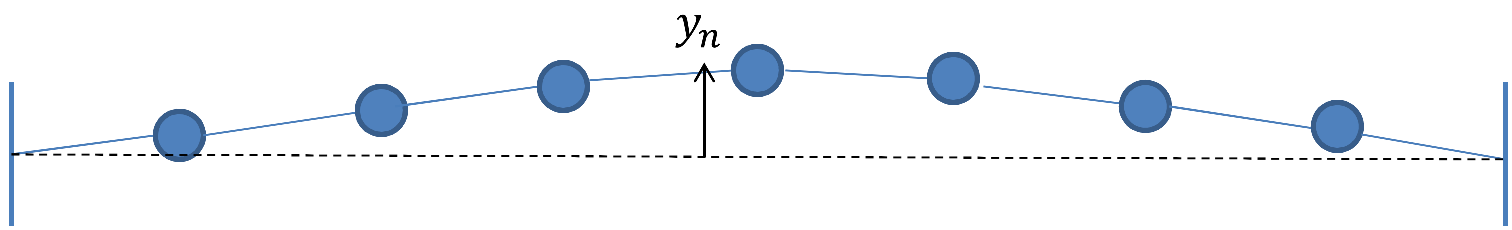
- General solution:

$$\vec{x}(t) = A\,\vec{u}_1 \cos(\omega_1 t + \alpha) + B\,\vec{u}_2 \cos(\omega_2 t + \beta)$$

- Initial conditions: $\vec{x}(0) = \vec{x}_0, \dot{\vec{x}}(0) = \vec{v}_0$

$$\vec{u}_1 \cdot \vec{x}_0 = A \cos\alpha$$
$$\vec{u}_2 \cdot \vec{x}_0 = B \cos\beta$$
$$\vec{u}_1 \cdot \vec{v}_0 = -A\omega_1 \sin\alpha$$
$$\vec{u}_2 \cdot \vec{v}_0 = -B\omega_2 \sin\beta$$

Many Coupled Oscillators



- Equation of motion for mass n:

$$m\,\ddot{y}_n = F_n = \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = \frac{T}{m\ell}$$

- We can construction solutions of the form $\vec{y}_k(t) = \vec{u}_k \cos \omega_k t$
- Frequencies of normal modes of oscillation:

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

Many Coupled Oscillators

- Eigenvalues:

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- Eigenvectors:

$$u_{kn} = \sin\left(\frac{nk\pi}{N+1}\right)$$

- Orthogonality:

$$\vec{u}_i \cdot \vec{u}_j = \frac{N}{2}\,\delta_{ij}$$

Many Coupled Oscillators

- General solution:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_k t - \theta_k)$$

- At time $t = 0$,

$$x_n(0) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\theta_k)$$

- Consider the expression:

$$\sum_{n=1}^{N} x_n(0)\sin\left(\frac{nk'\pi}{N+1}\right) = \sum_{n,k=1}^{N} a_k \cos\theta_k \sin\left(\frac{nk'\pi}{N+1}\right)\sin\left(\frac{nk\pi}{N+1}\right)$$

$$= \frac{N}{2}\sum_{k=1}^{N} a_k \cos\theta_k \,\delta_{k'k} = \frac{N}{2} a_{k'} \cos\theta_{k'}$$

Many Coupled Oscillators

- Likewise, consider the time derivatives:

$$\dot{x}_n(t) = -\sum_{k=1}^{N} a_k \omega_k \sin\left(\frac{nk\pi}{N+1}\right) \sin(\omega_k t - \theta_k)$$

$$\dot{x}_n(0) = \sum_{k=1}^{N} \underbrace{a_k \omega_k \sin\theta_k}_{\text{constants}} \sin\left(\frac{nk\pi}{N+1}\right)$$

$$\sum_{n=1}^{N} \dot{x}_n(0) \sin\left(\frac{nk'\pi}{N+1}\right) = \frac{N}{2} a_{k'} \omega_{k'} \sin\theta_{k'}$$

- If the initial velocities were all zero, then

$$\theta_k = 0 \text{ for } k = 1, \dots, N$$

Continuous Systems

- What happens when the number of masses goes to infinity, while the linear mass density remains constant?

$$m\,\ddot{y}_n = \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

$$\frac{m}{\ell} \to \mu$$

$$\frac{y_{n+1} - y_n}{\ell} \to \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \qquad \frac{(y_n - y_{n-1})}{\ell} \to \left(\frac{\partial y}{\partial x}\right)_x$$

$$\mu\ell\,\frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

Continuous Systems

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\ell}$$

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}$$

The Wave Equation: $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$ $$v = \sqrt{T/\mu}$$

Solutions

- When we had N masses, the solutions were
$$y_{n,k}(t) = A_{n,k}\cos(\omega_k t - \delta_k)$$
  - n labels the mass along the string
  - With a continuous system, n is replaced by x.
- Proposed solution to the wave equation for the continuous string:
$$y(x,t) = f(x)\cos\omega t$$
- Derivatives:
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)\cos\omega t$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}\cos\omega t$$

Solutions

- Substitute into the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\omega^2}{v^2} f(x)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{v^2} f(x) = 0$$

- This is the same differential equation as for the harmonic oscillator.
- Solutions are $f(x) = A\sin(\omega x/v) + B\cos(\omega x/v)$

Solutions

$$f(x) = A\sin(\omega x/v) + B\cos(\omega x/v)$$

- Boundary conditions at the ends of the string:

$$f(0) = f(L) = 0$$

$$f(x) = A\sin(\omega x/v) \text{ where } \omega L/v = n\pi$$

$$\omega_n = \frac{n\pi v}{L}$$

- Solutions can be written:

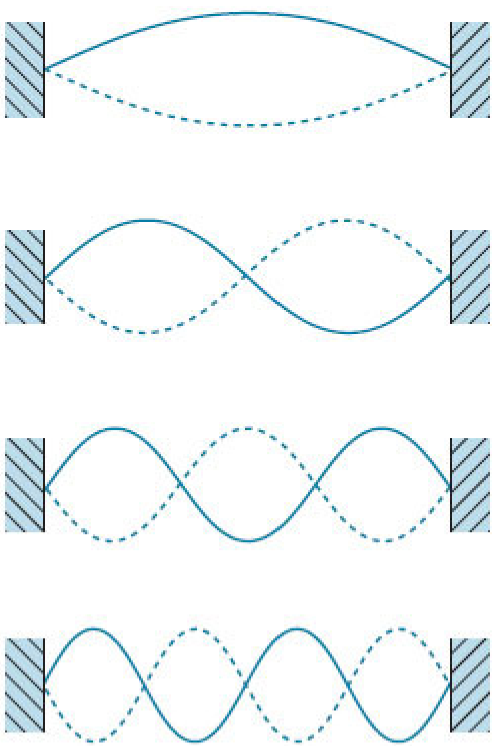
$$y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

- Normal modes of oscillation:

$$q_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t$$

Properties of the Solutions

$$q_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t$$



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

$$\lambda_n = \frac{2L}{n}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

Fourier Analysis

- In this case we define the "dot product" as an integral:

$$f \cdot y_n = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Are $y_n(x)$ orthogonal?

$$y_n \cdot y_m = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) dx - \frac{1}{2}\int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right) dx$$

$= 0$ when $n \neq m$

Fourier Analysis

- But when $n = m$,

$$y_n \cdot y_m = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) dx - \frac{1}{2}\int_0^L \cos\left(\frac{2n\pi x}{L}\right) dx$$

0

$$= \frac{1}{2}\int_0^L dx = \frac{L}{2}$$

- So we can write

$$y_n \cdot y_m = \frac{L}{2}\,\delta_{nm}$$

Initial Value Problem

$$f(x) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$y_n \cdot f = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \int_0^L \left[\sum_m a_m \sin\left(\frac{m\pi x}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \sum_m a_m \; y_m \cdot y_n = \frac{L}{2} \sum_m a_m \; \delta_{mn} = \frac{L}{2} a_n$$

Initial Value Problem

$$f(x) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Now we know how to calculate a_n from the initial conditions… we have solved the initial value problem.

Initial Value Problem

- The functions $y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ are like the eigenvectors
- They are orthogonal in the sense that

$$\int_0^L y_n(x) y_m(x) dx = \frac{L}{2}\delta_{nm}$$

- An arbitrary function $f(x)$ which satisfies $f(0) = f(L) = 0$ can be written:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

- How do we determine the coefficients a_k?

Initial Value Problem

- Multiply $f(x)$ by $y_n(x)$ and integrate:

$$\int_0^L f(x)y_n(x)dx = \int_0^L \sum_{m=1}^{\infty} a_m y_m(x) y_n(x) dx$$

$$= \sum_{m=1}^{\infty} a_m \left(\int_0^L y_m(x) y_n(x) dx \right)$$

$$= \sum_{m=1}^{L} a_m \left(\frac{L}{2} \delta_{mn} \right) = \frac{L}{2} a_n$$

- Therefore,

$$a_n = \frac{2}{L} \int_0^L f(x) y_n(x) dx$$

Example

- How to describe a square wave in terms of normal modes:

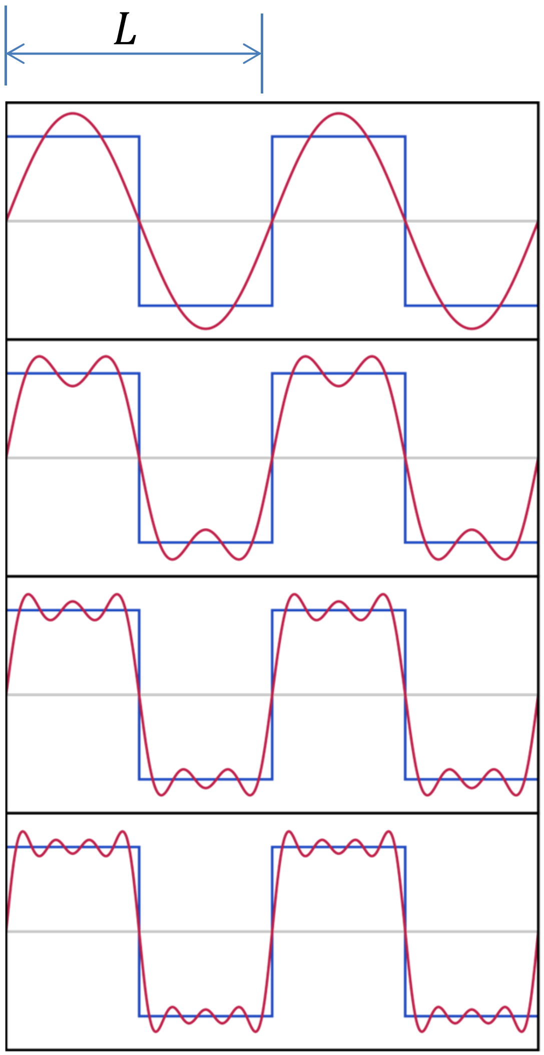
$$u(x) = \begin{cases} +1 \text{ when } 0 < x < L/2 \\ -1 \text{ when } L/2 < x < L \end{cases}$$

$$a_n = \frac{2}{L}\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right) dx - \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{n\pi}[1 - \cos(n\pi)]$$

$$a_1 = \frac{4}{\pi}, a_3 = \frac{4}{3\pi}, a_5 = \frac{4}{5\pi}, \cdots$$

Example



$$a_n = \frac{2}{n\pi}[1 - \cos(n\pi)]$$

$$a_1 = \frac{4}{\pi}, a_3 = \frac{4}{3\pi}, a_5 = \frac{4}{5\pi}, \cdots$$

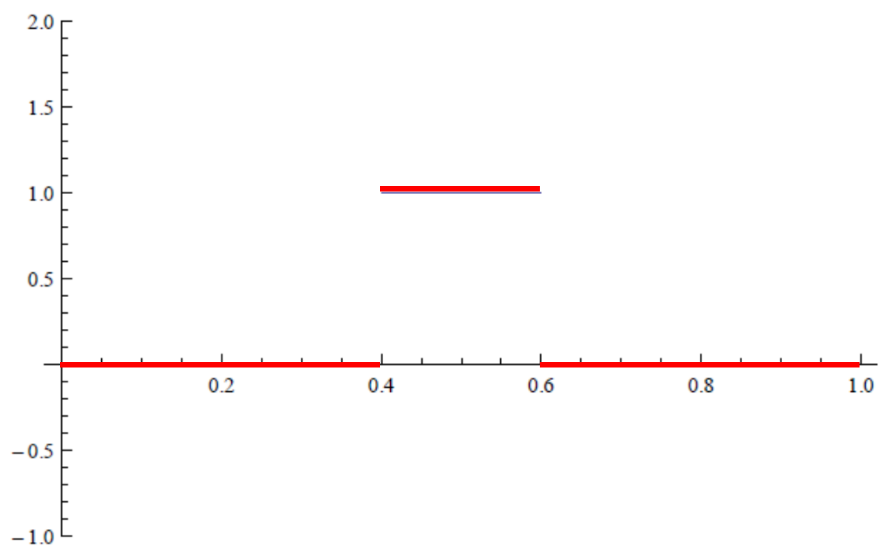
$$a_2 = 0, a_4 = 0, a_6 = 0, \cdots$$

The initial shape doesn't really satisfy the boundary conditions $y(0) = y(L) = 0$, but the approximation does.

Other Examples

- Consider an initial displacement in the middle of the string:

$$f(x) = \begin{cases} 0 \text{ when } x < 2L/5 \\ 1 \text{ when } 2L/5 < x < 3L/5 \\ 0 \text{ when } x > 3L/5 \end{cases}$$



Let's assume $L = 1$ and $v = 1$

Example

$$a_n = \frac{2}{L}\int_{2L/5}^{3L/5} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\texttt{a[n\_] := } \frac{2}{L}\int_0^L \texttt{f[x] Sin}\left[\frac{n\,\pi\,x}{L}\right] dx$$

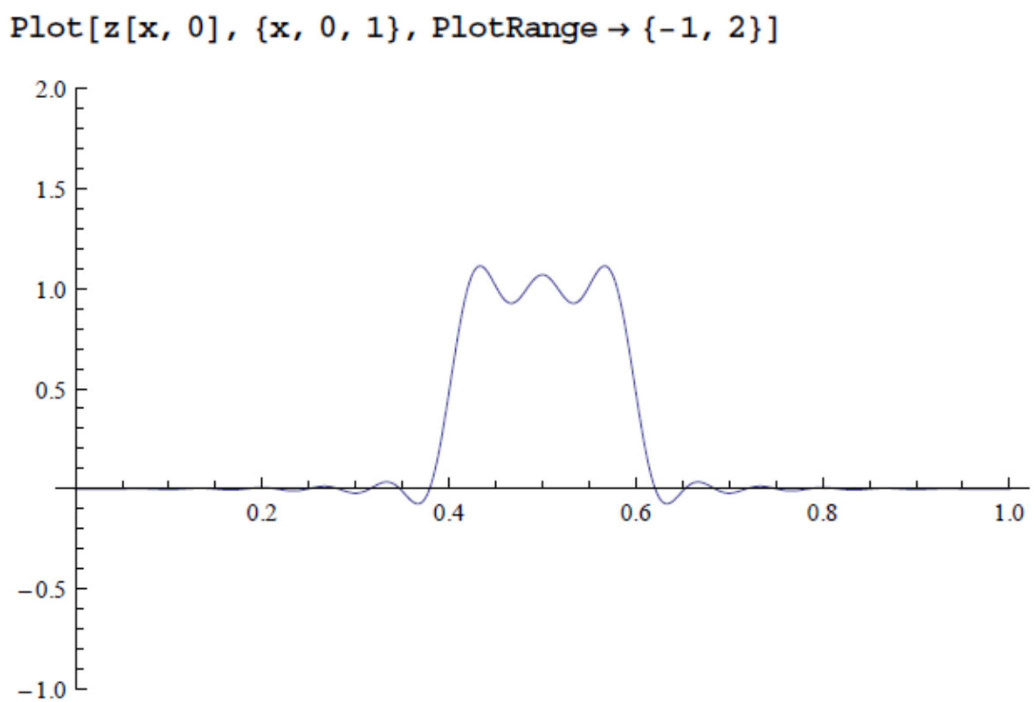
```
f[x_] = Piecewise[{{0, x < 2 / 5}, {1, x > 2 / 5 && x < 3 / 5}, {0, x > 3 / 5}}]
Table[a[n], {n, M}]
```

$$\left\{\frac{-1+\sqrt{5}}{\pi}, 0, \frac{-1-\sqrt{5}}{3\pi}, 0, \frac{4}{5\pi}, 0, \frac{-1-\sqrt{5}}{7\pi}, 0, \frac{-1+\sqrt{5}}{9\pi}, 0, \frac{-1+\sqrt{5}}{11\pi}, 0, \frac{-1-\sqrt{5}}{13\pi}, 0, \frac{4}{15\pi}, 0, \frac{-1-\sqrt{5}}{17\pi}, 0, \frac{-1+\sqrt{5}}{19\pi}, 0, \frac{-1+\sqrt{5}}{21\pi}, 0, \frac{-1-\sqrt{5}}{23\pi}, 0, \frac{4}{25\pi}, 0, \frac{-1-\sqrt{5}}{27\pi}, 0, \frac{-1+\sqrt{5}}{29\pi}, 0\right\}$$

- Now we know the first 30 values for a_n… we're done!

Example

- Is this a good approximation?



- A good description of sharp features require high frequencies (large n).

Example

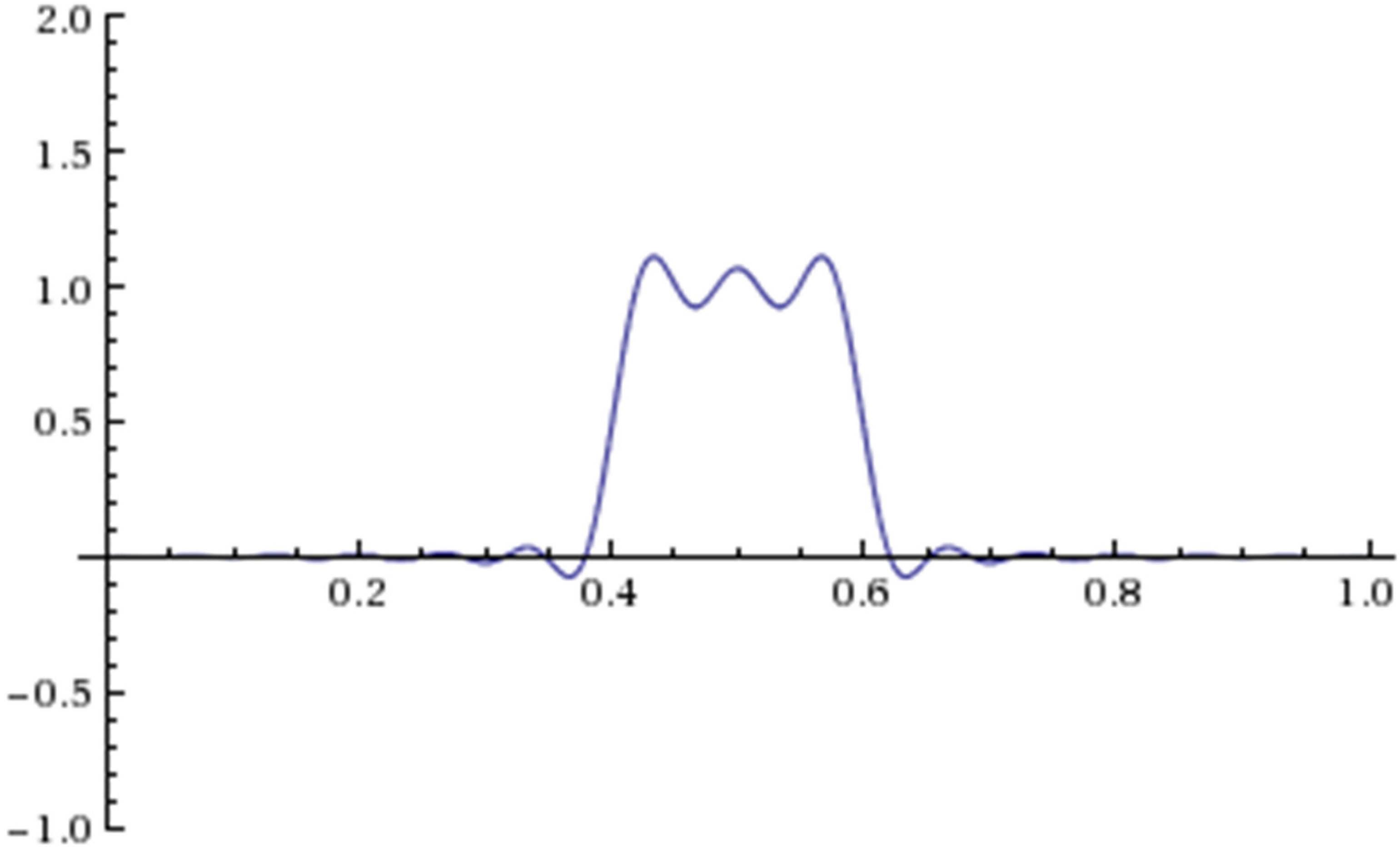
- The complete solution to the initial value problem is

$$y(x,t) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

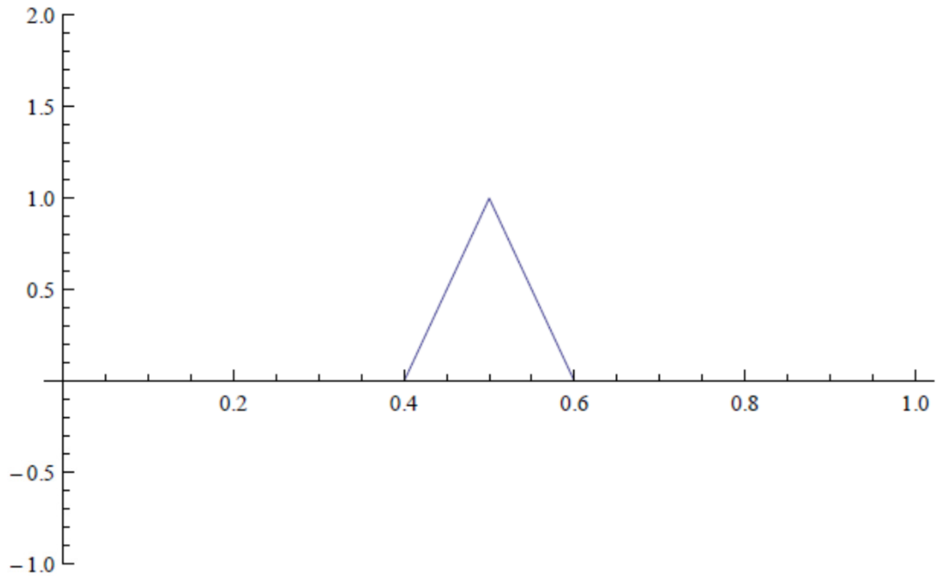
- What does this look like as a function of time?

Example



Another Example

- Consider a function that is a bit smoother:



$$f(x) = \begin{cases} 0 & x < \frac{2}{5} \\ 10\left(-\frac{2}{5} + x\right) & x > \frac{2}{5} \,\&\&\, x < \frac{1}{2} \\ 2 + 10\left(\frac{2}{5} - x\right) & x > \frac{1}{2} \,\&\&\, x < \frac{3}{5} \\ 0 & \text{True} \end{cases}$$

Example

- The integrals for the Fourier coefficients are of the form:

$$\int_a^b \sin\left(\frac{n\pi x}{L}\right) dx \text{ or } \int_a^b x \sin\left(\frac{n\pi x}{L}\right) dx$$

- These can be solved analytically, but it is a lot of work…

$$\left\{-\frac{10\left(-4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{\pi^2}, 0, \frac{10\left(-4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{9\pi^2}, 0, \frac{8}{5\pi^2}, 0, -\frac{10\left(4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{49\pi^2}, 0,\right.$$

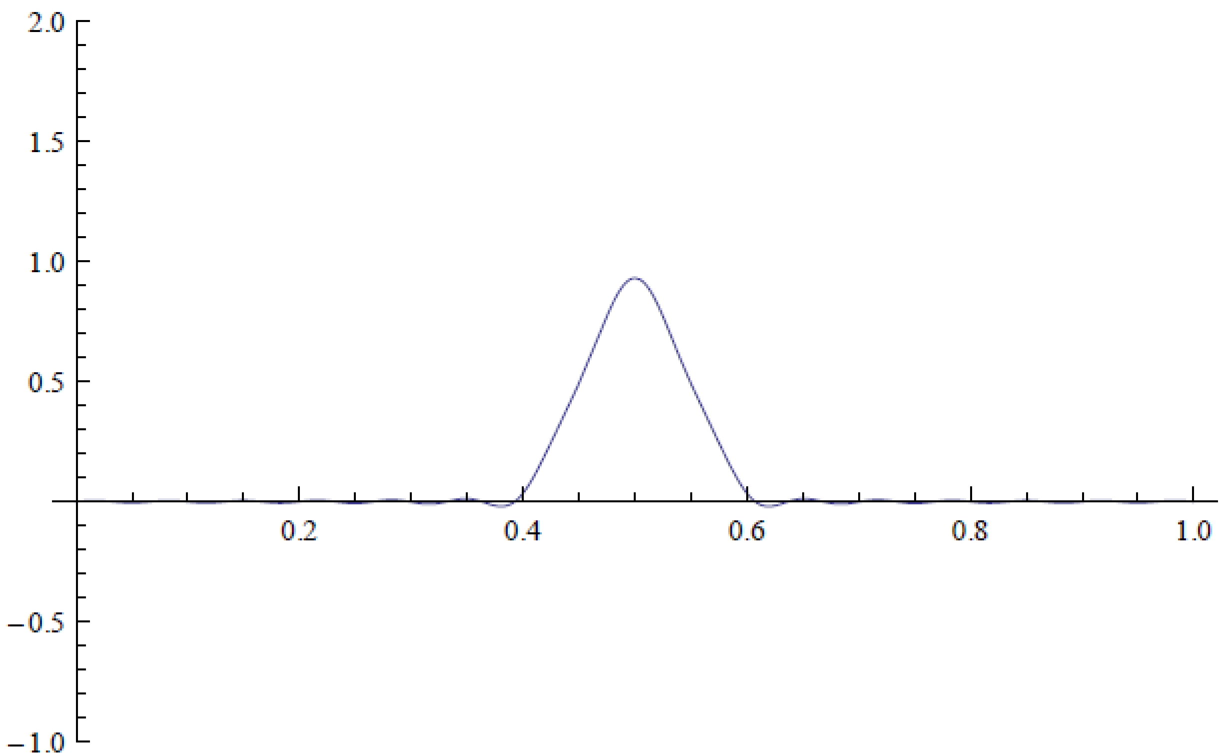
$$\frac{10\left(4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{81\pi^2}, 0, -\frac{10\left(4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{121\pi^2}, 0, \frac{10\left(4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{169\pi^2}, 0, -\frac{8}{45\pi^2},$$

$$0, -\frac{10\left(-4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{289\pi^2}, 0, \frac{10\left(-4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{361\pi^2}, 0, -\frac{10\left(-4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{441\pi^2}, 0,$$

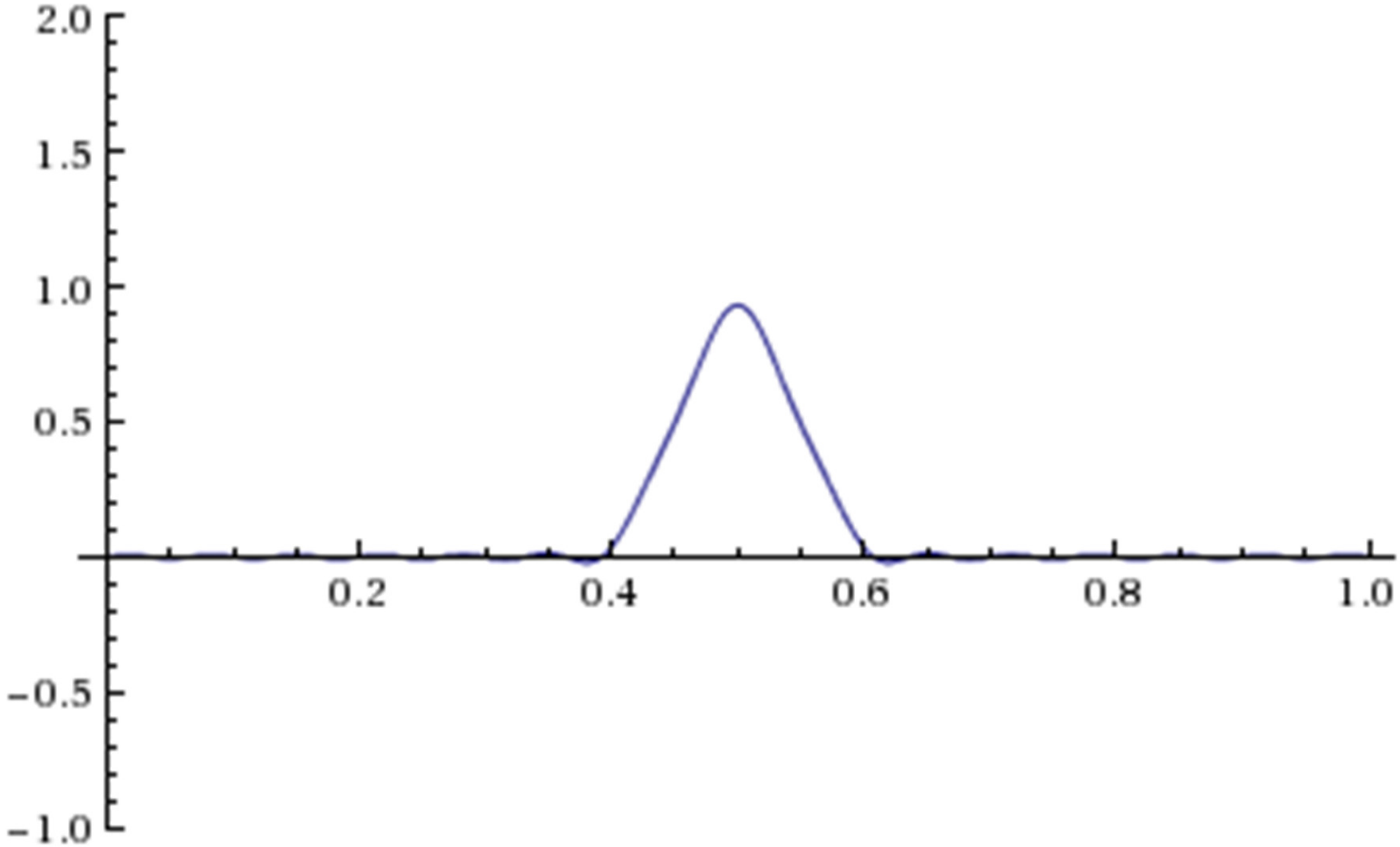
$$\left.\frac{10\left(-4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{529\pi^2}, 0, \frac{8}{125\pi^2}, 0, -\frac{10\left(4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{729\pi^2}, 0, \frac{10\left(4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{841\pi^2}, 0\right\}$$

Example

- The initial shape of the approximation with N=30 is better than for the square pulse.



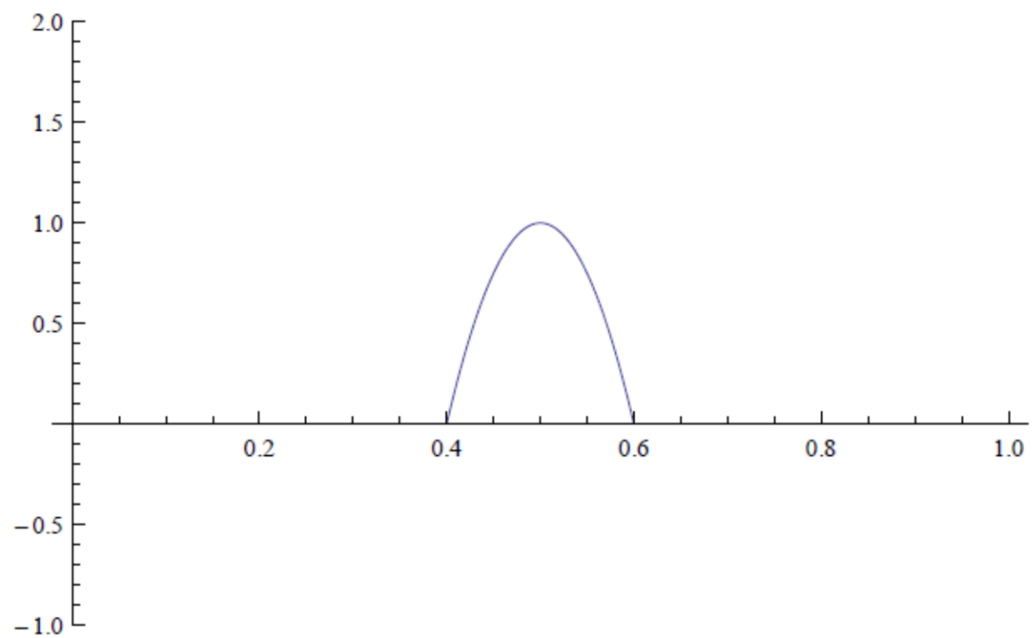
Example



Final Example

- An even smoother function:

$$f(x) = \begin{cases} 0 & x < \frac{2}{5} \\ 1 - 100\left(-\frac{1}{2} + x\right)^2 & x > \frac{2}{5} \;\&\&\; x < \frac{3}{5} \\ 0 & \text{True} \end{cases}$$



Example

- The integrals for the Fourier coefficients are of the form:

$$\int_a^b \sin\left(\frac{n\pi x}{L}\right) dx \text{ or } \int_a^b x \sin\left(\frac{n\pi x}{L}\right) dx \text{ or } \int_a^b x^2 \sin\left(\frac{n\pi x}{L}\right) dx$$

- These can be solved analytically, but it is a lot of work…

$$\Big\{-\frac{20\left(10-10\sqrt{5}+\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{\pi^3},\ 0,\ \frac{20\left(-10-10\sqrt{5}+3\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{27\pi^3},\ 0,\ \frac{32}{5\pi^3},$$

$$0,\ -\frac{20\left(10+10\sqrt{5}+7\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{343\pi^3},\ 0,\ \frac{20\left(-10+10\sqrt{5}+9\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{729\pi^3},\ 0,$$

$$-\frac{20\left(10-10\sqrt{5}+11\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{1331\pi^3},\ 0,\ \frac{20\left(-10-10\sqrt{5}+13\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{2197\pi^3},\ 0,\ \frac{32}{135\pi^3},$$

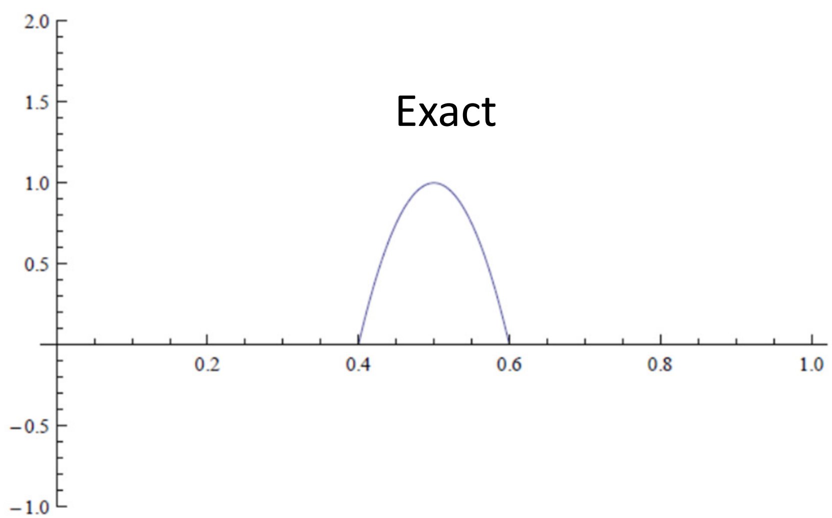
$$0,\ -\frac{20\left(10+10\sqrt{5}+17\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{4913\pi^3},\ 0,\ \frac{20\left(-10+10\sqrt{5}+19\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{6859\pi^3},\ 0,$$

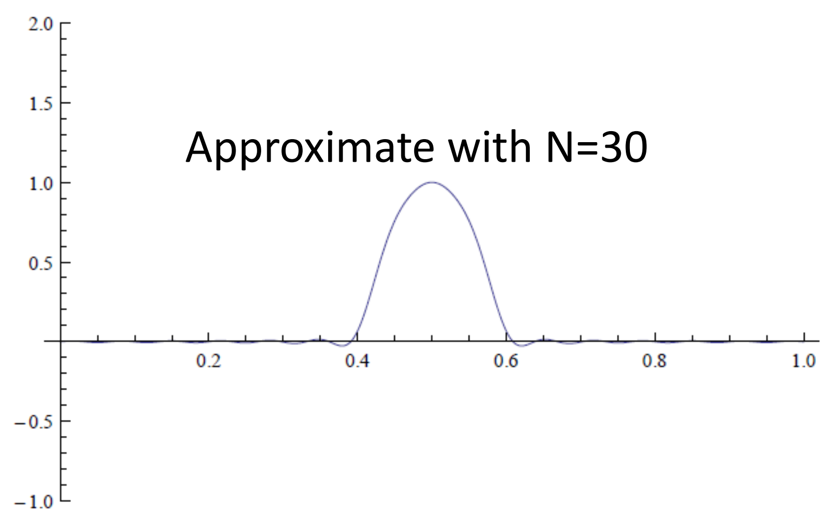
$$-\frac{20\left(10-10\sqrt{5}+21\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{9261\pi^3},\ 0,\ \frac{20\left(-10-10\sqrt{5}+23\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{12167\pi^3},\ 0,\ \frac{32}{625\pi^3},$$

$$0,\ -\frac{20\left(10+10\sqrt{5}+27\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{19683\pi^3},\ 0,\ \frac{20\left(-10+10\sqrt{5}+29\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{24389\pi^3},\ 0\Big\}$$

Example

- The initial shape of the approximation with N=30 is even better than the triangular pulse…





Example