Purdue University Department of Physics

Physics 42200
Waves & Oscillations

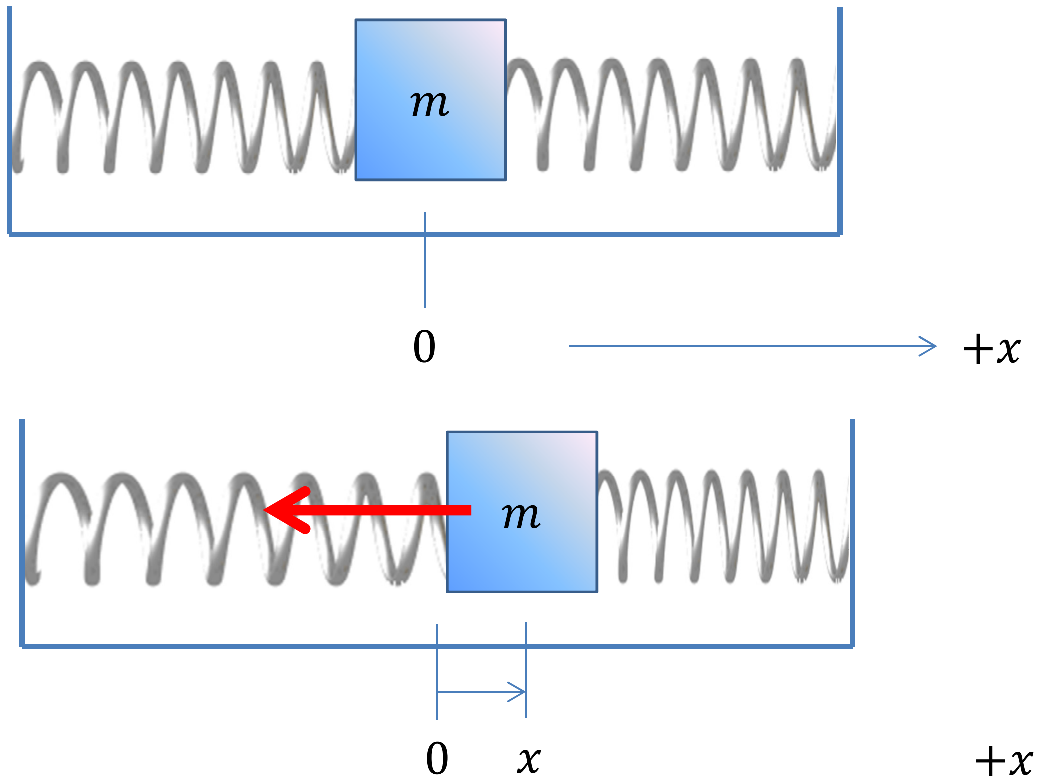
Lecture 13 – French, Chapter 5

Spring 2016 Semester

Matthew Jones

One Mass

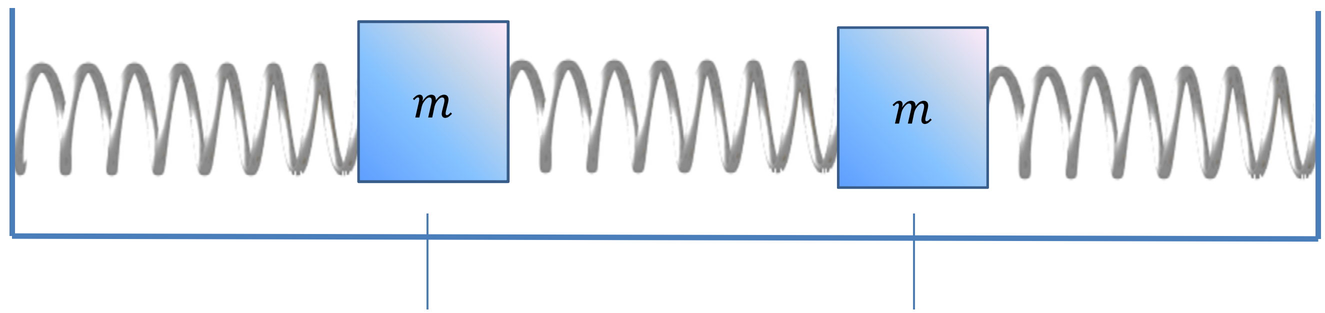
Consider one mass with two springs:



$$F = -2kx$$

Two Masses

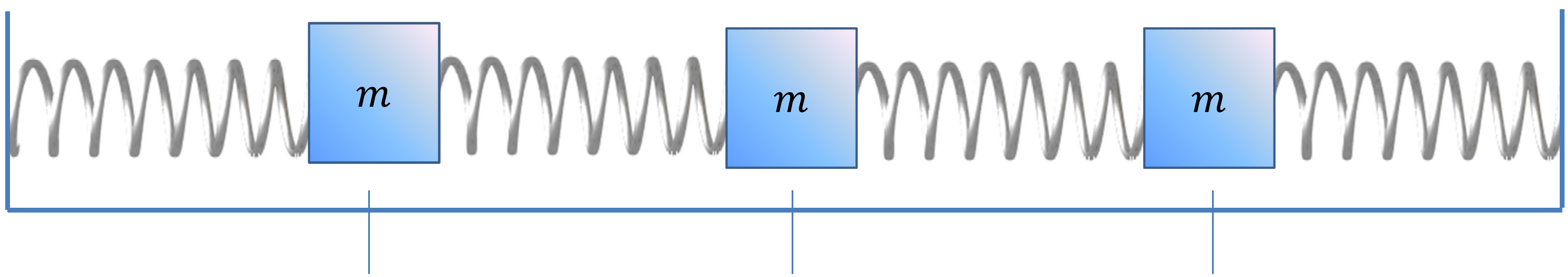
Consider two masses with three springs:



$$F_1 = -kx_1 - kx_1 + kx_2 = k(x_2 - 2x_1)$$
$$F_2 = kx_1 - kx_2 - kx_2 = k(x_1 - 2x_2)$$

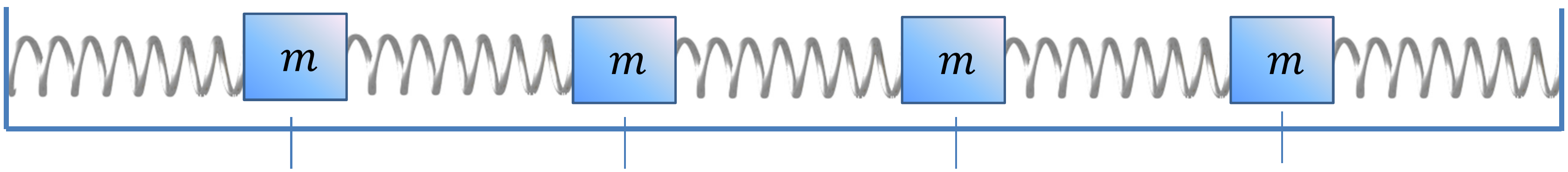
Three Masses

Consider three masses with four springs:



$$F_1 = -kx_1 - kx_1 + kx_2 = k(x_2 - 2x_1)$$
$$F_2 = -k(x_2 - x_1) - k(x_2 - x_3) = k(x_1 - 2x_2 + x_3)$$
$$F_3 = -kx_3 - kx_3 + kx_2 = k(x_2 - 2x_3)$$

Four Masses



$$F_1 = -kx_1 - kx_1 + kx_2 = k(x_2 - 2x_1)$$
$$F_2 = -k(x_2 - x_1) - k(x_2 - x_3) = k(x_1 - 2x_2 + x_3)$$
$$F_3 = -k(x_3 - x_2) - k(x_3 - x_4) = k(x_2 - 2x_3 + x_4)$$
$$F_4 = -kx_4 - kx_4 + kx_3 = k(x_3 - 2x_4)$$

- This pattern repeats for more and more masses.
- Except at the ends,

$$F_i = -k(x_i - x_{i-1}) - k(x_i - x_{i+1}) = k(x_{i-1} - 2x_i + x_{i+1})$$

- Equations of motion:

$$m\,\ddot{x}_i - k(x_{i-1} - 2x_i + x_{i+1}) = 0$$

Many Coupled Oscillators

$$m\,\ddot{x}_i - k(x_{i-1} - 2x_i + x_{i+1}) = 0$$
$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2(x_{i-1} + x_{i+1}) = 0$$

- Apply the same techniques we used before:
  - Suppose $x_i(t) = A_i \cos \omega t$
  - Then $\ddot{x}_i(t) = -\omega^2 A_i \cos \omega t$

$$(-\omega^2 + 2(\omega_0)^2)A_i - (\omega_0)^2(A_{i-1} + A_{i+1}) = 0$$
$$\frac{A_{i-1} + A_{i+1}}{A_i} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- Guess at a solution:

$$A_n = C\sin(n\Delta\theta)$$

- Will this work?

Many Coupled Oscillators

$$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- Proposed solution:

$$A_n = C \sin(n\Delta\theta)$$

- Boundary conditions: $A_0 = A_{N+1} = 0$
- This implies that $(N + 1)\Delta\theta = k\pi$

$$A_n = C \sin\left(\frac{nk\pi}{N+1}\right)$$

$$A_{n-1} + A_{n+1} = C \sin\left(\frac{(n-1)k\pi}{N+1}\right) + C \sin\left(\frac{(n+1)k\pi}{N+1}\right)$$

$$= 2C \sin\left(\frac{nk\pi}{N+1}\right) \cos\left(\frac{k\pi}{N+1}\right)$$

$$\frac{A_{n-1} + A_{n+1}}{A_n} = 2\cos\left(\frac{k\pi}{N+1}\right) = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

Many Coupled Oscillators

$$\frac{A_{n-1} + A_{n+1}}{A_n} = 2\cos\left(\frac{k\pi}{N+1}\right) = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- Solve for ω:

$$\omega^2 = 2(\omega_0)^2\left(1 - \cos\left(\frac{k\pi}{N+1}\right)\right)$$

$$= 4(\omega_0)^2\sin^2\left(\frac{k\pi}{2(N+1)}\right)$$

$$\omega_k = 2\omega_0\sin\left(\frac{k\pi}{2(N+1)}\right)$$

- There are N possible frequencies of oscillation.

Many Coupled Oscillators

- The motion of the masses depends on both the position of the mass (n) and the mode number (k):

$$A_{n,k} = C_n \sin\left(\frac{nk\pi}{N+1}\right)$$

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- When all the particles oscillate in the k^{th} normal mode, the n^{th} particle's position is:

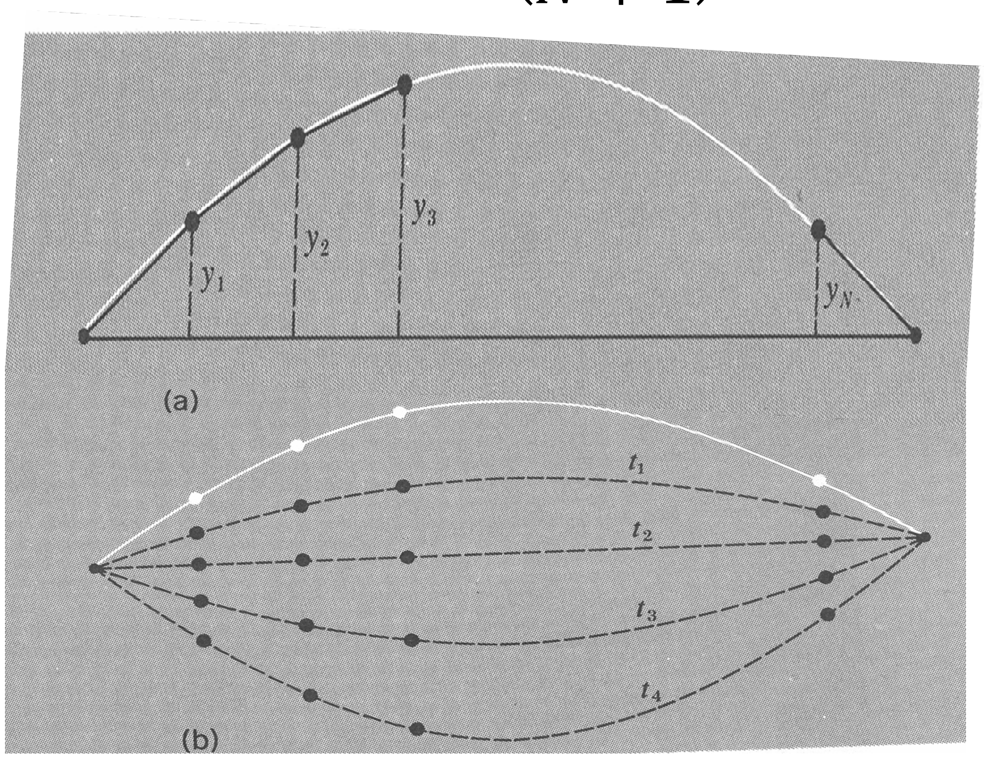
$$x_{n,k}(t) = A_{n,k} \cos(\omega_k t + \delta_k)$$

Many Coupled Oscillators

What do these modes look like?

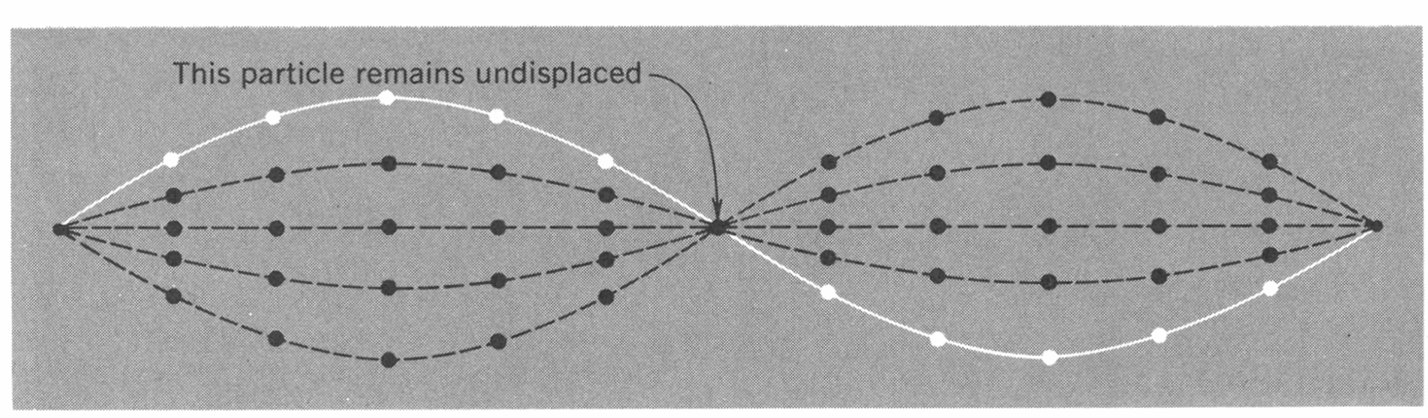
- Lowest order mode has $k = 1$…

$$x_{n,1}(t) = C_1 \sin\left(\frac{n\pi}{N+1}\right) \cos \omega_1 t$$

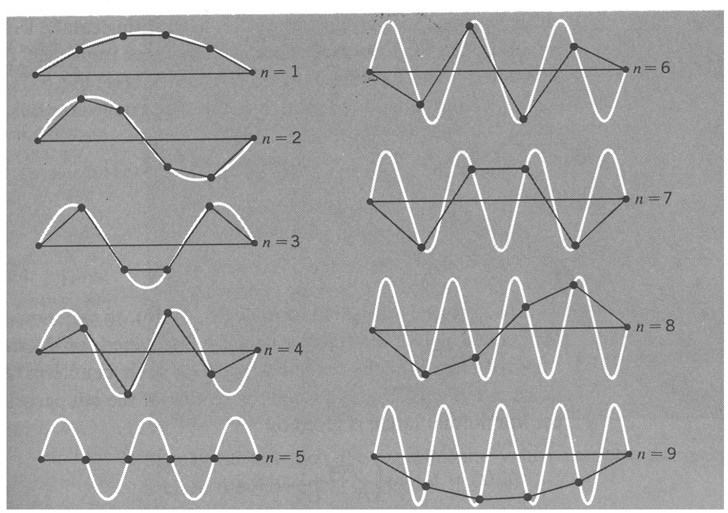


Many Coupled Oscillators

- Positions of masses in the second mode:



- Positions for 4 particles in modes $k = 1,2,3,4$:



Vibrations of Continuous Systems

- Amplitude of mass n for normal mode k:

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

- Frequency of normal mode k:

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- Solution for normal modes:

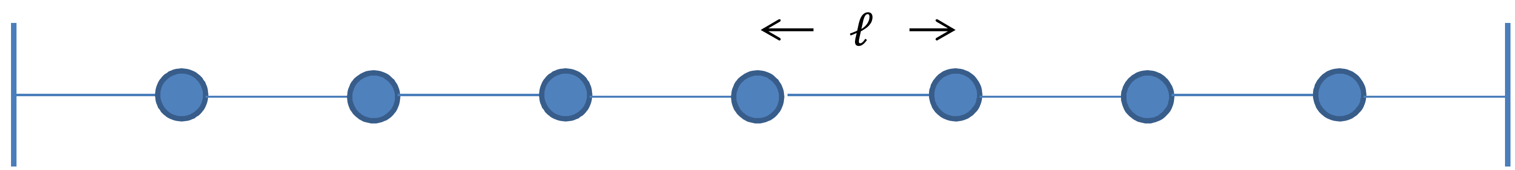
$$x_n(t) = A_{n,k} \cos \omega_k t$$

- General solution:

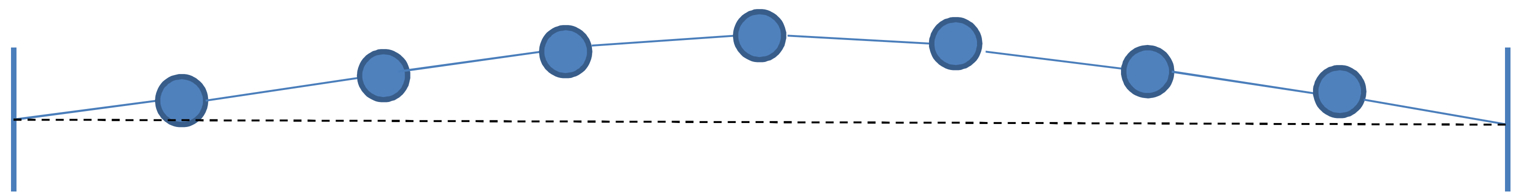
$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

Another Example

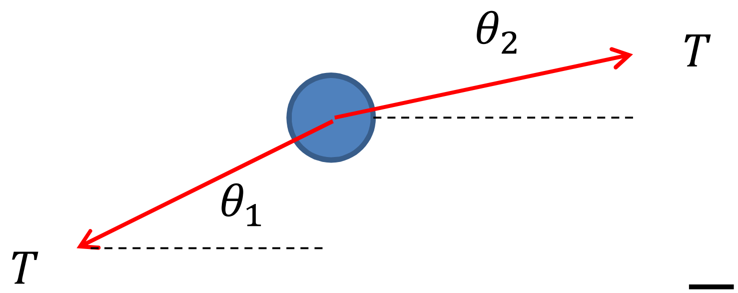
- Discrete masses on an elastic string with tension T:



- Consider transverse displacements:

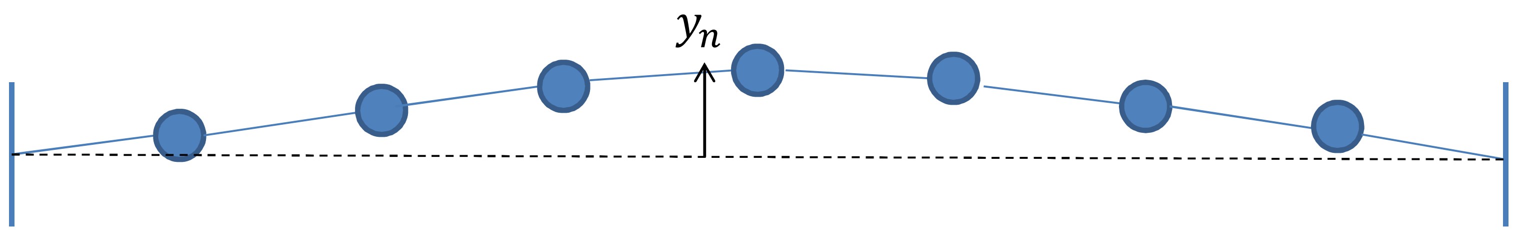


- Vertical force on one mass:



$$F_n = T \sin\theta_2 - T \sin\theta_1$$
$$= T(\theta_2 - \theta_1)$$
$$= \frac{T}{\ell}[(y_{n+1} - y_n) - (y_n - y_{n-1})]$$

Another Example



- Equation of motion for mass n:

$$m\,\ddot{y}_n = F_n = \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

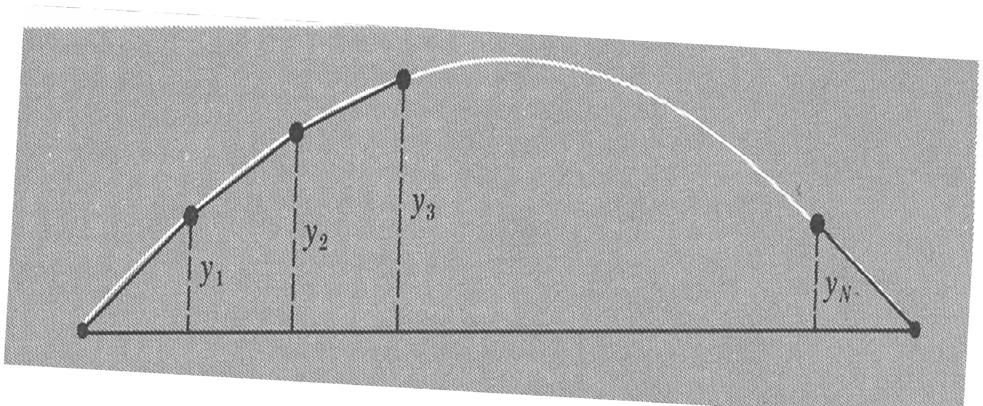
$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

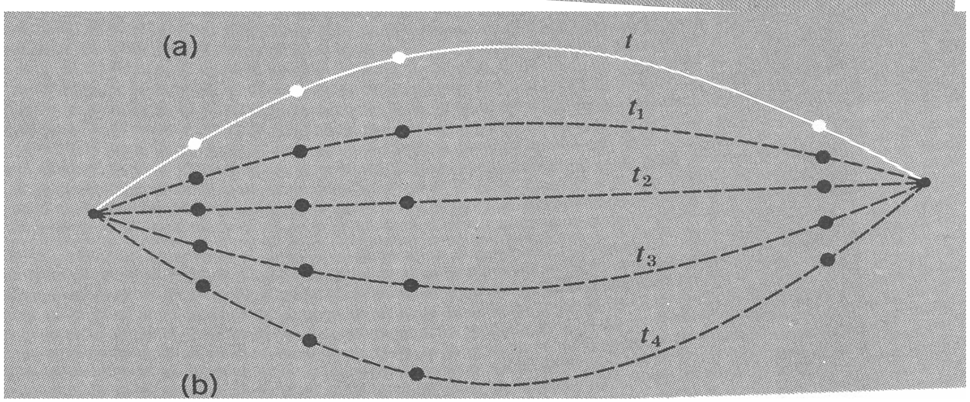
$$(\omega_0)^2 = \frac{T}{m\ell}$$

- Normal modes:

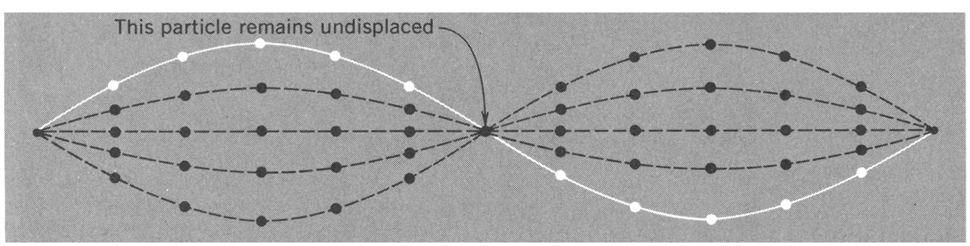
$$y_{n,k}(t) = A_{n,k}\cos(\omega_k t - \delta_k)$$

Masses on a String





First normal mode



Second normal mode

Continuous Systems

- What happens when the number of masses goes to infinity, while the linear mass density remains constant?

$$m\,\ddot{y}_n = \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

$$\frac{m}{\ell} \to \mu$$

$$\frac{y_{n+1} - y_n}{\ell} \to \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \qquad \frac{(y_n - y_{n-1})}{\ell} \to \left(\frac{\partial y}{\partial x}\right)_x$$

$$\mu\ell\,\frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

Continuous Systems

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\ell}$$

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}$$

The Wave Equation: $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$ $$v = \sqrt{T/\mu}$$

Solutions

- When we had N masses, the solutions were
$$y_{n,k}(t) = A_{n,k} \cos(\omega_k t - \delta_k)$$
  - n labels the mass along the string
  - With a continuous system, n is replaced by x.
- Proposed solution to the wave equation for the continuous string:
$$y(x,t) = f(x) \cos \omega t$$
- Derivatives:
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x) \cos \omega t$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} \cos \omega t$$

Solutions

- Substitute into the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\omega^2}{v^2} f(x)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{v^2} f(x) = 0$$

- This is the same differential equation as for the harmonic oscillator.
- Solutions are $f(x) = A\sin(\omega x/v) + B\cos(\omega x/v)$

Solutions

$$f(x) = A\sin(\omega x/v) + B\cos(\omega x/v)$$

- Boundary conditions at the ends of the string:

$$f(0) = f(L) = 0$$

$$f(x) = A\sin(\omega x/v) \text{ where } \omega L/v = n\pi$$

- Solutions can be written:

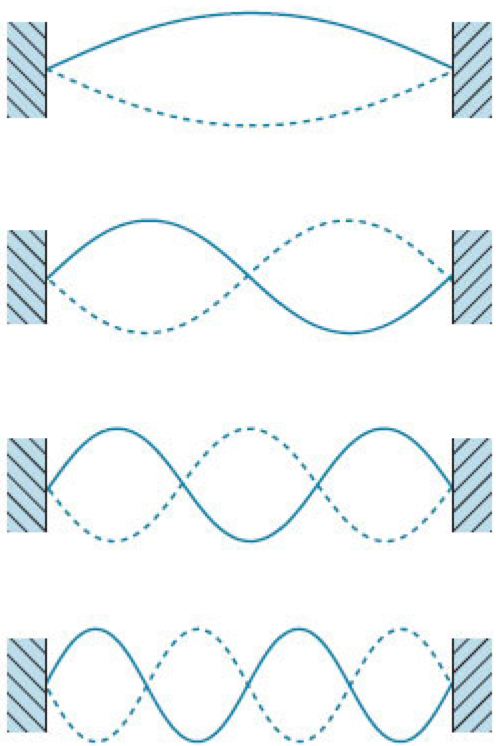
$$f_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right)$$

- Complete solution describing the motion of the whole string:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

Properties of the Solutions

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

$$\lambda_n = \frac{2L}{n}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$