PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

# Waves & Oscillations

Lecture 12 – French, Chapter 5

Spring 2016 Semester

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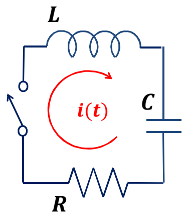
## The Eigenvalue Problem

- If $\boldsymbol{A}$ is an $n \times n$ matrix and $\vec{\boldsymbol{u}}$ is a vector, find the numbers $\lambda$ that satisfy
$$\boldsymbol{A}\,\vec{\boldsymbol{u}} = \lambda\,\vec{\boldsymbol{u}}$$
- Re-write the equation this way:
$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\,\vec{\boldsymbol{u}} = 0$$
- This is true only if
$$\det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = 0$$
- For a $2 \times 2$ matrix, this is:
$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$
- This is a second order polynomial in $\lambda$. Use the quadratic formula to find the roots.

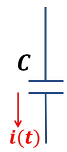
## The Eigenvalue Problem

- The eigenvectors are vectors $\vec{\boldsymbol{u}}_i$ such that
$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\vec{\boldsymbol{u}}_i = 0$$
- There are $n$ eigenvalues and $n$ eigenvectors
- If $\vec{\boldsymbol{u}}_i$ is an eigenvector, then $\alpha\vec{\boldsymbol{u}}_i$ is also an eigenvector.
- Sometimes it is convenient to choose the eigenvectors so that they have unit length:
$$\hat{\boldsymbol{u}}_i \cdot \hat{\boldsymbol{u}}_i = 1$$
- Eigenvectors are orthogonal:
$$\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_j = 0 \text{ when } i \neq j$$
- An arbitrary vector $\vec{\boldsymbol{v}}$ can be written as a linear combination of the eigenvectors:
$$\vec{\boldsymbol{v}} = a_1\hat{\boldsymbol{u}}_1 + a_2\hat{\boldsymbol{u}}_2 + \cdots$$

## A Circuit with One Loop



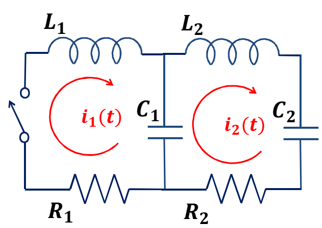

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

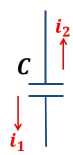



$$-\frac{1}{C}\int i(t)\,dt$$

$$\frac{d^2i}{dt^2} + \gamma\frac{di}{dt} + (\omega_0)^2 i(t) = 0$$

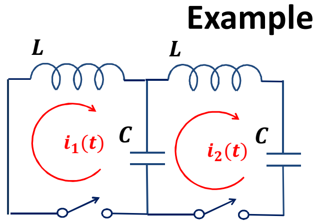
## A Circuit with Two Loops




$$-\frac{1}{C}\int (i_1 - i_2)\,dt$$

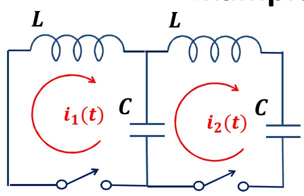
## Example




$$-L\frac{di_1}{dt} - \frac{1}{C}\int (i_1 - i_2)dt = 0$$

$$-L\frac{di_2}{dt} - \frac{1}{C}\int i_2 dt - \frac{1}{C}\int (i_2 - i_1)dt = 0$$

## Example




$$\frac{d^2 i_1}{dt^2} + (\omega_0)^2 (i_1 - i_2) = 0$$
$$\frac{d^2 i_2}{dt^2} + (\omega_0)^2 (2i_2 - i_1) = 0$$

## Normal Modes of Oscillation

- What are the frequencies of the normal modes of oscillation?
  - Let $\vec{\imath}(t) = \vec{\imath}\cos\omega t$
  - Then $\frac{d^2\vec{\imath}}{dt^2} = -\omega^2 \vec{\imath}(t)$
- Substitute into the pair of differential equations:

$$(-\omega^2 + (\omega_0)^2) i_1 - (\omega_0)^2 i_2 = 0$$
$$(-\omega^2 + 2(\omega_0)^2) i_2 - (\omega_0)^2 i_1 = 0$$

- Write it as a matrix:

$$\begin{pmatrix} (\omega_0)^2 - \omega^2 & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

## Eigenvalue Problem

$$\begin{pmatrix} (\omega_0)^2 - \omega^2 & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

- For simplicity, let $\lambda = \omega^2$ and calculate the determinant:

$$\begin{vmatrix} (\omega_0)^2 - \lambda & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \lambda \end{vmatrix} = (\lambda - (\omega_0)^2)(\lambda - 2(\omega_0)^2) - (\omega_0)^4$$
$$= \lambda^2 - 3\lambda(\omega_0)^2 + (\omega_0)^4 = 0$$

- Roots of the polynomial:

$$\lambda = \frac{3}{2}(\omega_0)^2 \pm \frac{1}{2}\sqrt{9(\omega_0)^4 - 4(\omega_0)^4}$$
$$\omega^2 = (\omega_0)^2 \left(\frac{3 \pm \sqrt{5}}{2}\right)$$

## Eigenvalue Problem

- The eigenvectors are obtained by substituting in each eigenvalue.
  - When $\omega^2 = (\omega_0)^2\left(\frac{3+\sqrt{5}}{2}\right)$

$$\frac{(\omega_0)^2}{2}\begin{pmatrix}-1-\sqrt{5} & -2 \\ -2 & 1-\sqrt{5}\end{pmatrix}\begin{pmatrix}i_1 \\ i_2\end{pmatrix} = 0$$

$$i_1 = \left(\frac{1-\sqrt{5}}{2}\right)i_2$$

  - First normal mode of oscillation:

$$\vec{q}_1 = \boldsymbol{A}\begin{pmatrix}1-\sqrt{5} \\ 2\end{pmatrix}\cos(\omega_1 t + \boldsymbol{\alpha})$$

## Eigenvalue Problem

- The eigenvectors are obtained by substituting in each eigenvalue.
  - When $\omega^2 = (\omega_0)^2\left(\frac{3-\sqrt{5}}{2}\right)$

$$\frac{(\omega_0)^2}{2}\begin{pmatrix}-1+\sqrt{5} & -2 \\ -2 & 1+\sqrt{5}\end{pmatrix}\begin{pmatrix}i_1 \\ i_2\end{pmatrix} = 0$$

$$i_1 = \left(\frac{1+\sqrt{5}}{2}\right)i_2$$

  - Second normal mode of oscillation:

$$\vec{q}_2 = \boldsymbol{B}\begin{pmatrix}1+\sqrt{5} \\ 2\end{pmatrix}\cos(\omega_2 t + \boldsymbol{\beta})$$

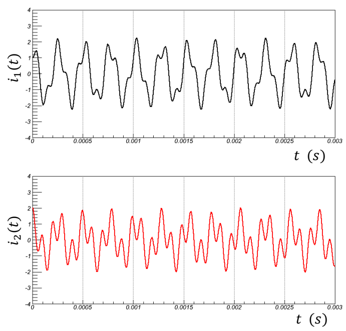
## Eigenvalue Problem

- The original “coordinates” are the sum of the normal modes of oscillation:

$$i_1(t) = A(1-\sqrt{5})\cos(\omega_1 t + \alpha) + B(1+\sqrt{5})\cos(\omega_2 t + \beta)$$
$$i_2(t) = 2A\cos(\omega_1 t + \alpha) + 2B\cos(\omega_2 t + \beta)$$

- The constants of integration can be chosen to satisfy the initial conditions
  - For example, suppose that $i_1(0) = i_o$ and $i_2(0) = 0$
  - Then $A = -B, 2A = i_o$ ➔ $A = \frac{i_0}{2}, B = -\frac{i_0}{2}$

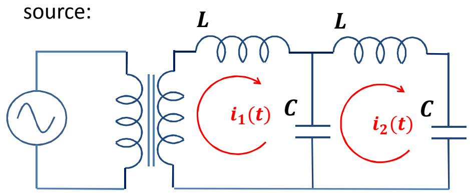
## Two Loop Circuit




$$f_0 = \frac{\omega_0}{2\pi} = 1\ kHz$$
$$i_0 = 1\ A$$

## Forced Coupled Circuit

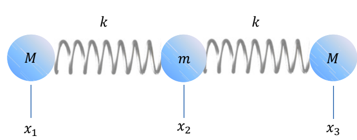
- If the two loops were driven with a sinusoidal voltage source:




- Resonance would occur at the frequency of each normal mode:

$$\omega^2 = (\omega_0)^2 \left(\frac{3 \pm \sqrt{5}}{2}\right)$$

## Three Masses




- Forces on each mass:

$$F_1 = -k(x_1 - x_2)$$
$$F_2 = k(x_1 - x_2) - k(x_2 - x_3) = kx_1 - 2kx_2 + kx_3$$
$$F_3 = -k(x_3 - x_2)$$

## Three Masses

- Equations of motion:

$$F_1 = M\ddot{x}_1 = -k(x_1 - x_2)$$
$$F_2 = m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3$$
$$F_3 = M\ddot{x}_3 = -k(x_3 - x_2)$$

- Let $\omega_0^2 = \frac{k}{M}$ and ${\omega_0'}^2 = \frac{k}{m}$
- Then,

$$\ddot{x}_1 + \omega_0^2 x_1 - \omega_0^2 x_2 = 0$$
$$\ddot{x}_2 - {\omega_0'}^2 x_1 + 2{\omega_0'}^2 x_2 - {\omega_0'}^2 x_3 = 0$$
$$\ddot{x}_3 - \omega_0^2 x_2 + \omega_0^2 x_3 = 0$$

- Write this as a matrix…

## Three Masses

- Assume that solutions are of the form

$$x_i(t) = A_i \cos(\omega t + \varphi)$$

- Then $\ddot{x}_i = -\omega^2 x_i$ and

$$\begin{pmatrix} -\omega^2 + \omega_0^2 & -\omega_0^2 & 0 \\ -{\omega_0'}^2 & -\omega^2 + 2{\omega_0'}^2 & -{\omega_0'}^2 \\ 0 & -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

- Let $\lambda = \omega^2$. Then this will be true if

$$\begin{vmatrix} \lambda - \omega_0^2 & \omega_0^2 & 0 \\ {\omega_0'}^2 & \lambda - 2{\omega_0'}^2 & {\omega_0'}^2 \\ 0 & \omega_0^2 & \lambda - \omega_0^2 \end{vmatrix} = 0$$

## Three Masses

- Expand the determinant:

$$\begin{vmatrix} \lambda - \omega_0^2 & \omega_0^2 & 0 \\ {\omega_0'}^2 & \lambda - 2{\omega_0'}^2 & {\omega_0'}^2 \\ 0 & \omega_0^2 & \lambda - \omega_0^2 \end{vmatrix}$$
$$= (\lambda - \omega_0^2)\left[(\lambda - 2{\omega_0'}^2)(\lambda - \omega_0^2) - {\omega_0'}^2\omega_0^2\right]$$
$$- \omega_0^2{\omega_0'}^2(\lambda - \omega_0^2)$$
$$= (\lambda - \omega_0^2)\left[(\lambda - 2{\omega_0'}^2)(\lambda - \omega_0^2) - 2{\omega_0'}^2\omega_0^2\right]$$
$$= \lambda(\lambda - \omega_0^2)(\lambda - \omega_0^2 - 2{\omega_0'}^2)$$

## Three Masses

$$\lambda(\lambda-\omega_0^2)\left(\lambda-\omega_0^2-2{\omega_0'}^2\right)=0$$

- The roots are

$$\lambda=\omega^2=0$$
$$\lambda=\omega^2=\omega_0^2$$
$$\lambda=\omega^2=\omega_0^2+2{\omega_0'}^2$$

- What motion does this correspond to?
- Calculate the eigenvectors…

## Three Masses

- When $\lambda=0$:

$$\begin{pmatrix}\omega_0^2 & -\omega_0^2 & 0\\ -{\omega_0'}^2 & 2{\omega_0'}^2 & -{\omega_0'}^2\\ 0 & -\omega_0^2 & \omega_0^2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

- This tells us that

$$x_2=x_1$$
$$x_3=x_2$$

- All masses move in the same direction at once.
- This is just a translation of the entire system.

## Three Masses

- When $\lambda=\omega_0^2$

$$\begin{pmatrix}0 & -\omega_0^2 & 0\\ -{\omega_0'}^2 & 2{\omega_0'}^2-\omega_0^2 & -{\omega_0'}^2\\ 0 & -\omega_0^2 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

- This tells us that

$$x_2=0$$
$$x_1=-x_3$$

- Motion is described by

$$x_1(t)=A\cos(\omega_0 t+\varphi)$$
$$x_2(t)=0$$
$$x_3(t)=-A\cos(\omega_0 t+\varphi)$$

## Three Masses

- When $\lambda = \omega_0^2 + 2{\omega_0'}^2$

$$\begin{pmatrix} -2{\omega_0'}^2 & -\omega_0^2 & 0 \\ -{\omega_0'}^2 & \omega_0^2 & -{\omega_0'}^2 \\ 0 & -\omega_0^2 & -2{\omega_0'}^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

- This tells us that

$$2{\omega_0'}^2 x_1 = -\omega_0^2 x_2$$
$$\omega_0^2 x_2 = -2{\omega_0'}^2 x_2$$

- Which means that

$$x_1 = x_3$$
$$x_2 = -\frac{2{\omega_0'}^2}{\omega_0^2} x_1$$