PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 11 – French, Chapter 5

Spring 2016 Semester

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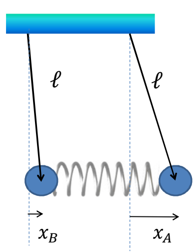
Two Coupled Oscillators



- The spring is stretched by the amount $x_A - x_B$
- Restoring force on pendulum A:
  $F_A = -k(x_A - x_B)$
- Restoring force on pendulum B:
  $F_B = k(x_A - x_B)$

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$
$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

Two Coupled Oscillators

$$\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = 0$$
$$\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0$$
$$\omega_0 = \sqrt{g/\ell},\ \omega_c = \sqrt{k/m}$$

- Add equations for A and B together:
$$\frac{d^2}{dt^2}(x_A + x_B) + (\omega_0)^2(x_A + x_B) = 0$$
- Subtract equations A and B:
$$\frac{d^2}{dt^2}(x_A - x_B) + [(\omega_0)^2 + 2(\omega_c)^2](x_A - x_B) = 0$$

Two Coupled Oscillators

$$\frac{d^2}{dt^2}(x_A + x_B) + (\omega_0)^2(x_A + x_B) = 0$$
$$\frac{d^2}{dt^2}(x_A - x_B) + (\omega')^2(x_A - x_B) = 0$$
$$\omega_0 = \sqrt{g/\ell}\ , \omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2}$$

- Normal coordinates:

$$q_1 = x_A + x_B$$
$$q_2 = x_A - x_B$$

- Decoupled equations:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = 0$$
$$\ddot{q}_2 + (\omega')^2 q_2 = 0$$

Two Coupled Oscillators

- Decoupled equations:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = 0$$
$$\ddot{q}_2 + (\omega')^2 q_2 = 0$$

- Solutions are

$$q_1(t) = A\cos(\omega_0 t + \alpha)$$
$$q_2(t) = B\cos(\omega' t + \beta)$$

- The variables q_1 and q_2 are called “normal coordinates”.

Initial Conditions

- Suppose we had the initial conditions:

$$x_A = A_0 \qquad \dot{x}_A = 0$$
$$x_B = 0 \qquad \dot{x}_B = 0$$

- These can be satisfied with $\alpha = \beta = 0$:

$$x_A(t) = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}A\cos\omega_0 t + \frac{1}{2}B\cos\omega' t$$
$$x_B(t) = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}A\cos\omega_0 t - \frac{1}{2}B\cos\omega' t$$

- At time $t = 0$,

$$\frac{1}{2}(A + B) = A_0 \qquad \frac{1}{2}(A - B) = 0$$

- Now we know that $A = B = A_0$.

Initial Conditions

- Velocity:

$$\dot{x}_A(t) = -\frac{1}{2}A_0\omega_0 \sin\omega_0 t - \frac{1}{2}A_0\omega' \sin\omega' t$$
$$\dot{x}_B(t) = -\frac{1}{2}A_0\omega_0 \sin\omega_0 t + \frac{1}{2}A_0\omega' \sin\omega' t$$

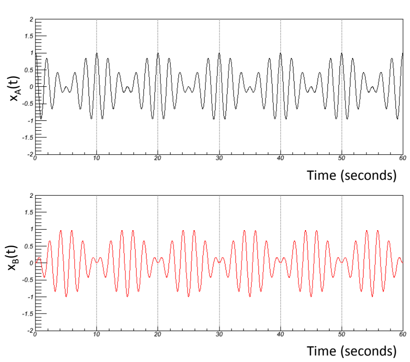
- Initial conditions are satisfied at $t = 0$.

Initial Conditions

- Complete solution:

$$x_A(t) = \frac{1}{2}A_0(\cos\omega_0 t + \cos\omega' t)$$
$$= A_0 \cos\left(\frac{\omega' - \omega_0}{2}t\right)\cos\left(\frac{\omega' + \omega_0}{2}t\right)$$
$$x_B(t) = \frac{1}{2}A_0(\cos\omega_0 t - \cos\omega' t)$$
$$= A_0 \sin\left(\frac{\omega' - \omega_0}{2}t\right)\sin\left(\frac{\omega' + \omega_0}{2}t\right)$$

Coupled Oscillators



$$f_0 = \frac{\omega_0}{2\pi} = 0.5\ Hz$$
$$f_c = \frac{\omega_c}{2\pi} = 0.6\ Hz$$

Coupled Oscillators

- This procedure worked, but the problem was very simple. How can we apply this in general?
- Procedure:
  1. Construct the set of coupled differential equations
  2. Assume solutions are of the form
  $$q_i(t) = A_i \cos(\omega t + \varphi_i)$$
  3. Substitute into the differential equations
  4. Find the values of ω that satisfy the resulting matrix equation (eigenvalues).
  5. Solve for constants of integration

Coupled Oscillators

$$\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = 0$$
$$\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0$$
$$\omega_0 = \sqrt{g/\ell},\ \omega_c = \sqrt{k/m}$$

- Let $x_A(t) = A\cos(\omega t + \alpha)$
- Let $x_B(t) = B\cos(\omega t + \beta)$

- Second derivatives:
  - $\ddot{x}_A(t) = -A\omega^2 \cos(\omega t + \alpha) = -\omega^2 x_A$
  - $\ddot{x}_B(t) = -B\omega^2 \cos(\omega t + \beta) = -\omega^2 x_B$

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

Coupled Oscillators

- A very important result:

  If the matrix equation
  $$\boldsymbol{A}\vec{v} = 0$$
  for any vector $\vec{v}$, then
  $$\det \boldsymbol{A} = 0$$
- You are expected to be able to calculate the determinant of an arbitrary 2x2 or 3x3 matrix!

Coupled Oscillators

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

- The determinant of the matrix is:
$$[(\omega_0)^2 + (\omega_c)^2 - \omega^2]^2 - (\omega_c)^4 = 0$$
- Expand the polynomial in $\lambda = \omega^2$:
$$\lambda^2 - 2\big((\omega_0)^2 + (\omega_c)^2\big) + [(\omega_0)^2 + (\omega_c)^2]^2 - (\omega_c)^4 = 0$$
- Use the quadratic formula:
$$\lambda = \big((\omega_0)^2 + (\omega_c)^2\big) \pm \sqrt{(\omega_c)^4}$$
- Oscillation frequencies are
$$\omega^2 = (\omega_0)^2$$
$$\omega'^2 = (\omega_0)^2 + 2(\omega_c)^2$$

Coupled Oscillators

- Eigenvalue problem:
$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \omega^2 \begin{pmatrix} x_A \\ x_B \end{pmatrix}$$
- First eigenvector: substitute $\omega^2 = (\omega_0)^2$
$$\begin{pmatrix} (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_B = x_A$$
- Second eigenvector: substitute $\omega'^2 = (\omega_0)^2 + 2(\omega_c)^2$
$$\begin{pmatrix} -(\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & -(\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_B = -x_A$$

Normal Coordinates

- The first normal mode of vibration corresponds to the first eigenvector:
$$\vec{q}_1(t) = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t + \alpha)$$
- The second normal mode of vibration corresponds to the second eigenvector:
$$\vec{q}_2(t) = B \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega' t + \beta)$$
- Arbitrary motion:
$$\begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t + \boldsymbol{\alpha}) + \boldsymbol{B} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega' t + \boldsymbol{\beta})$$
- Initial conditions determine constants of integration.

Forced Coupled Oscillator

- What happens when a driving force is applied to one of the oscillators?

$$\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = F_0/m \cos\omega t$$
$$\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0$$

- Normal coordinates:

$$q_1 = x_A + x_B$$
$$q_2 = x_A - x_B$$

- Equations of motion:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = F_0/m \cos\omega t$$
$$\ddot{q}_2 + (\omega')^2 q_2 = F_0/m \cos\omega t$$

- Decoupled equations which we know how to solve.

Forced Coupled Oscillators

- Steady state amplitudes:

$$A_1(\omega) = \frac{F_0/m}{(\omega_0)^2 - \omega^2}$$
$$A_2(\omega) = \frac{F_0/m}{(\omega')^2 - \omega^2}$$

- Motion of individual masses:

$$x_A(t) = \frac{1}{2}(q_1(t) + q_2(t))$$

- Amplitude of steady state oscillations:

$$A(\omega) = \frac{F_0}{2m}\left(\frac{1}{(\omega_0)^2 - \omega^2} + \frac{1}{(\omega')^2 - \omega^2}\right)$$
$$= \frac{F_0}{2m}\frac{(\omega')^2 + (\omega_0)^2 - 2\omega^2}{\left((\omega_0)^2 - \omega^2\right)\left((\omega')^2 - \omega^2\right)} = \frac{F_0}{m}\frac{\left((\omega_0)^2 + (\omega_c)^2\right) - \omega^2}{\left((\omega_0)^2 - \omega^2\right)\left((\omega')^2 - \omega^2\right)}$$

Forced Coupled Oscillators

- Motion of individual masses:

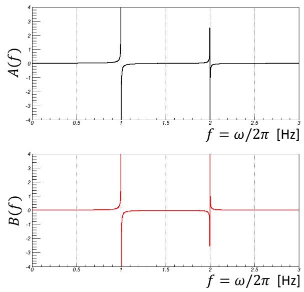
$$x_B(t) = \frac{1}{2}(q_1(t) - q_2(t))$$

- Amplitude of steady state oscillations:

$$B(\omega) = \frac{F_0}{2m}\left(\frac{1}{(\omega_0)^2 - \omega^2} - \frac{1}{(\omega')^2 - \omega^2}\right)$$
$$= \frac{F_0}{2m}\frac{(\omega')^2 - (\omega_0)^2}{\left((\omega_0)^2 - \omega^2\right)\left((\omega')^2 - \omega^2\right)}$$
$$= \frac{F_0}{m}\frac{(\omega_c)^2}{\left((\omega_0)^2 - \omega^2\right)\left((\omega')^2 - \omega^2\right)}$$

- Now the system resonates at two frequencies: ω_o and ω'.

Forced Coupled Oscillators



Phase reverses when crossing each of the resonant frequencies.