

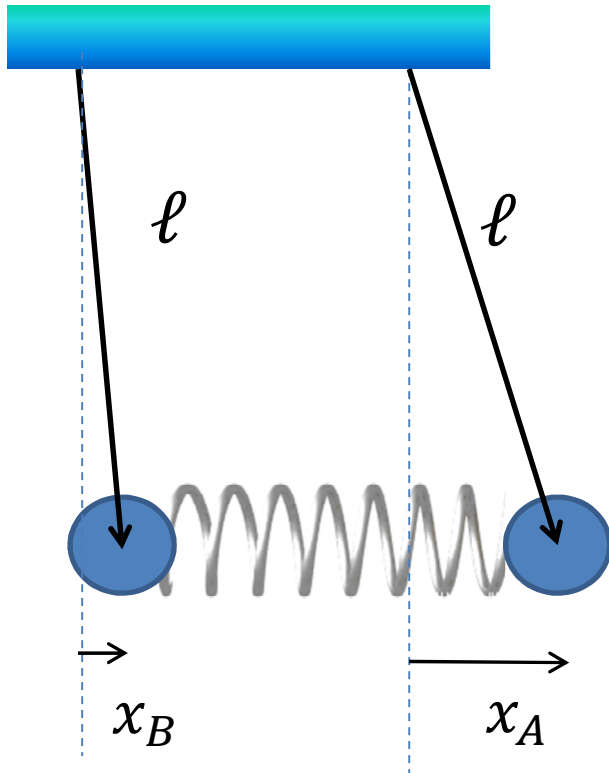
Physics 42200  
**Waves & Oscillations**

Lecture 11 – French, Chapter 5

Spring 2016 Semester

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# Two Coupled Oscillators



- The spring is stretched by the amount  $x_A - x_B$

- Restoring force on pendulum A:

$$F_A = -k(x_A - x_B)$$

- Restoring force on pendulum B:

$$F_B = k(x_A - x_B)$$

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$
$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

# Two Coupled Oscillators

$$\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2x_B = 0$$

$$\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2x_A = 0$$

$$\omega_0 = \sqrt{g/\ell}, \quad \omega_c = \sqrt{k/m}$$

- Add equations for A and B together:

$$\frac{d^2}{dt^2}(x_A + x_B) + (\omega_0)^2(x_A + x_B) = 0$$

- Subtract equations A and B:

$$\frac{d^2}{dt^2}(x_A - x_B) + [(\omega_0)^2 + 2(\omega_c)^2](x_A - x_B) = 0$$

# Two Coupled Oscillators

$$\frac{d^2}{dt^2}(x_A + x_B) + (\omega_0)^2(x_A + x_B) = 0$$

$$\frac{d^2}{dt^2}(x_A - x_B) + (\omega')^2(x_A - x_B) = 0$$

$$\omega_0 = \sqrt{g/\ell}, \quad \omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2}$$

- Normal coordinates:

$$q_1 = x_A + x_B$$

$$q_2 = x_A - x_B$$

- Decoupled equations:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = 0$$

$$\ddot{q}_2 + (\omega')^2 q_2 = 0$$

# Two Coupled Oscillators

- Decoupled equations:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = 0$$

$$\ddot{q}_2 + (\omega')^2 q_2 = 0$$

- Solutions are

$$q_1(t) = A \cos(\omega_0 t + \alpha)$$

$$q_2(t) = B \cos(\omega' t + \beta)$$

- The variables  $q_1$  and  $q_2$  are called “normal coordinates”.

# Initial Conditions

- Suppose we had the initial conditions:

$$x_A = A_0 \quad \dot{x}_A = 0$$

$$x_B = 0 \quad \dot{x}_B = 0$$

- These can be satisfied with  $\alpha = \beta = 0$ :

$$x_A(t) = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}A \cos \omega_0 t + \frac{1}{2}B \cos \omega' t$$

$$x_B(t) = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}A \cos \omega_0 t - \frac{1}{2}B \cos \omega' t$$

- At time  $t = 0$ ,

$$\frac{1}{2}(A + B) = A_0 \quad \frac{1}{2}(A - B) = 0$$

- Now we know that  $A = B = A_0$ .

# Initial Conditions

- Velocity:

$$\dot{x}_A(t) = -\frac{1}{2}A_0\omega_0 \sin \omega_0 t - \frac{1}{2}A_0\omega' \sin \omega' t$$
$$\dot{x}_B(t) = -\frac{1}{2}A_0\omega_0 \sin \omega_0 t + \frac{1}{2}A_0\omega' \sin \omega' t$$

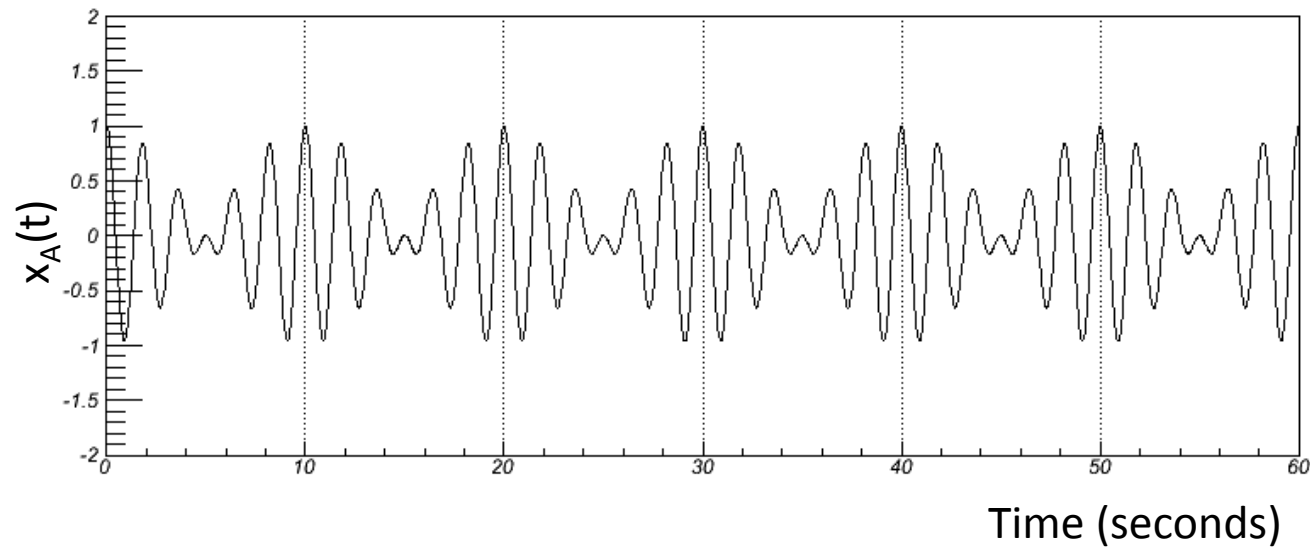
- Initial conditions are satisfied at  $t = 0$ .

# Initial Conditions

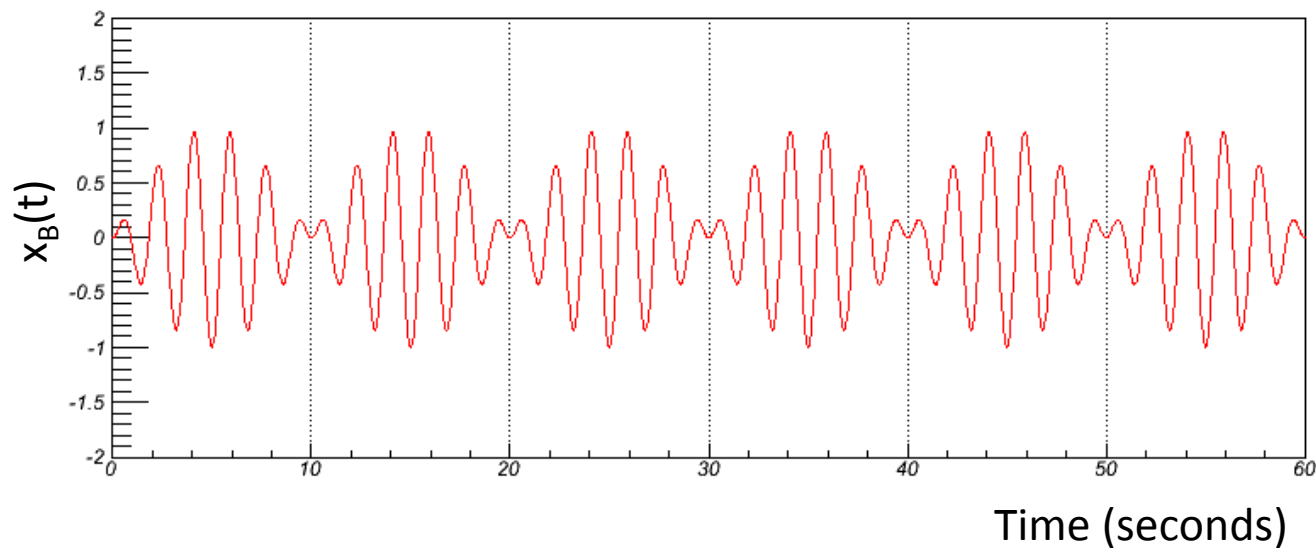
- Complete solution:

$$\begin{aligned}x_A(t) &= \frac{1}{2} A_0 (\cos \omega_0 t + \cos \omega' t) \\&= A_0 \cos \left( \frac{\omega' - \omega_0}{2} t \right) \cos \left( \frac{\omega' + \omega_0}{2} t \right) \\x_B(t) &= \frac{1}{2} A_0 (\cos \omega_0 t - \cos \omega' t) \\&= A_0 \sin \left( \frac{\omega' - \omega_0}{2} t \right) \sin \left( \frac{\omega' + \omega_0}{2} t \right)\end{aligned}$$

# Coupled Oscillators



$$f_0 = \frac{\omega_0}{2\pi} = 0.5 \text{ Hz}$$
$$f_c = \frac{\omega_c}{2\pi} = 0.6 \text{ Hz}$$



# Coupled Oscillators

- This procedure worked, but the problem was very simple. How can we apply this in general?

- Procedure:

1. Construct the set of coupled differential equations
2. Assume solutions are of the form

$$q_i(t) = A_i \cos(\omega t + \varphi_i)$$

3. Substitute into the differential equations
4. Find the values of  $\omega$  that satisfy the resulting matrix equation (eigenvalues).
5. Solve for constants of integration

# Coupled Oscillators

$$\begin{aligned}\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2x_B &= 0 \\ \ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2x_A &= 0\end{aligned}$$

$$\omega_0 = \sqrt{g/\ell}, \quad \omega_c = \sqrt{k/m}$$

➤ Let  $x_A(t) = A \cos(\omega t + \alpha)$

➤ Let  $x_B(t) = B \cos(\omega t + \beta)$

- Second derivatives:

➤  $\ddot{x}_A(t) = -A\omega^2 \cos(\omega t + \alpha) = -\omega^2 x_A$

➤  $\ddot{x}_B(t) = -B\omega^2 \cos(\omega t + \beta) = -\omega^2 x_B$

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

# Coupled Oscillators

- A very important result:

If the matrix equation

$$\mathbf{A}\vec{v} = 0$$

for any vector  $\vec{v}$ , then

$$\det \mathbf{A} = 0$$

- You are expected to be able to calculate the determinant of an arbitrary 2x2 or 3x3 matrix!

# Coupled Oscillators

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

- The determinant of the matrix is:

$$[(\omega_0)^2 + (\omega_c)^2 - \omega^2]^2 - (\omega_c)^4 = 0$$

- Expand the polynomial in  $\lambda = \omega^2$ :

$$\lambda^2 - 2((\omega_0)^2 + (\omega_c)^2) + [(\omega_0)^2 + (\omega_c)^2]^2 - (\omega_c)^4 = 0$$

- Use the quadratic formula:

$$\lambda = ((\omega_0)^2 + (\omega_c)^2) \pm \sqrt{(\omega_c)^4}$$

- Oscillation frequencies are

$$\begin{aligned} \omega^2 &= (\omega_0)^2 \\ \omega'^2 &= (\omega_0)^2 + 2(\omega_c)^2 \end{aligned}$$

# Coupled Oscillators

- Eigenvalue problem:

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \omega^2 \begin{pmatrix} x_A \\ x_B \end{pmatrix}$$

- First eigenvector: substitute  $\omega^2 = (\omega_0)^2$

$$\begin{pmatrix} (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_B = x_A$$

- Second eigenvector: substitute  $\omega'^2 = (\omega_0)^2 + 2(\omega_c)^2$

$$\begin{pmatrix} -(\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & -(\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_B = -x_A$$

# Normal Coordinates

- The first normal mode of vibration corresponds to the first eigenvector:

$$\vec{q}_1(t) = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t + \alpha)$$

- The second normal mode of vibration corresponds to the second eigenvector:

$$\vec{q}_2(t) = B \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega' t + \beta)$$

- Arbitrary motion:

$$\begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = \mathbf{A} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t + \alpha) + \mathbf{B} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega' t + \beta)$$

- Initial conditions determine constants of integration.

# Forced Coupled Oscillator

- What happens when a driving force is applied to one of the oscillators?

$$\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2x_B = F_0/m \cos \omega t$$
$$\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2x_A = 0$$

- Normal coordinates:

$$q_1 = x_A + x_B$$

$$q_2 = x_A - x_B$$

- Equations of motion:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = F_0/m \cos \omega t$$

$$\ddot{q}_2 + (\omega')^2 q_2 = F_0/m \cos \omega t$$

- Decoupled equations which we know how to solve.

# Forced Coupled Oscillators

- Steady state amplitudes:

$$A_1(\omega) = \frac{F_0/m}{(\omega_0)^2 - \omega^2}$$

$$A_2(\omega) = \frac{F_0/m}{(\omega')^2 - \omega^2}$$

- Motion of individual masses:

$$x_A(t) = \frac{1}{2} (q_1(t) + q_2(t))$$

- Amplitude of steady state oscillations:

$$\begin{aligned} A(\omega) &= \frac{F_0}{2m} \left( \frac{1}{(\omega_0)^2 - \omega^2} + \frac{1}{(\omega')^2 - \omega^2} \right) \\ &= \frac{F_0}{2m} \frac{(\omega')^2 + (\omega_0)^2 - 2\omega^2}{((\omega_0)^2 - \omega^2)((\omega')^2 - \omega^2)} = \frac{F_0}{m} \frac{((\omega_0)^2 + (\omega_c)^2) - \omega^2}{((\omega_0)^2 - \omega^2)((\omega')^2 - \omega^2)} \end{aligned}$$

# Forced Coupled Oscillators

- Motion of individual masses:

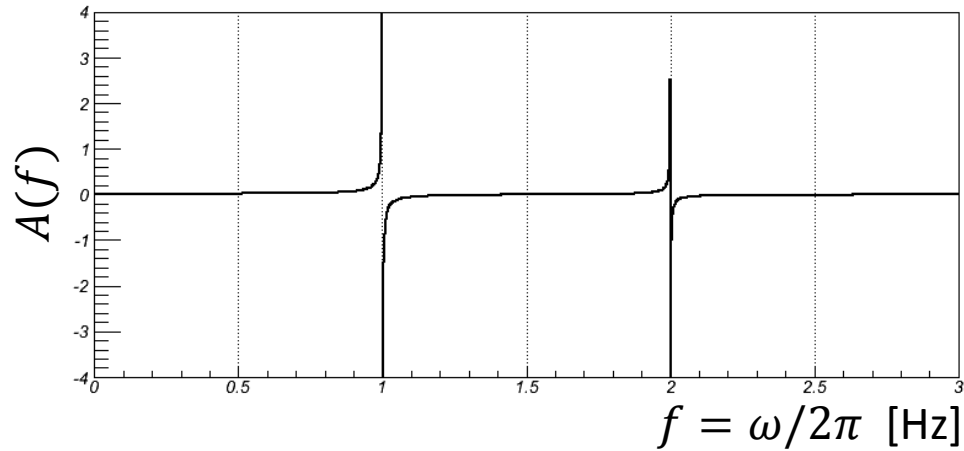
$$x_B(t) = \frac{1}{2} (q_1(t) - q_2(t))$$

- Amplitude of steady state oscillations:

$$\begin{aligned} B(\omega) &= \frac{F_0}{2m} \left( \frac{1}{(\omega_0)^2 - \omega^2} - \frac{1}{(\omega')^2 - \omega^2} \right) \\ &= \frac{F_0}{2m} \frac{(\omega')^2 - (\omega_0)^2}{((\omega_0)^2 - \omega^2)((\omega')^2 - \omega^2)} \\ &= \frac{F_0}{m} \frac{(\omega_c)^2}{((\omega_0)^2 - \omega^2)((\omega')^2 - \omega^2)} \end{aligned}$$

- Now the system resonates at two frequencies:  $\omega_0$  and  $\omega'$ .

# Forced Coupled Oscillators



Phase reverses when crossing each of the resonant frequencies.

