Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 10 – French, Chapter 5

Spring 2016 Semester

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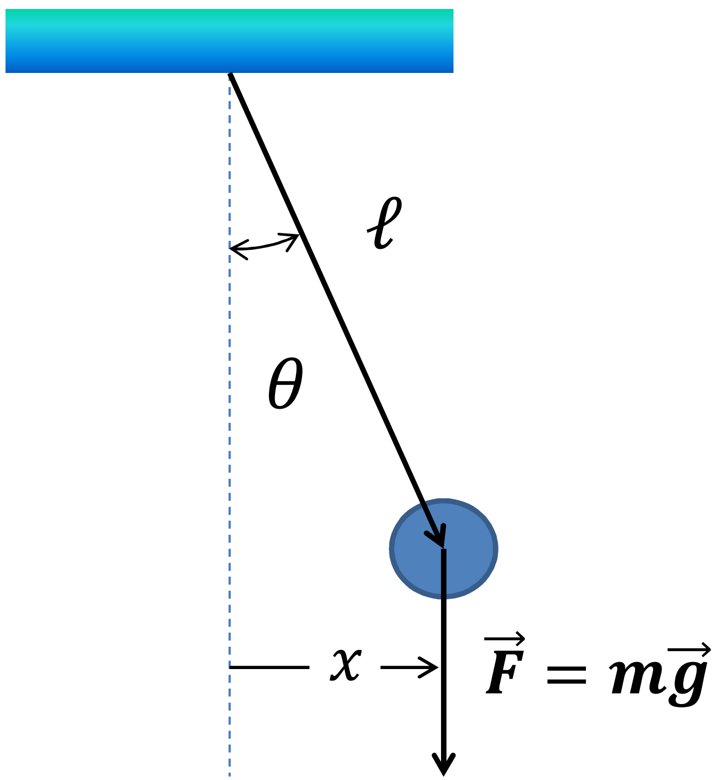
Coupled Oscillators

- Simple pendulum:



$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

$$\ddot{\theta} + \omega^2\theta \approx 0$$

$$\omega = \sqrt{\frac{\ell}{g}}$$

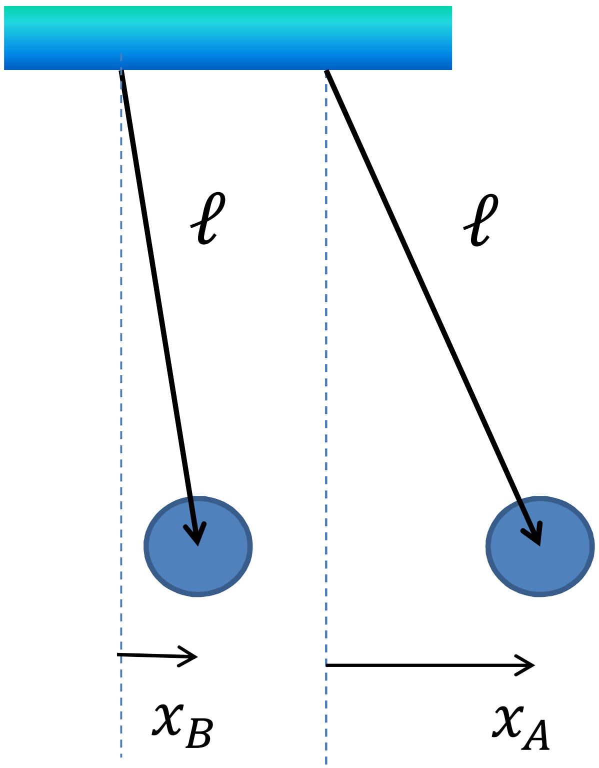
$$x \approx \ell\theta$$

$$\ddot{x} + \omega^2 x \approx 0$$

$$x(t) = \boldsymbol{A}\cos(\omega t + \boldsymbol{\alpha})$$

Two Independent Oscillators

- Two simple pendula:

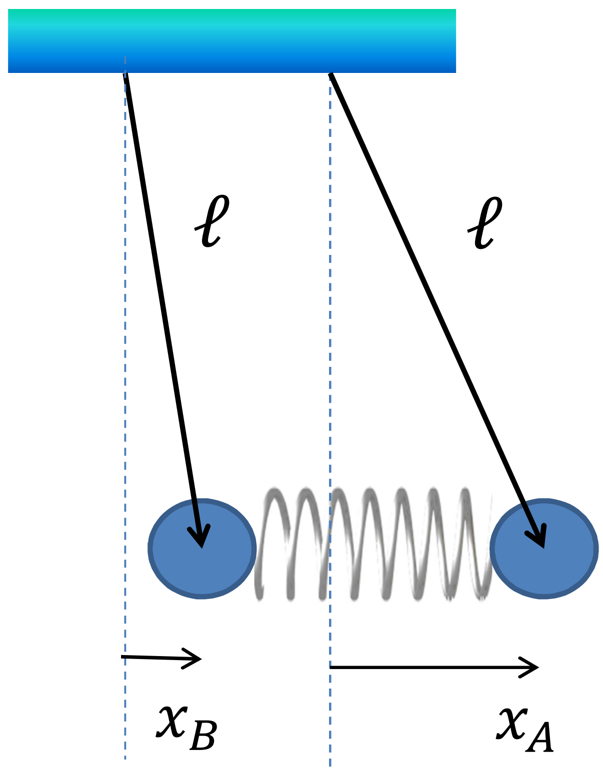


$$\ddot{x}_A + \omega^2 x_A \approx 0$$
$$\ddot{x}_B + \omega^2 x_B \approx 0$$

$$x_A(t) = \boldsymbol{A}\cos(\omega t + \boldsymbol{\alpha})$$
$$x_B(t) = \boldsymbol{B}\cos(\omega t + \boldsymbol{\beta})$$

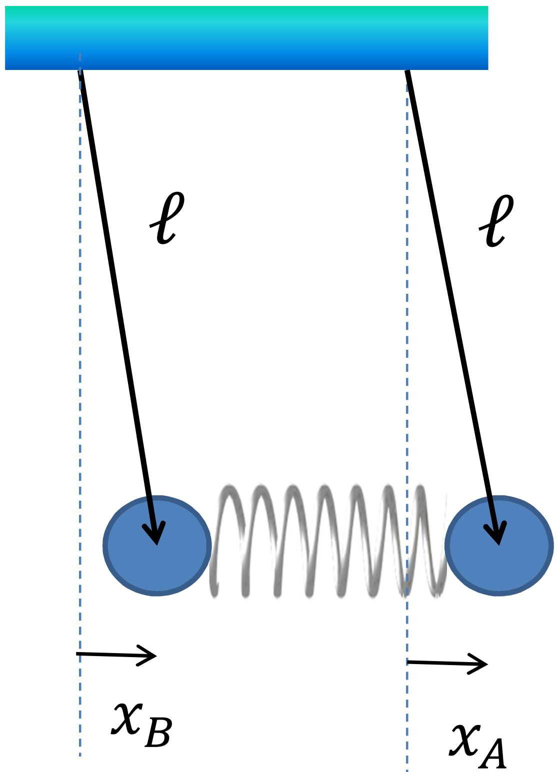
Two Coupled Oscillators

- Two simple pendula connected to a spring:



- There are many types of motion possible now.
- The solutions are not independent
- We can consider two “modes” of oscillation.

Two Coupled Oscillators



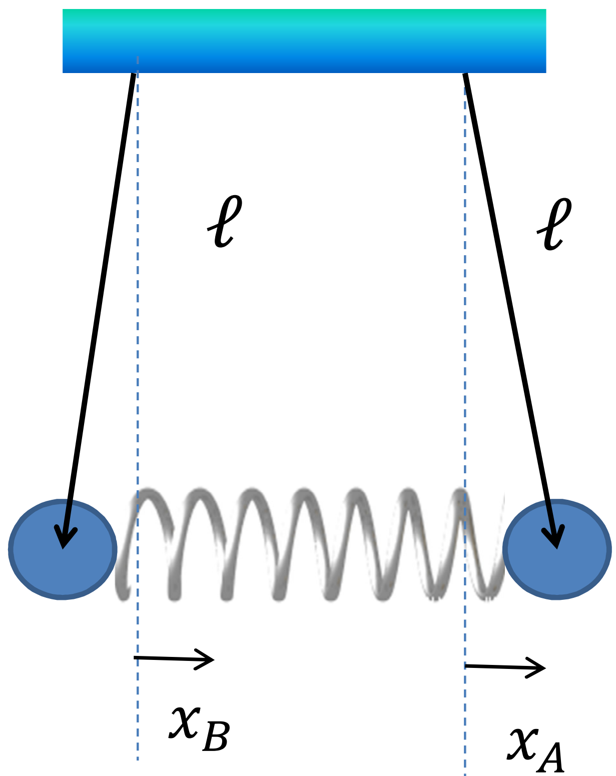
- The spring is at its relaxed length and exerts no force on A or B.
- Each pendulum oscillates at its natural frequency

$$\omega_0 = \sqrt{g/\ell}$$

$$x_A(t) = x_B(t)$$
$$= \boldsymbol{A}\cos(\omega t + \boldsymbol{\alpha})$$

One differential equation describes both pendula.

Two Coupled Oscillators



- In this case,

$$x_A = -x_B$$

- The spring is stretched or compressed and produces

$$F_A = -k(x_A - x_B) = -2kx_A$$

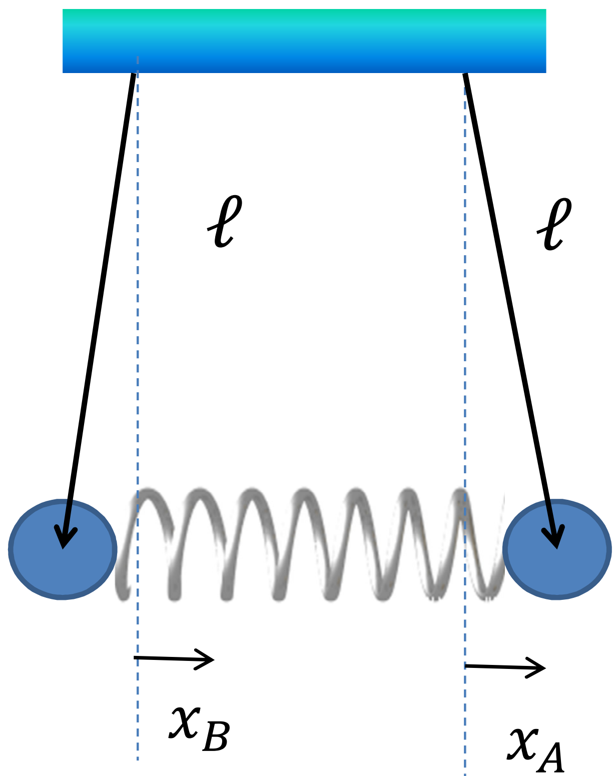
- Differential equation for A:

$$\ddot{x}_A + \left[(\omega_0)^2 + \frac{2k}{m}\right] x_A = 0$$

- Differential equation for B:

$$\ddot{x}_B + \left[(\omega_0)^2 + \frac{2k}{m}\right] x_B = 0$$

Two Coupled Oscillators



$$\ddot{x}_A + [(\omega_0)^2 + 2(\omega_c)^2]x_A = 0$$

- This is just the differential equation for simple harmonic motion:

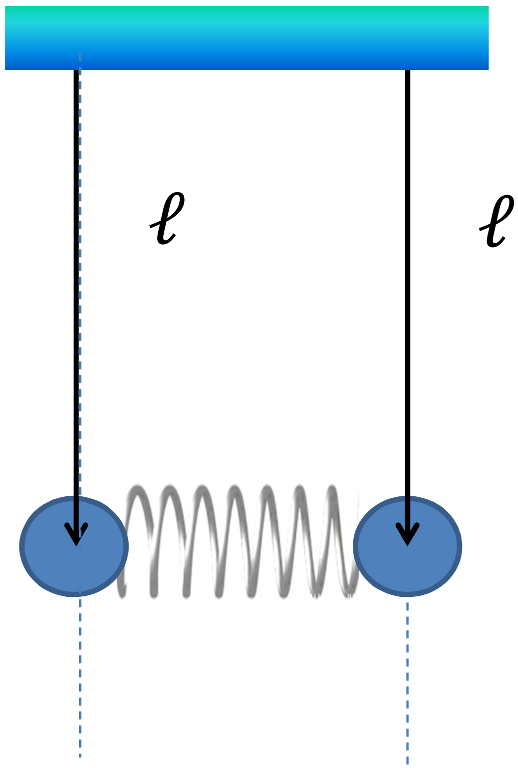
$$\ddot{x}_A + \omega'^2 x_A = 0$$

- Oscillation frequency is

$$\omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2}$$

- The spring increases the restoring force and increases the frequency.

Two Coupled Oscillators



- We have identified two modes of the system:
  - One oscillates with frequency

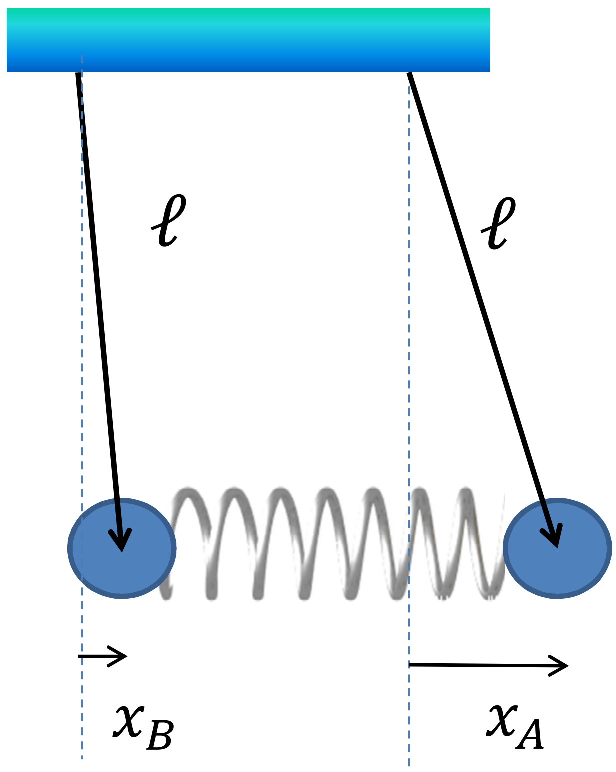
    $$\omega_0 = \sqrt{g/\ell}$$

  - The other with frequency

    $$\omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2}$$

- These are the only two normal modes of the system.
- But we can superimpose the solutions to describe arbitrary motion.

Two Coupled Oscillators



- The spring is stretched by the amount $x_A - x_B$
- Restoring force on pendulum A:
  $$F_A = -k(x_A - x_B)$$
- Restoring force on pendulum B:
  $$F_B = k(x_A - x_B)$$

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$

$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

Two Coupled Oscillators

$$\ddot{x}_A + (\omega_0)^2 x_A + \frac{k}{m}(x_A - x_B) = 0$$
$$\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = 0$$

$$\ddot{x}_B + (\omega_0)^2 x_B - k(x_A - x_B) = 0$$
$$\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0$$

- Each equation contains a term in the other coordinate
- The motion of A affects B and the motion of B affects A
- They must be solved simultaneously

Two Coupled Oscillators

$$\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = 0$$
$$\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0$$

- Add equations for A and B together:

$$\frac{d^2}{dt^2}(x_A + x_B) + (\omega_0)^2(x_A + x_B) = 0$$

- Subtract equations A and B:

$$\frac{d^2}{dt^2}(x_A - x_B) + [(\omega_0)^2 + 2(\omega_c)^2](x_A - x_B) = 0$$

Two Coupled Oscillators

- We have successfully "decoupled" the differential equations:

$$\frac{d^2}{dt^2}(x_A + x_B) + (\omega_0)^2(x_A + x_B) = 0$$
$$\frac{d^2}{dt^2}(x_A - x_B) + (\omega')^2(x_A - x_B) = 0$$

where $\omega_0 = \sqrt{g/\ell}$ and $\omega' = \sqrt{(\omega_0)^2 + 2(\omega_c)^2}$

- We just need to re-label the coordinates:

$$q_1 = x_A + x_B$$
$$q_2 = x_A - x_B$$

Two Coupled Oscillators

- Decoupled equations:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = 0$$
$$\ddot{q}_2 + (\omega')^2 q_2 = 0$$

- Solutions are

$$q_1(t) = A\cos(\omega_0 t + \alpha)$$
$$q_2(t) = B\cos(\omega' t + \beta)$$

- The variables q_1 and q_2 are called “normal coordinates”.

Initial Conditions

- Suppose we had the initial conditions:

$$x_A = A_0 \qquad \dot{x}_A = 0$$
$$x_B = 0 \qquad \dot{x}_B = 0$$

- Try to satisfy these when $\alpha = \beta = 0$:

$$x_A(t) = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}A\cos\omega_0 t + \frac{1}{2}B\cos\omega' t$$
$$x_B(t) = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}A\cos\omega_0 t - \frac{1}{2}B\cos\omega' t$$

- At time $t = 0$,

$$\frac{1}{2}(A + B) = A_0 \qquad \frac{1}{2}(A - B) = 0$$

- Now we know that $A = B = A_0$.

Initial Conditions

- Velocity:

$$\dot{x}_A(t) = -\frac{1}{2}A_0\omega_0 \sin \omega_0 t - \frac{1}{2}A_0\omega' \sin \omega' t$$
$$\dot{x}_B(t) = -\frac{1}{2}A_0\omega_0 \sin \omega_0 t + \frac{1}{2}A_0\omega' \sin \omega' t$$

- Initial conditions are satisfied at $t = 0$.

Initial Conditions

- Complete solution:

$$x_A(t) = \frac{1}{2}A_0(\cos\omega_0 t + \cos\omega' t)$$

$$= A_0 \cos\left(\frac{\omega' - \omega_0}{2}t\right)\cos\left(\frac{\omega' + \omega_0}{2}t\right)$$

$$x_B(t) = \frac{1}{2}A_0(\cos\omega_0 t - \cos\omega' t)$$

$$= A_0 \sin\left(\frac{\omega' - \omega_0}{2}t\right)\sin\left(\frac{\omega' + \omega_0}{2}t\right)$$

Review of Linear Algebra

- What if we had more than 2 masses?
- We need a systematic way to analyze systems with any number of masses.
- We will formulate a way to analyze these using matrices and eigenvalues.
- First examples will be with 2x2 matrices, but this can be generalized to systems of arbitrary size.
- You will need to know some basic linear algebra…

Things you Need to Know

- Given a system of linear equations, write them using matrices:

$$a\,x + b\,y = p$$
$$cx + d\,y = q$$

- Write this as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Things you Need to Know

- You need to know how to calculate the determinant of a matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- For a 3x3 matrix,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1)$$
$$+ a_3(b_1c_2 - b_2c_1)$$

Things you Need to Know

- You need to be able to solve (at least) 2x2 systems of equations.
- Use Kramer's rule!

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$x = \begin{vmatrix} p & b \\ q & d \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$y = \begin{vmatrix} a & p \\ c & q \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Things you Need to Know

- Eigenvalues of a matrix:
- What value(s) of λ will satisfy this equation?

$$\det(A + \lambda I) = 0$$

- Example with a 2x2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$= (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - \lambda(a + d) - bc = 0$$

- Solve this using the quadratic equation.
- But since you did the first assignment, this is nothing new!