

Physics 42200
Waves & Oscillations

Lecture 7 – French, Chapter 4

Spring 2015 Semester

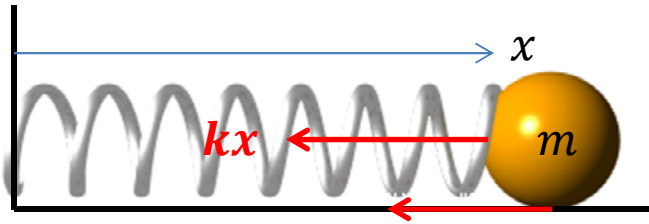
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Forced Oscillations and Resonance

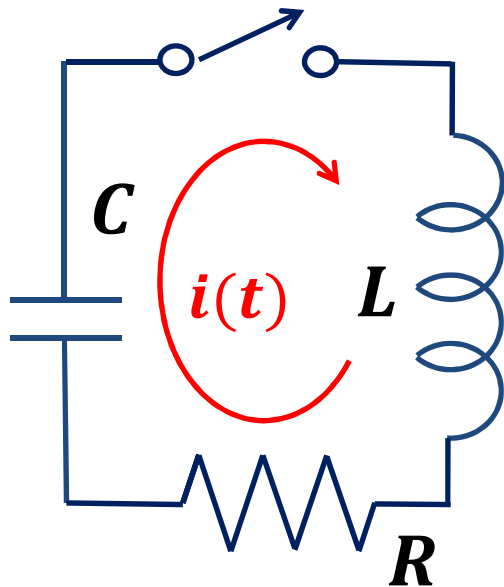


This is why you should pay attention.

Simple Harmonic Motion



$$m\ddot{x} + b\dot{x} + kx = 0$$



$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0$$

Second-order, homogeneous, linear differential equations with constant coefficients.

Forced Harmonic Motion

- Homogeneous equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

- Solutions are, for example,

$$x(t) = \mathbf{A}e^{-\frac{\gamma}{2}t} \sin \omega t + \mathbf{B}e^{-\frac{\gamma}{2}t} \cos \omega t$$

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$


- Non-homogeneous equation:

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

- There is an additional time-dependent force that does not depend on x .
- Periodic forcing: $F(t) = F_0 \cos \omega t$ where ω and F_0 are the frequency and amplitude of the applied force.

Periodic Forcing

- We are talking about two frequencies:

On Monday, $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ was the natural frequency of oscillations with no external force

Now, ω is the frequency of the applied force and is independent of k, m, b .

- What will the solution, $x(t)$, look like?

Periodic Forcing

- For short times, the initial conditions might influence the motion, but this dies away because of the $e^{-\gamma t/2}$ terms (transient motion).
- For longer times, after the transient behavior has died out, the system undergoes “steady-state” motion which should continue indefinitely.
- What is the form of the “steady-state” solution?
- Two scenarios:

$$\omega \ll \omega_0$$

$$\omega \gg \omega_0$$

Periodic Forcing

- When $\omega \ll \omega_0$, the motion has the same ***frequency*** and ***phase*** as the driving force.
- When $\omega \gg \omega_0$, the motion has the same ***frequency*** but is ***180° out of phase***.
- Maybe the form of the steady-state solution should look something like

$$x(t) = \textcolor{red}{A} \cos(\omega t + \textcolor{red}{\varphi})$$

- We have to solve for ***A*** and ***φ*** .
- These are *not* determined from the initial conditions... this solution only describes the motion for $\gamma t/2 \gg 1$.

Periodic Forcing

- Consider the simpler equation (no viscous damping):

$$m\ddot{x} + kx = F_0 \cos \omega t$$
$$\ddot{x} + (\omega_0)^2 x = \frac{F_0}{m} \cos \omega t$$

- Proposed solution when $\omega \ll \omega_0$:

$$x(t) = \mathbf{A} \cos \omega t$$

- What value of $\mathbf{A}$ will satisfy the differential equation?

$$\dot{x}(t) = -\mathbf{A}\omega \sin \omega t$$

$$\ddot{x}(t) = -\mathbf{A}\omega^2 \cos \omega t$$

- Substitute into the equation:

$$(-\omega^2 + (\omega_0)^2)\mathbf{A} \cos \omega t = \frac{F_0}{m} \cos \omega t$$

$$\mathbf{A} = \frac{F_0/m}{(\omega_0)^2 - \omega^2}$$

Periodic Forcing

- Consider the simpler equation (no viscous damping):

$$m\ddot{x} + kx = F_0 \cos \omega t$$
$$\ddot{x} + (\omega_0)^2 x = \frac{F_0}{m} \cos \omega t$$

- Proposed solution when $\omega \gg \omega_0$:

$$x(t) = \mathbf{A} \cos(\omega t + \pi) = -\mathbf{A} \cos \omega t$$

- What value of $\mathbf{A}$ will satisfy the differential equation?

$$\dot{x}(t) = \mathbf{A}\omega \sin \omega t$$

$$\ddot{x}(t) = \mathbf{A}\omega^2 \cos \omega t$$

- Substitute into the equation:

$$(\omega^2 - (\omega_0)^2)\mathbf{A} \cos \omega t = \frac{F_0}{m} \cos \omega t$$

$$\mathbf{A} = \frac{F_0/m}{\omega^2 - (\omega_0)^2}$$

Periodic Forcing

- In both cases, the solution is of the form

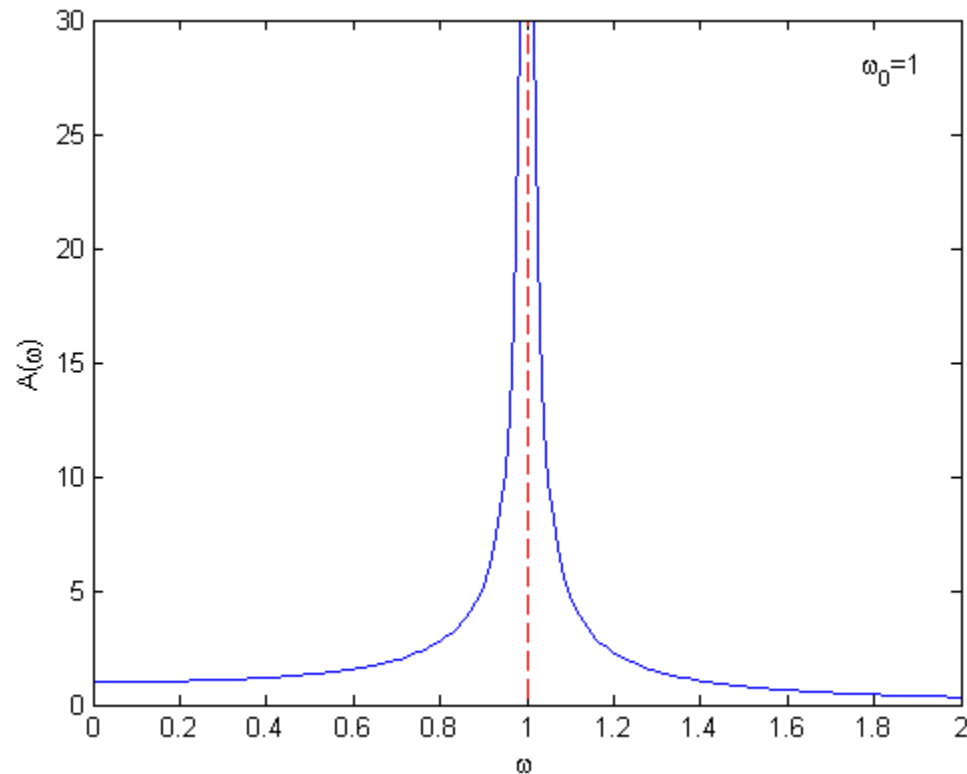
$$x(t) = A \cos(\omega t + \varphi)$$

$$\varphi = \begin{cases} 0 & \text{when } \omega \ll \omega_0 \\ \pi & \text{when } \omega \gg \omega_0 \end{cases}$$

$$A = \frac{F_0/m}{|\omega^2 - (\omega_0)^2|}$$

- What happens when $\omega \approx \omega_0$?
 - Probably nothing good: $A \rightarrow \infty$ which is unphysical.

Periodic Forcing



Amplitude gets very large when the frequency of the driving force is close to the natural oscillation frequency.

Periodic Forcing

- Let's derive the form of the solution without any assumptions about ω .

- Assume $x(t)$ is of the form
$$x(t) = Ae^{i(\omega t - \delta)} = \underbrace{Ae^{-i\delta}}_{\text{Just a constant}} e^{i\omega t}$$

Real numbers

- Derivatives:

$$\dot{x}(t) = i\omega x(t)$$

$$\ddot{x}(t) = -\omega^2 x(t)$$

- Substitute into the differential equation:

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}$$

Periodic Forcing

- Rewrite the differential equation slightly:

$$\ddot{x} + \gamma\dot{x} + (\omega_0)^2 x = \frac{F_0}{m} e^{i\omega t}$$

- Substitute in the solution:

$$[(-\omega^2 + i\omega\gamma + (\omega_0)^2)Ae^{-i\delta}]e^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

- True for any t provided that

$$A = \frac{e^{i\delta} F_0/m}{(\omega_0)^2 - \omega^2 + i\omega\gamma} = \frac{((\omega_0)^2 - \omega^2 - i\omega\gamma)e^{i\delta} F_0/m}{((\omega_0)^2 - \omega^2)^2 + (\omega\gamma)^2}$$

Periodic Forcing

- We said that A was a real number... its magnitude is

$$A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\gamma)^2}}$$

- The phase, δ , must be

$$\delta = \tan^{-1} \left(\frac{\omega\gamma}{(\omega_0)^2 - \omega^2} \right)$$

- Is this consistent with the expected limits?

Limiting Behavior

- When $\omega \ll \omega_0$ then $\left((\omega_0)^2 - \omega^2\right)^2 \gg (\omega\gamma)^2$

$$A \rightarrow \frac{F_0/m}{(\omega_0)^2 - \omega^2}$$

$$\delta = \tan^{-1}(\text{"small, positive"}) \rightarrow 0$$

- When $\omega \gg \omega_0$ then $\left((\omega_0)^2 - \omega^2\right)^2 \gg (\omega\gamma)^2$

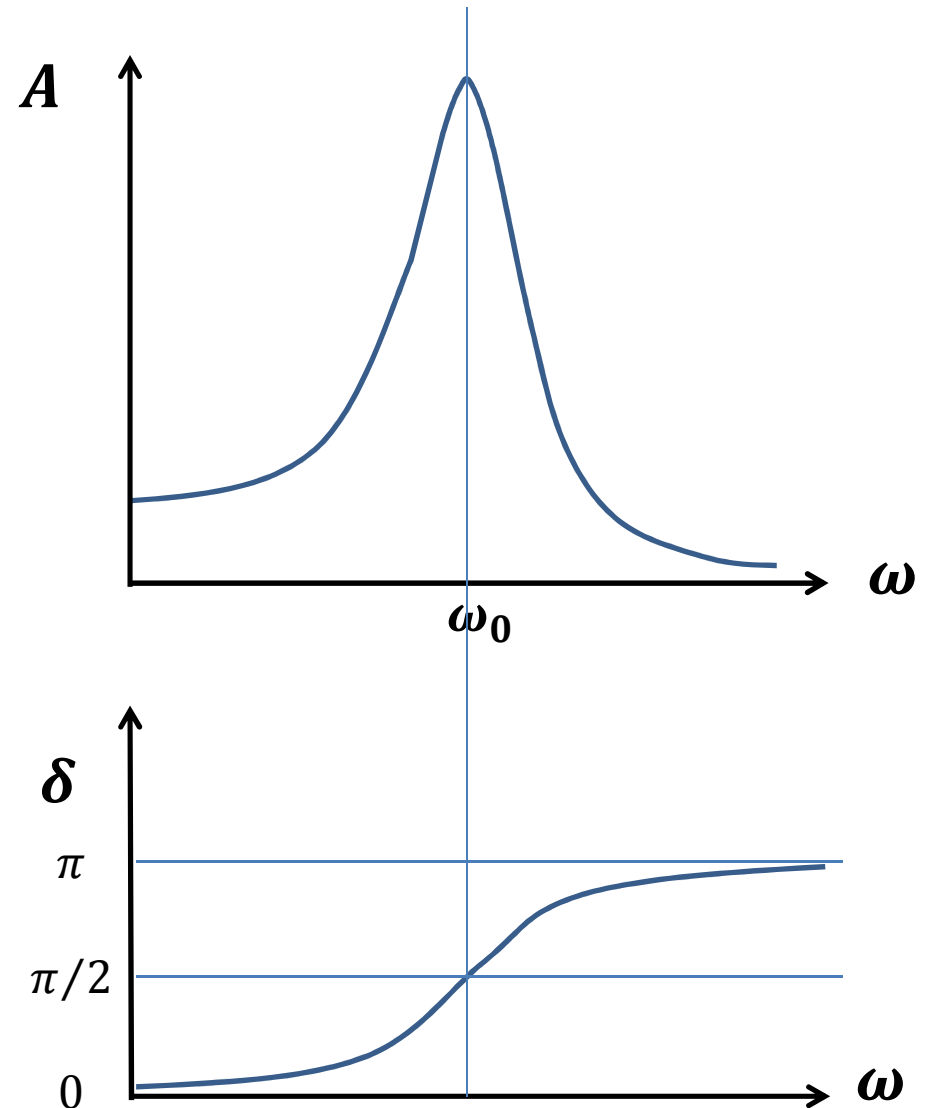
$$A \rightarrow \frac{F_0/m}{\omega^2 - (\omega_0)^2}$$

$$\delta = \tan^{-1}(\text{"small, negative"}) \rightarrow \pi$$

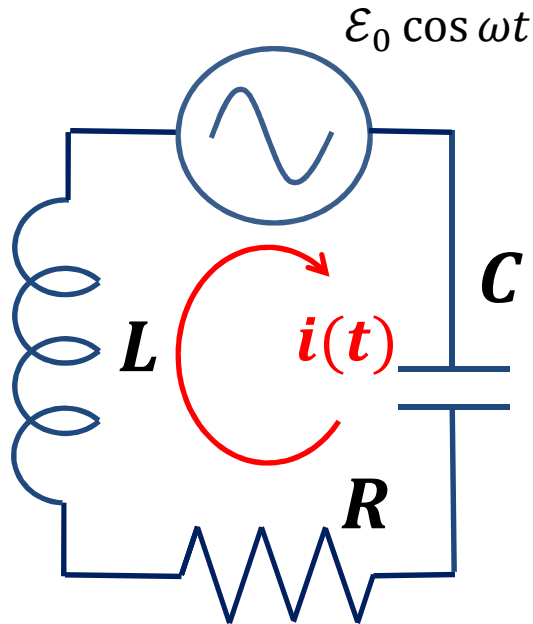
- When $\omega \rightarrow 0$ then $A \rightarrow F_0/k$.

Resonance

- The peak occurs at a frequency that is close to, but not exactly equal to ω_0 .
- At resonance, the phase shift is $\delta = \pi/2$.
- The force pushes the mass in the direction it is already moving adding energy to the system.



Example



$$L \frac{di}{dt} + Ri(t) + \frac{1}{C} \int i(t) dt + \mathcal{E}_0 \cos \omega t = 0$$

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) - \mathcal{E}_0 \omega \sin \omega t = 0$$

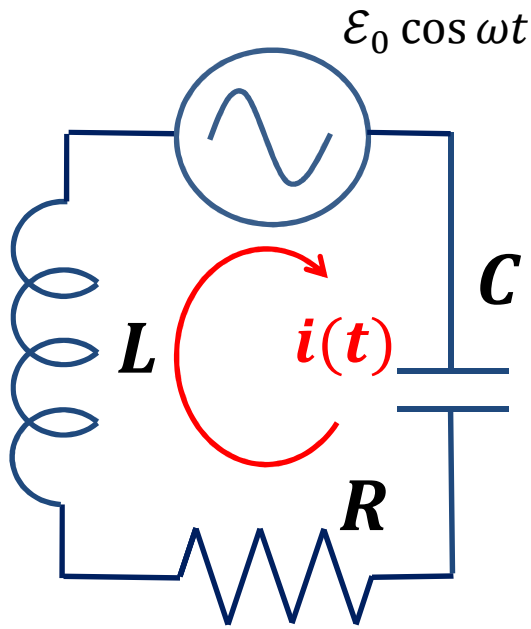
$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = \mathcal{E}_0 \omega \sin \omega t$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i(t) = \frac{\mathcal{E}_0 \omega}{L} \sin \omega t$$

$$\frac{d^2 i}{dt^2} + \gamma \frac{di}{dt} + \omega_0^2 i(t) = \frac{\mathcal{E}_0 \omega}{L} \sin \omega t$$

$$\frac{d^2 i}{dt^2} + \gamma \frac{di}{dt} + \omega_0^2 i(t) = \frac{F_0}{m} \sin \omega t$$

Compare with the
mechanical analog:

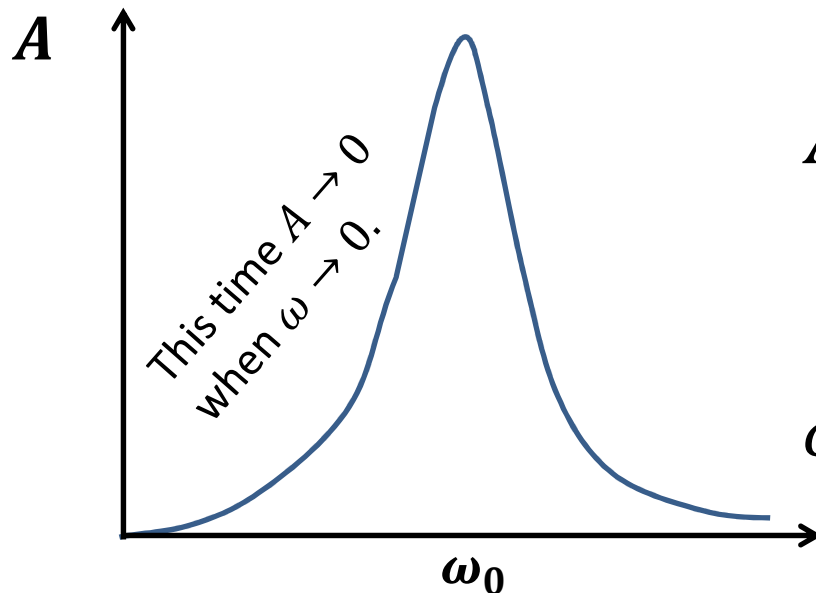


Example

$$\frac{d^2 i}{dt^2} + \gamma \frac{di}{dt} + \omega_0^2 i(t) = \frac{\mathcal{E}_0 \omega}{L} \sin \omega t$$

This time we can assume the steady state solutions are of the form

$$i(t) = A \sin(\omega t - \delta)$$



$$A = \frac{\mathcal{E}_0 \omega / L}{\sqrt{((\omega_0)^2 - \omega^2)^2 + (\omega \gamma)^2}}$$

$$\delta = \tan^{-1} \left(\frac{\omega \gamma}{(\omega_0)^2 - \omega^2} \right)$$