

Physics 42200
Waves & Oscillations

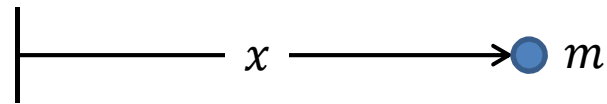
Lecture 5 – French, Chapter 3

Spring 2015 Semester

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Mechanics Lesson: Circular Motion

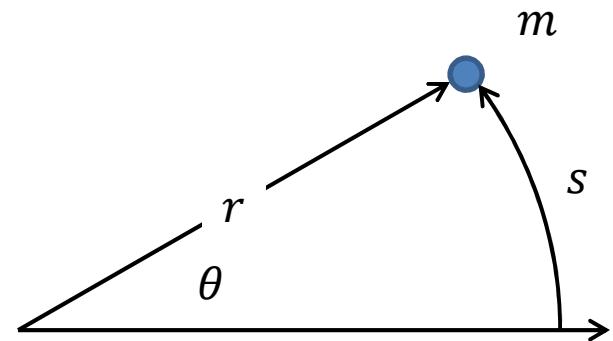
- Linear motion:



- Mass: m
- Position: $x(t)$
- Velocity: $v = \dot{x} = dx/dt$
- Momentum: $p = m \dot{x}$
- Acceleration: $a = \ddot{x} = d^2x/dt^2$
- Force: $F = m \ddot{x}$
- Kinetic energy: $T = \frac{1}{2} m \dot{x}^2$

Mechanics Lesson: Circular Motion

- Circular motion:



- Moment of inertia: $I = mr^2$
- Angle: $\theta(t)$, arc length: $s(t) = r \theta(t)$
- Angular velocity: $\omega = \dot{\theta}$, linear velocity: $v = r \dot{\theta}$
- Angular momentum: $L = I \dot{\theta}$
- Angular acceleration: $\alpha = \ddot{\theta}$
- Torque (or “moment”): $N = r F = I \ddot{\theta}$
- Kinetic energy: $T = \frac{1}{2} m \dot{s}^2 = \frac{1}{2} m r^2 \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2$

(in 3 dimensions, $\vec{\omega}$, $\vec{\alpha}$, $\vec{L}$, $\vec{N}$ are vectors...)

Free Vibrations of Physical Systems

- Mass + spring system: $m\ddot{x} - kx = 0$
- Stretched elastic material: $m\ddot{x} - \frac{YA}{L}x = 0$
- Floating objects: $m\ddot{x} - \rho gAx = 0$
- Twisted elastic material: $I\ddot{\theta} - \frac{\pi nR^4}{2\ell}\theta = 0$
- What should you have learned from the last lecture?
 - A bunch of formulas?
 - Fundamental physical laws?

Stretched Elastic Material

- Physical concepts:
 - Stuff stretches when you pull on it
 - If it is longer to begin with, it will stretch more, when subjected to the same force
 - Definition: $strain \equiv \Delta l_0 / l_0$
 - But it won't stretch as much if it is thicker
 - Definition: $stress = F/A$
 - Assertion: $strain \propto stress$
 - Limits of applicability? When strain is $\sim few\%$
$$F = \frac{YA}{l_0} x$$
 - This defines the constant of proportionality, Y .

Floating Objects

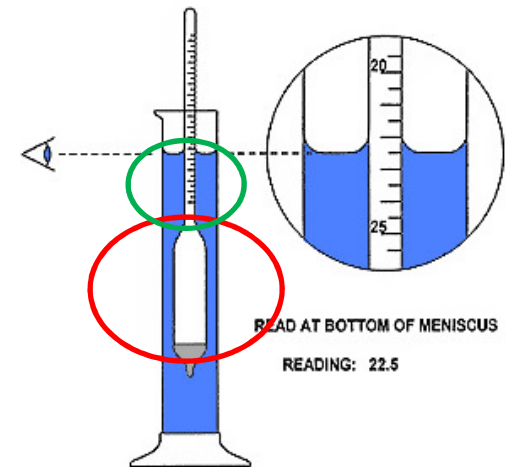
Warning! People forgot this on the midterm last year!

- Physical concepts:

- Archimedes' principle: buoyant force is equal to the weight of displaced liquid.
- Static equilibrium:

$$\rho g \left(V_0 + \pi h \left(\frac{d}{2} \right)^2 \right) = mg$$

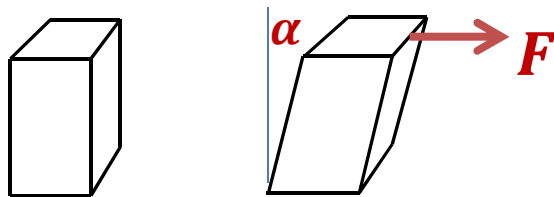
- This expression is not worth memorizing...
- But you should understand what the pieces mean.
- The details are only specific to this problem



Twisted Elastic Material

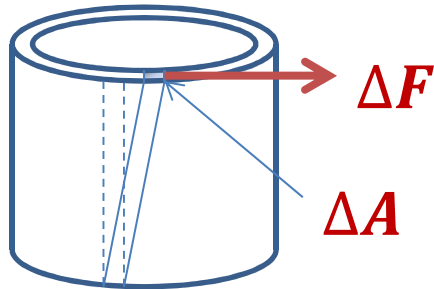
- Physical concept:

- Shear modulus...



$$\frac{F}{A} = n\alpha$$

- Applied to a specific geometry:



$$M = \frac{\pi n R^4 \theta}{2l}$$

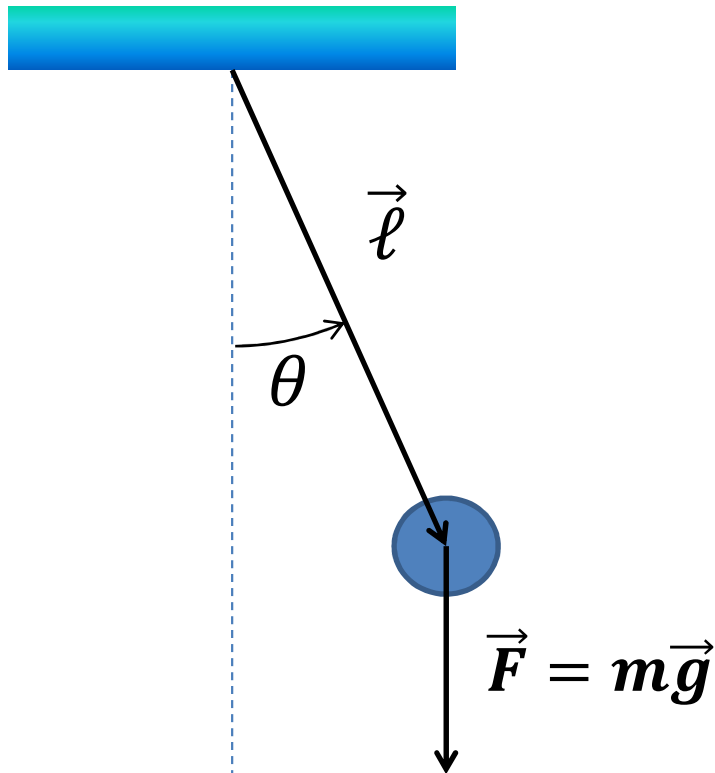
- Again, the formula is specific to one specific geometry. If the object were rectangular, instead of round, the formula would be different.

More general advice

- Study the examples...
 - Which physical principles are being used?
 - Do you agree with the translation from the physical concepts into algebraic relations?
 - Did the solution require looking at the problem in a different way?
 - Do you understand the geometry?
 - Do you understand the algebra?
 - Could you use the same ideas and techniques to analyze a similar problem?

Almost Linear Systems

- Consider a simple pendulum:



Physical concepts:

- torque produces an angular acceleration.
- definition of torque:

$$\vec{N} = \vec{\ell} \times \vec{F}$$

$$N = -\ell F \sin \theta$$

- angular acceleration:

$$N = I\ddot{\theta}$$

- moment of inertia:

$$I = m\ell^2$$

Almost Linear Systems

$$N = I\ddot{\theta}$$

$$N = -\ell mg \sin \theta$$

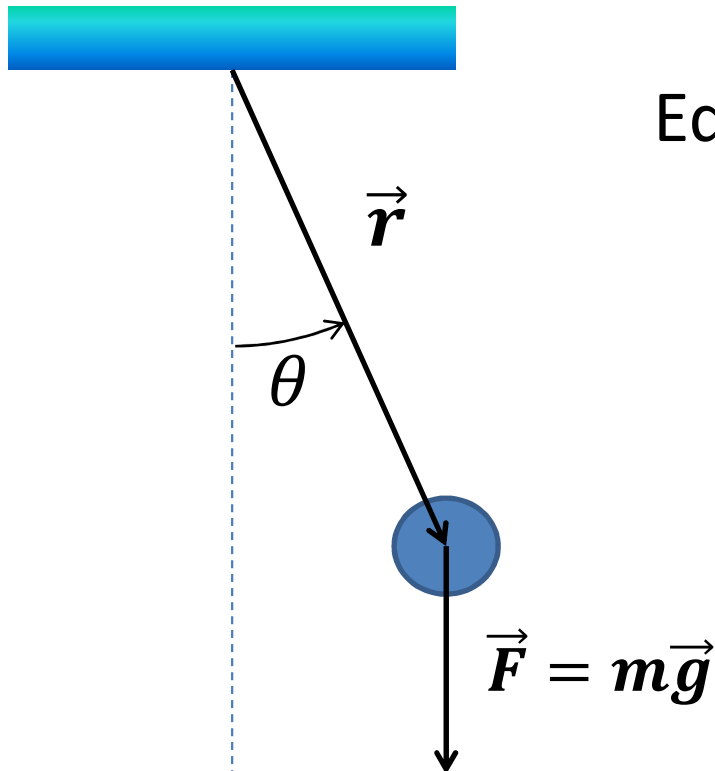
$$I = m\ell^2$$

Equation of motion:

$$m\ell^2\ddot{\theta} = -mg\ell \sin \theta$$

$$\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0$$

$$\ddot{\theta} + \omega^2 \sin \theta = 0$$



But this is not of the form $\ddot{\theta} + \omega^2\theta = 0$

Almost Linear Systems

$$\ddot{\theta} + \omega^2 \sin \theta = 0$$

- The solution is *not* $\theta(t) = A \cos(\omega t + \varphi)$ but it is close...
- Recall that one way to write $\sin \theta$ is as a power series in θ :

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

- When $\theta \ll 1$, $\sin \theta \approx \theta$
- How good is this approximation?
- Suppose we want it to be within 1%

$$\sin(0.3925) - 0.3925 = -0.0100005 \dots$$

- In degrees, $22.49^\circ = 0.3925$ radians

Almost Linear Systems

- Provided θ is sufficiently small (ie, $\theta < 22^\circ$),

$$\ddot{\theta} + \omega^2 \theta \approx \ddot{\theta} + \omega^2 \sin \theta = 0$$

- The solution is approximately

$$\theta(t) = A \cos(\omega t + \varphi)$$

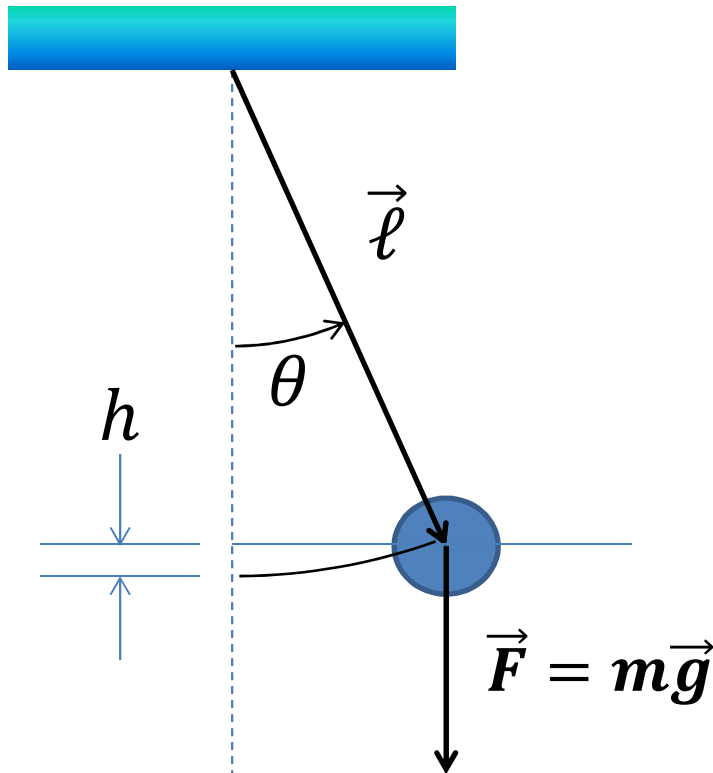
- The frequency is approximately

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{\ell}}$$

- The approximation is better when A is even smaller

Potential Energy Functions

- Same system analyzed using energy:



Kinetic energy:

$$T = \frac{1}{2} I \dot{\theta}^2$$

Potential energy:

$$V = mgh = mg\ell(1 - \cos \theta)$$

Total energy:

$$E = T + V = \text{const.}$$

Does this resemble the mass+spring problem?

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = E$$

Potential Energy Functions

- Recall that one way to write $\cos \theta$ is as a power series in θ :

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

- Energy for a simple pendulum:

$$E = T + V = \frac{1}{2}I\dot{\theta}^2 + mg\ell(1 - \cos \theta)$$

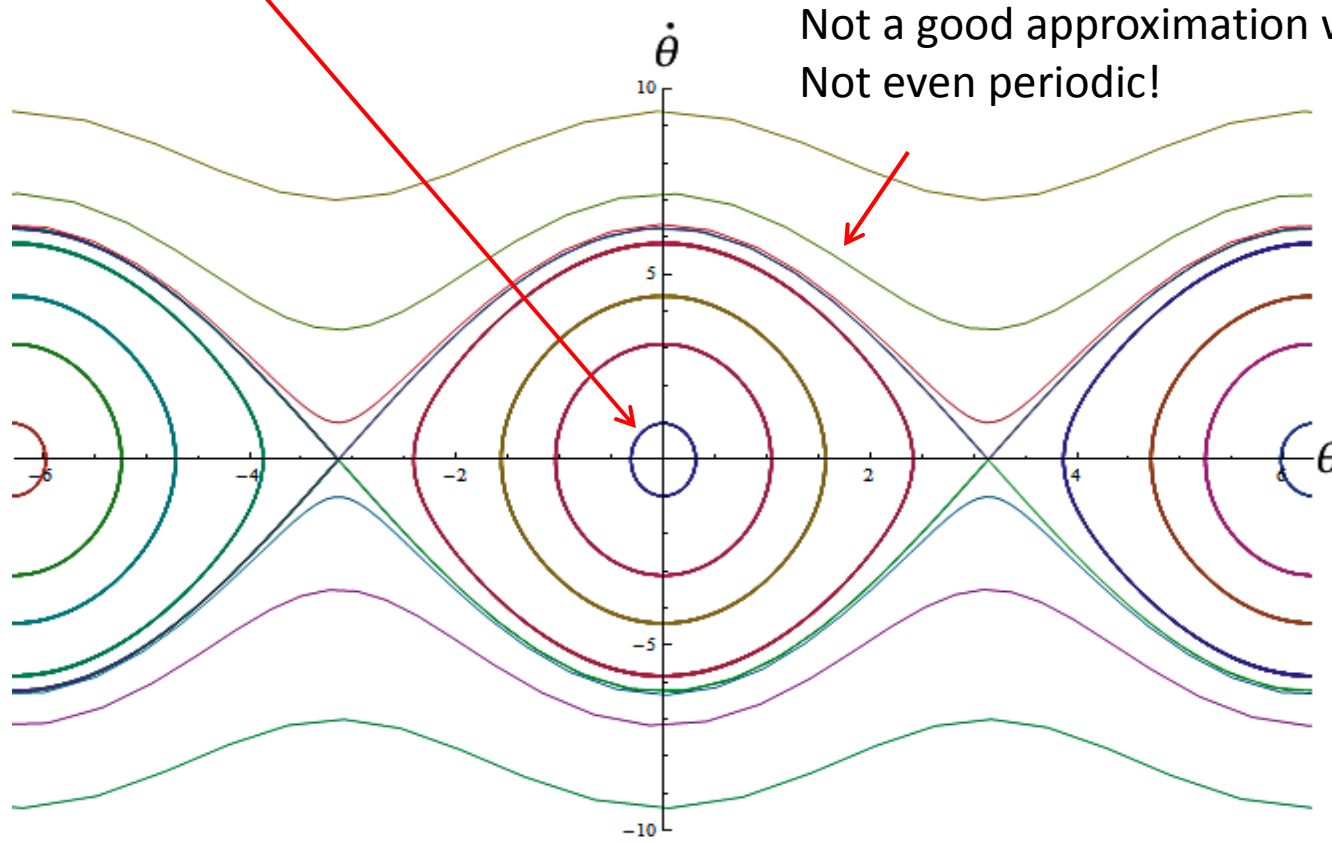
$$T + V \approx \frac{1}{2}I\dot{\theta}^2 + mg\ell \left(1 - \left(1 - \frac{\theta^2}{2!} \right) \right)$$

$$E = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mg\ell\theta^2$$

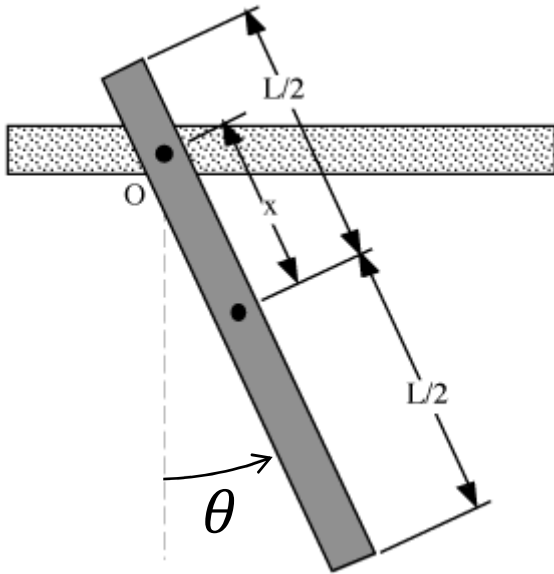
- Now, this is in the same form as for the mass + spring system.
- Interpretation?

Phase Diagram

Almost elliptical
when E is small. } $E = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mg\ell\theta^2 = \text{const}$



Physical Pendulum



No new physical concepts – just a different geometry.

$$N = I\ddot{\theta}$$

Gravitational force acts through the center of gravity:

$$N = -Mgx \sin \theta$$

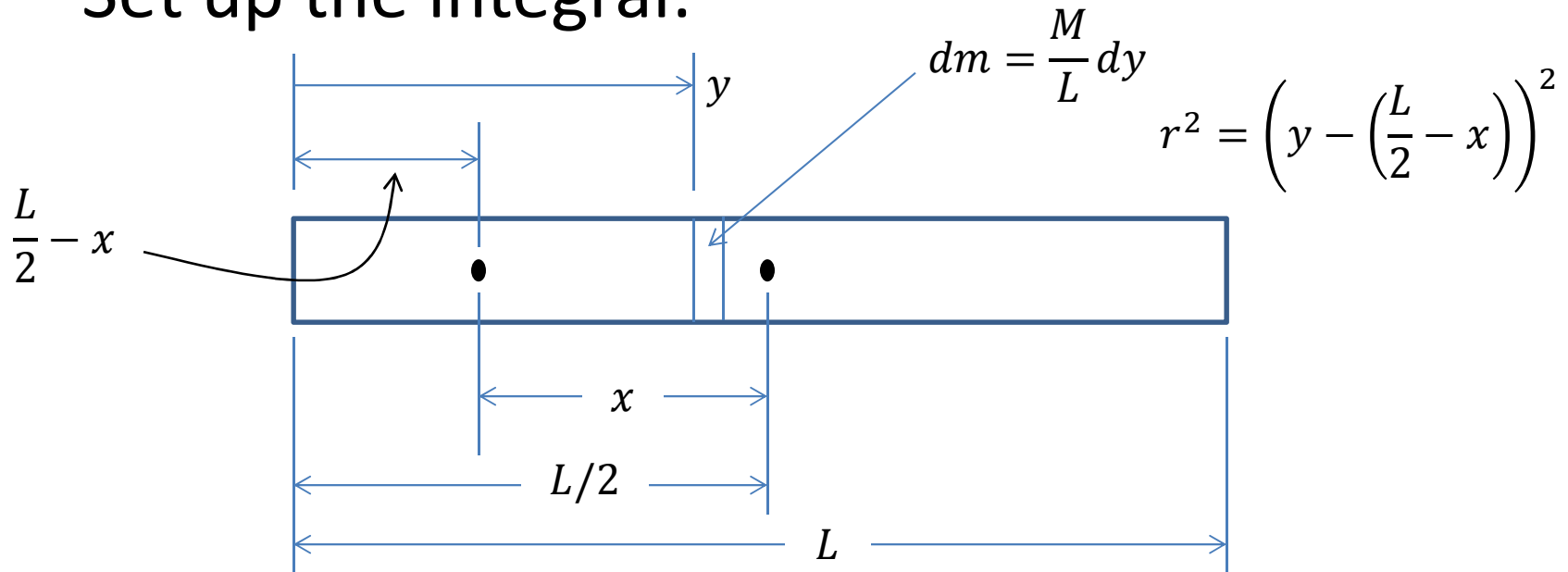
What is the moment of inertia?

Recall that

$$I \equiv \sum_i m_i (r_i)^2 \quad \text{or} \quad I \equiv \int r^2 dm$$

Moment of Inertia of a Stick

- Set up the integral:



$$I = \frac{M}{L} \int_0^L (y + x - L/2)^2 dy$$

Let $u = y + x - \frac{L}{2}$
Then $du = dy$

$$I = \frac{M}{L} \int_{x-L/2}^{x+L/2} u^2 du = \frac{M}{3L} u^3 \Big|_{x-L/2}^{x+L/2}$$

Moment of Inertia of a Stick

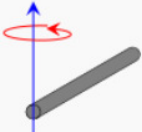
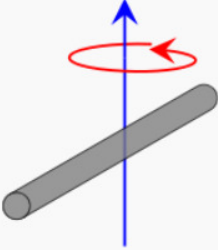
$$\begin{aligned} I &= \frac{M}{3L} [(x + L/2)^3 - (x - L/2)^3] \\ &= \frac{M}{3L} [3x^2L + L^3/4] \\ &= M \left[x^2 + \frac{L^2}{12} \right] \end{aligned}$$

Check the limiting cases:

$$x = 0 \Rightarrow I = \frac{ML^2}{12} \qquad x = \frac{L}{2} \Rightarrow I = \frac{ML^2}{3}$$

Moment of Inertia of a Stick

Wikipedia: List of moments of inertia

Description	Figure	Moment(s) of inertia
Point mass m at a distance r from the axis of rotation.		$I = mr^2$
Two point masses, M and m , with reduced mass μ and separated by a distance, x .		$I = \frac{Mm}{M+m}x^2 = \mu x^2$
Rod of length L and mass m (Axis of rotation at the end of the rod)		$I_{\text{end}} = \frac{mL^2}{3}$ ^[1]
Rod of length L and mass m		$I_{\text{center}} = \frac{mL^2}{12}$ ^[1]

Physical Pendulum

- Equation of motion:

$$I\ddot{\theta} + Mgx \sin \theta = 0$$
$$\ddot{\theta} + \omega^2 \theta \approx 0$$

where $\omega = \sqrt{\frac{Mgx}{I}} = \sqrt{\frac{gx}{x^2 + \frac{L^2}{12}}}$

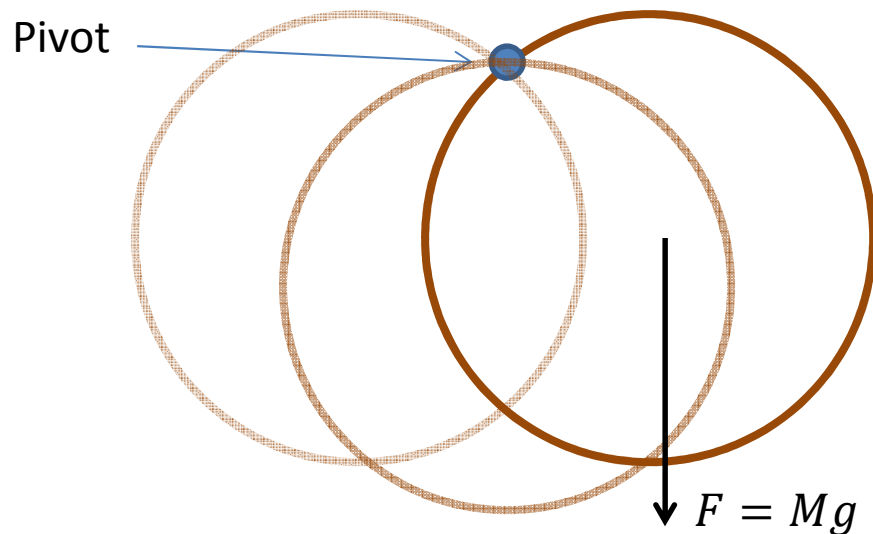
- When $x = L/2$ (suspended from one end)

$$\omega = \sqrt{\frac{3g}{2\ell}}$$

(same frequency as a simple pendulum with 2/3 the length)

One More Physical Pendulum

- The “ring pendulum”:



$$I\ddot{\theta} + MgR \sin \theta = 0$$
$$I = 2MR^2$$

$$\omega \approx \sqrt{\frac{MgR}{2MR^2}} = \sqrt{\frac{g}{2R}}$$

You should recognize that the problem is the same as in the case of the stick. The only difference is the moment of inertia.