PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 37 – Interference

Spring 2015 Semester

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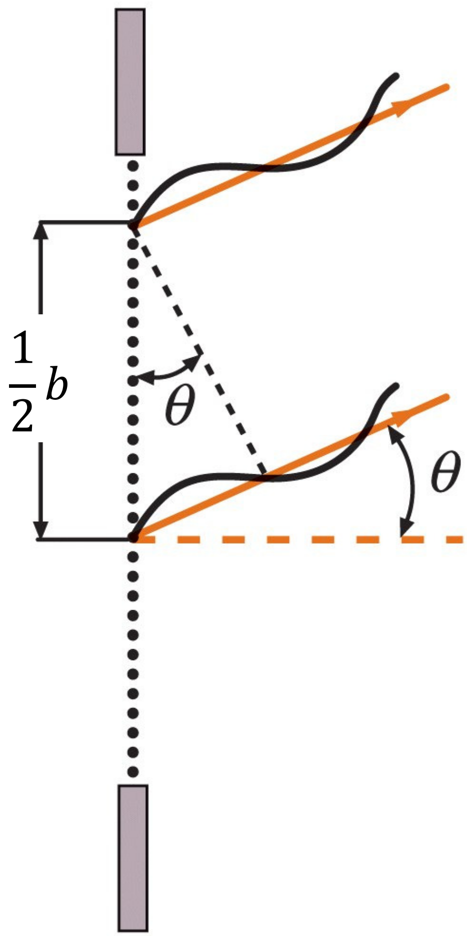
Single Slit Diffraction



Think of the slit as a number of point sources with equal amplitude. Divide the slit into two pieces and think of the interference between light in the upper half and light in the lower half.

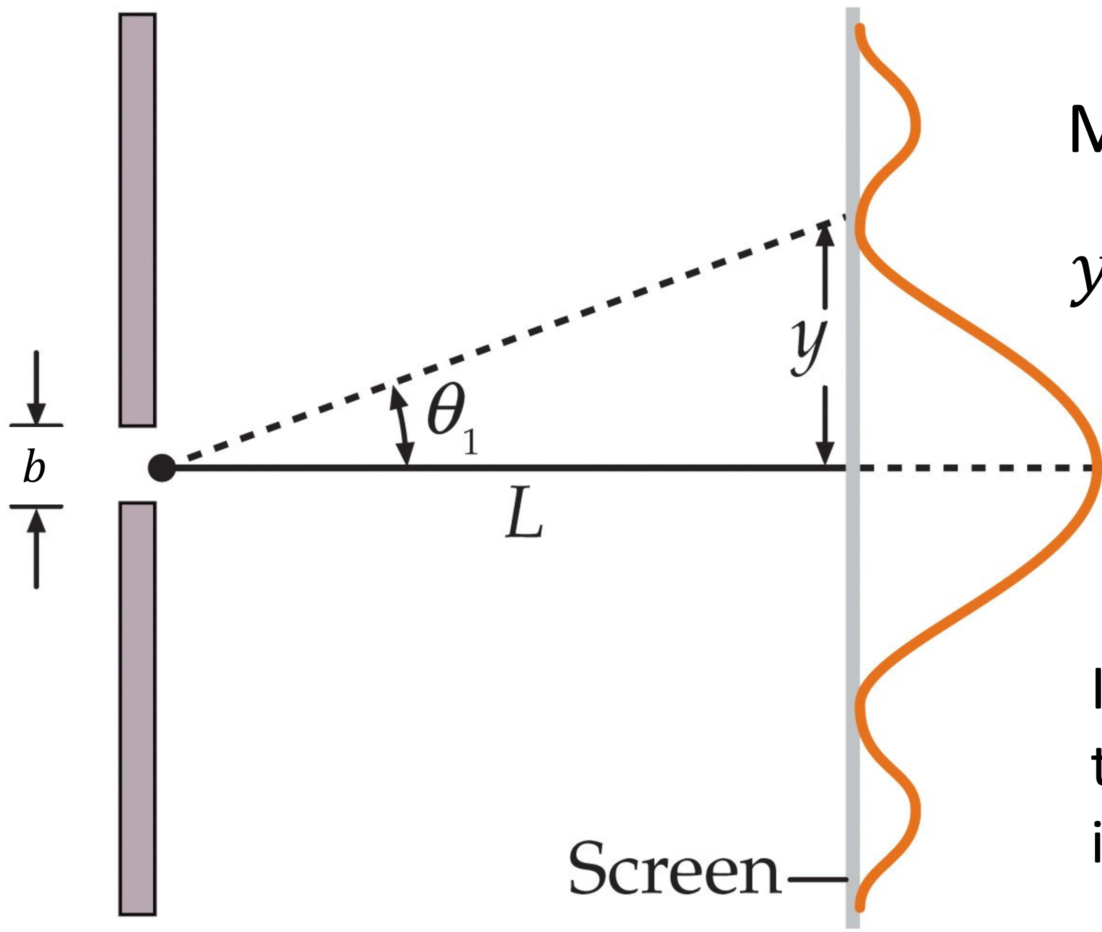
Destructive interference when

$$\frac{b}{2}\sin\theta = \frac{\lambda}{2}$$

Minima when

$$\sin\theta = {}^{\lambda}/_{b}$$

Single Slit Diffraction



$$\sin\theta \approx \tan\theta = y/L$$

Minima located at

$$y = \frac{mL\lambda}{b}, m = 1,2,3, \ldots$$

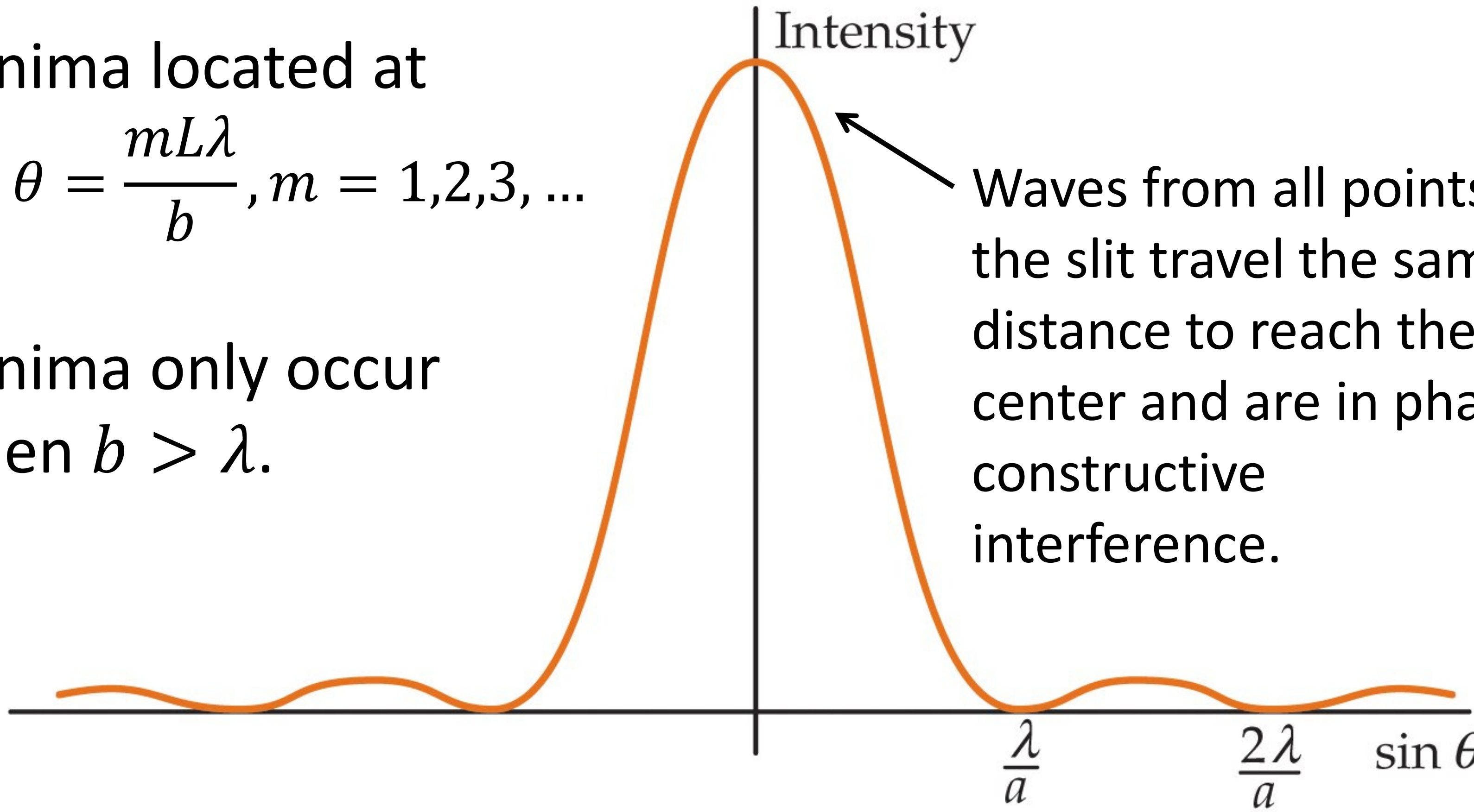
In general, the "width" of the image on the screen is not even close to a.

Single Slit Diffraction

Minima located at

$$\sin\theta = \frac{mL\lambda}{b}, m = 1,2,3, \ldots$$

Minima only occur when $b > \lambda$.

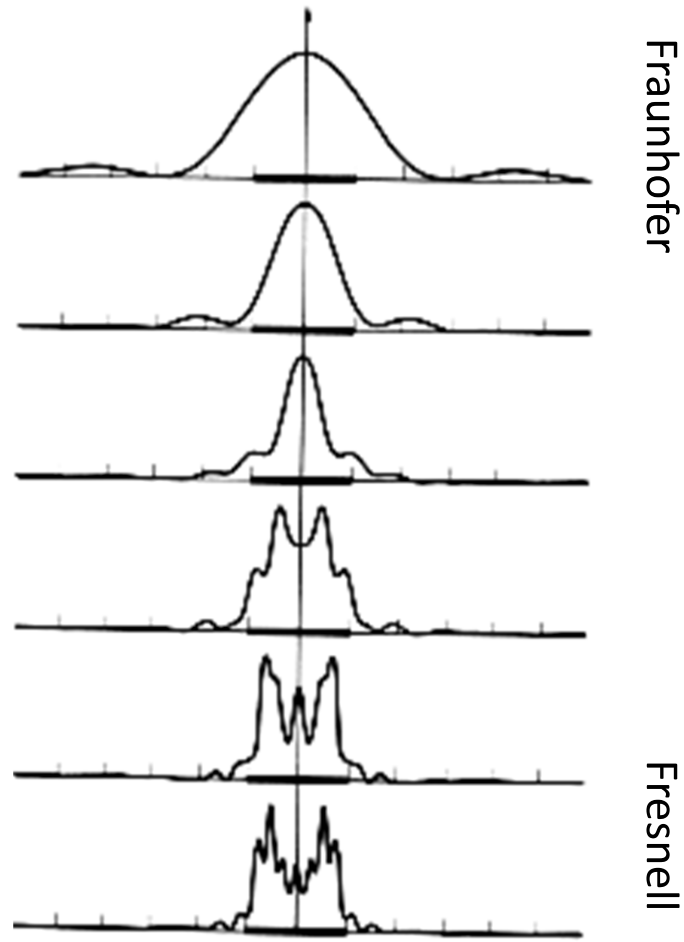


Waves from all points in the slit travel the same distance to reach the center and are in phase: constructive interference.

Fresnel and Fraunhofer Diffraction

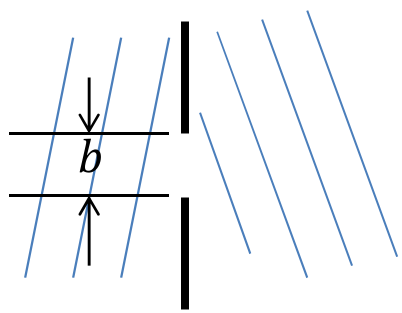
Assumptions about the wave front that impinges on the slit:

- When it's a plane, the phase varies linearly across the slit: Fraunhofer diffraction
- When the phase of the wave front has significant curvature: Fresnel diffraction



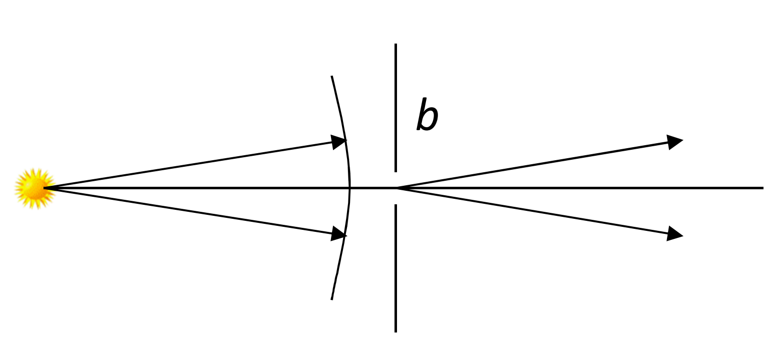
Fresnel and Fraunhofer Diffraction

- Fraunhofer diffraction
  - Far field: $R \gg b^2/\lambda$



  - R is the smaller of the distance to the source or to the screen

- Fresnel Diffraction:
  - Near field: wave front is not a plane at the aperture



Single-Slit Fraunhofer Diffraction

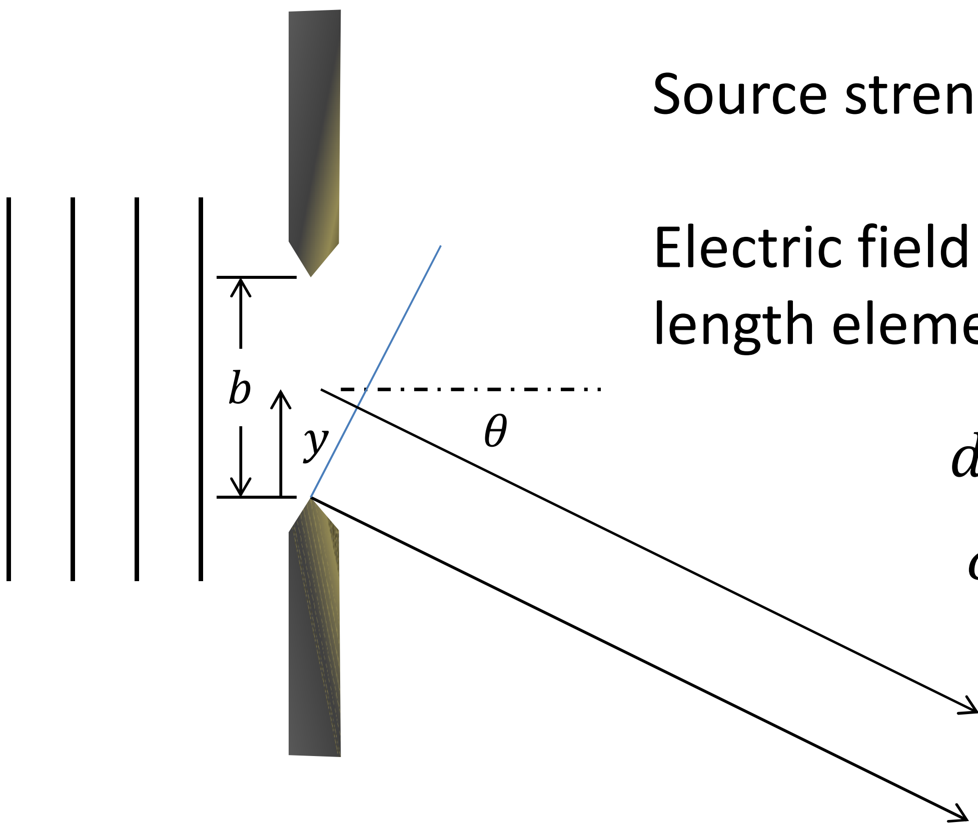
Light with intensity I_0 impinges on a slit with width b

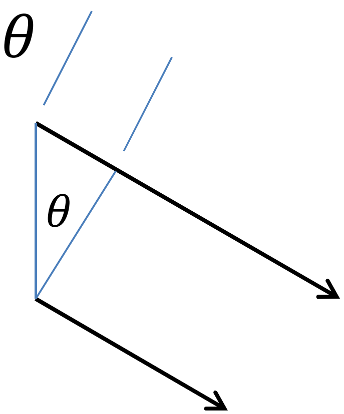
Source strength per unit length: $\mathcal{E}_L$

Electric field at a distance R due to the length element dy:

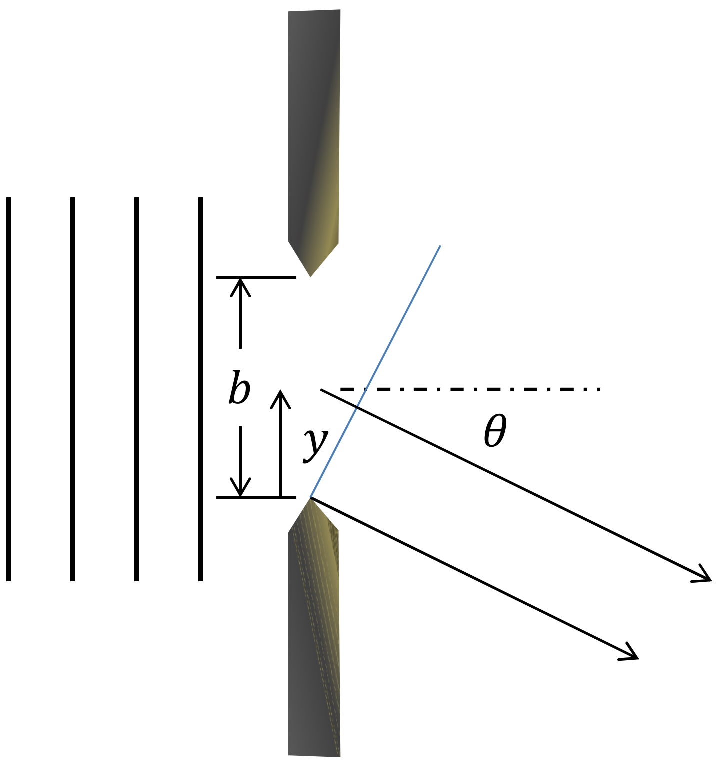
$$dE = \frac{\mathcal{E}_L e^{i\delta(y)} dy}{R}$$

$$\delta(y) = ky \sin\theta$$





Single-Slit Fraunhofer Diffraction



$$dE = \frac{\mathcal{E}_L e^{iky\sin\theta} dy}{R}$$

Let $y = 0$ be at the center of the slit.

Integrate from $-b/2$ to $+b/2$:

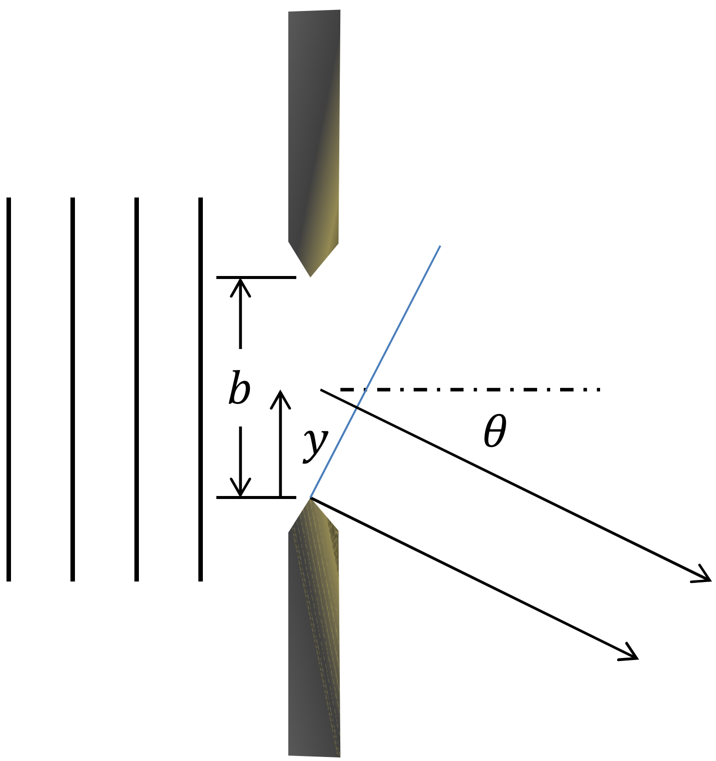
Total electric field:

$$E = \frac{\mathcal{E}_L}{R}\int_{-b/2}^{+b/2} e^{iky\sin\theta} dy$$

$$= \frac{\mathcal{E}_L}{R}\frac{e^{i(kb/2)\sin\theta} - e^{-i(kb/2)\sin\theta}}{ik\sin\theta}$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\left(\frac{1}{2}kb\sin\theta\right)}{\frac{1}{2}kb\sin\theta}$$

Single-Slit Fraunhofer Diffraction



$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta}$$

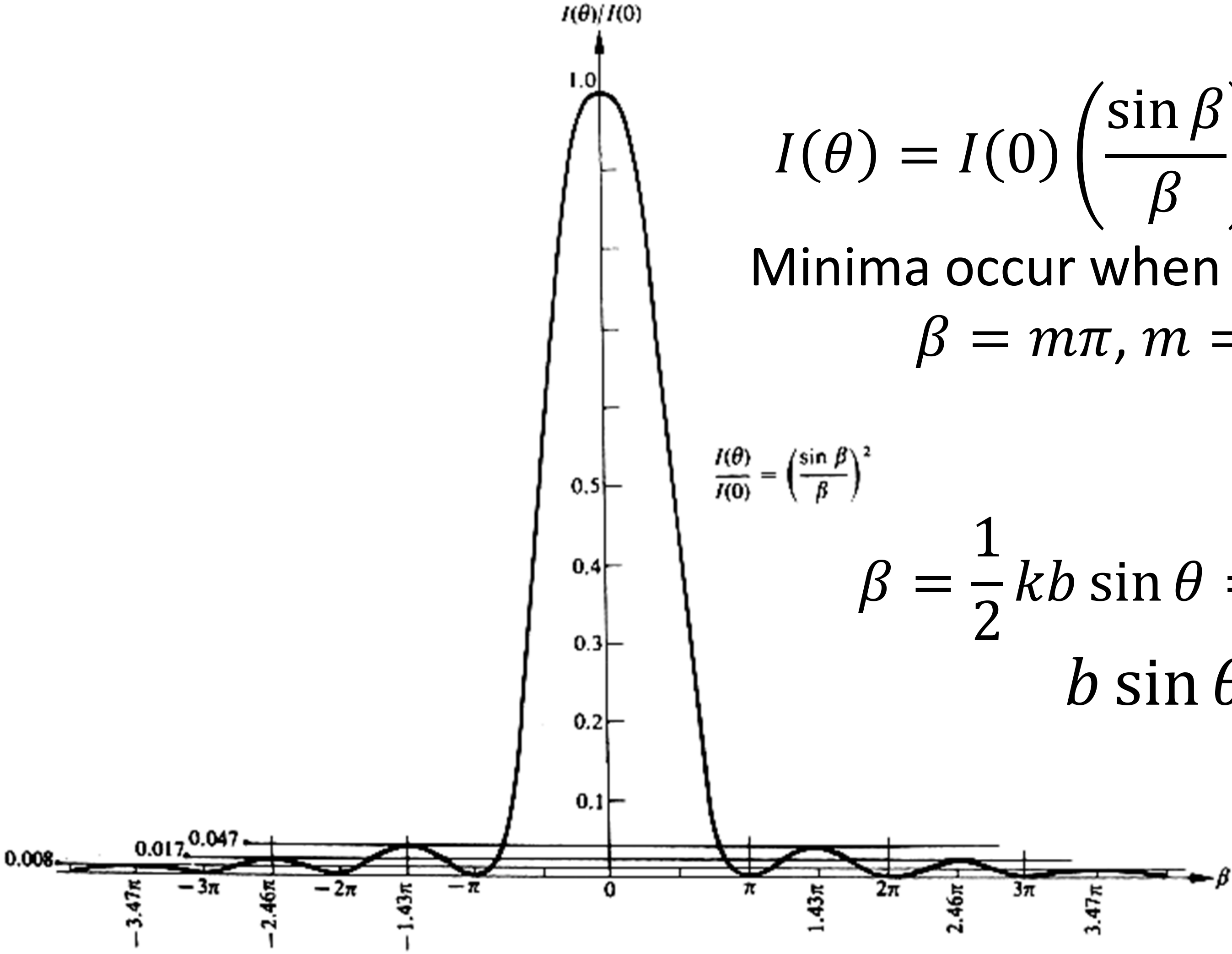
where

$$\beta = \frac{1}{2} kb \sin\theta$$

The intensity of the light will be

$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$$
$$= I(0)\,\mathrm{sinc}^2\,\beta$$

Single-Slit Fraunhofer Diffraction



$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2 = I(0)\,\mathrm{sinc}^2\,\beta$$

Minima occur when

$$\beta = m\pi, m = \pm 1, \pm 2, \cdots$$

$$\beta = \frac{1}{2}kb\sin\theta = \frac{\pi b}{\lambda}\sin\theta = m\pi$$

$$b\sin\theta = m\lambda$$

Fourier Transforms

$$E = \frac{\mathcal{E}_L}{R} \int_{-b/2}^{+b/2} e^{iky \sin\theta} dy$$

- Limits of integration can be expressed using

$$U(y) = \begin{cases} 1 \text{ when } |y| < b/2 \\ 0 \text{ otherwise} \end{cases}$$

- Then, the transmitted field is:

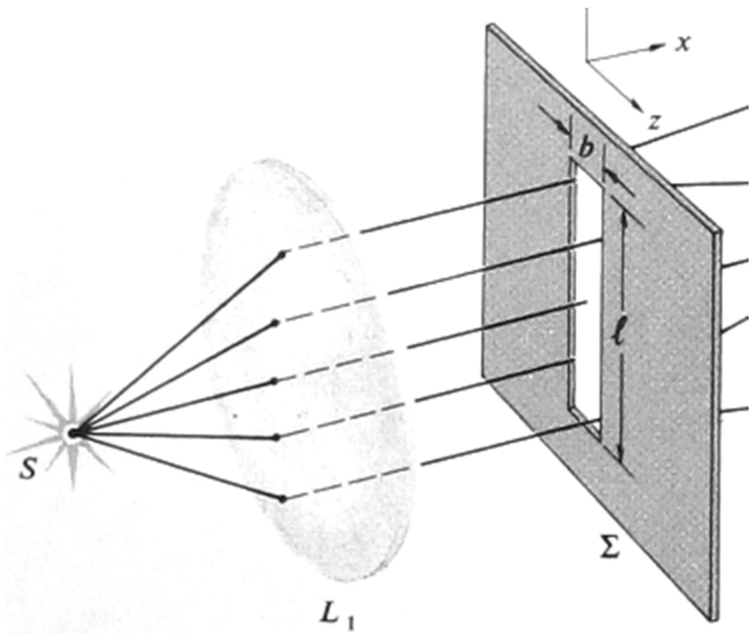
$$E = \frac{\mathcal{E}_L}{R} \int_{-\infty}^{+\infty} U(y) e^{ik'y} dy$$

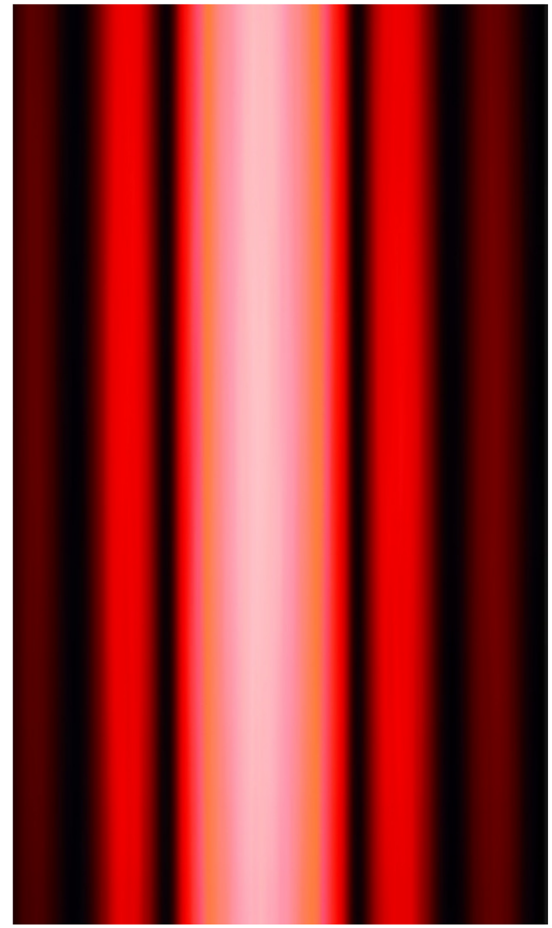
$$k' = k \sin\theta$$

- You might recognize that this is just the Fourier transform of $U(y)$…

Single slit: Fraunhofer diffraction

Adding dimension: long narrow slit
Diffraction most prominent in the narrow direction.





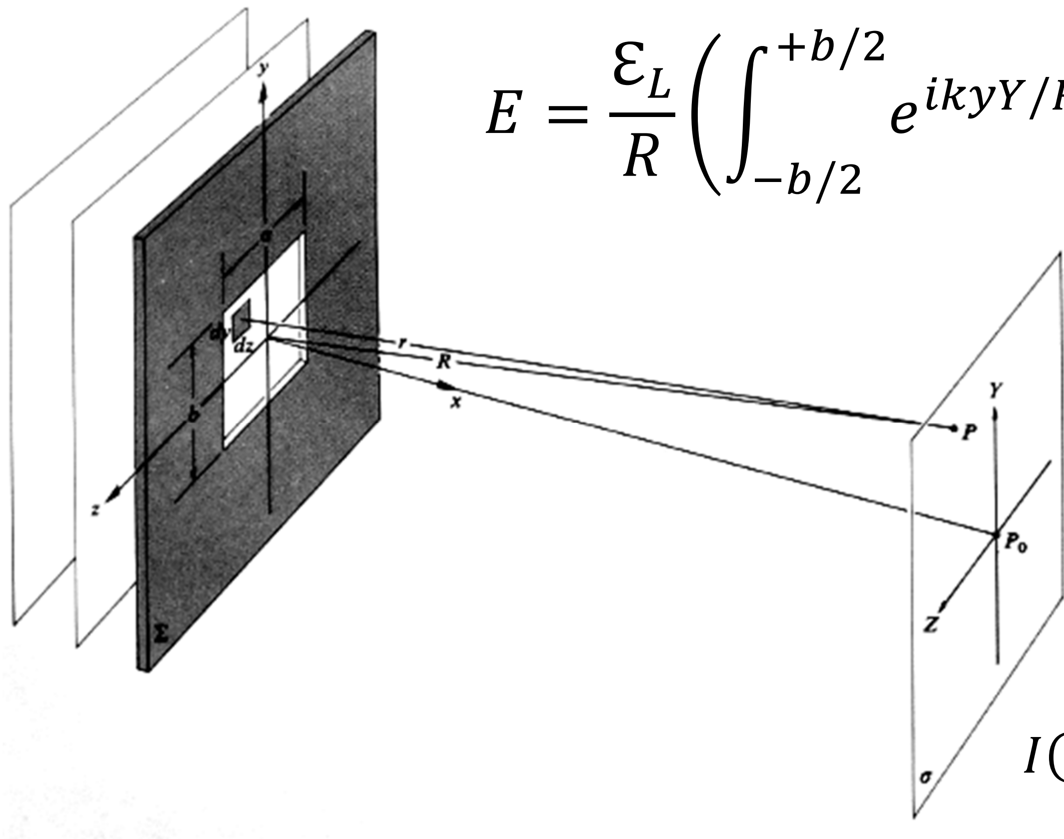
Emerging light has cylindrical symmetry

Rectangular Aperture Fraunhofer Diffraction

Source strength per unit area: $\mathcal{E}_A$

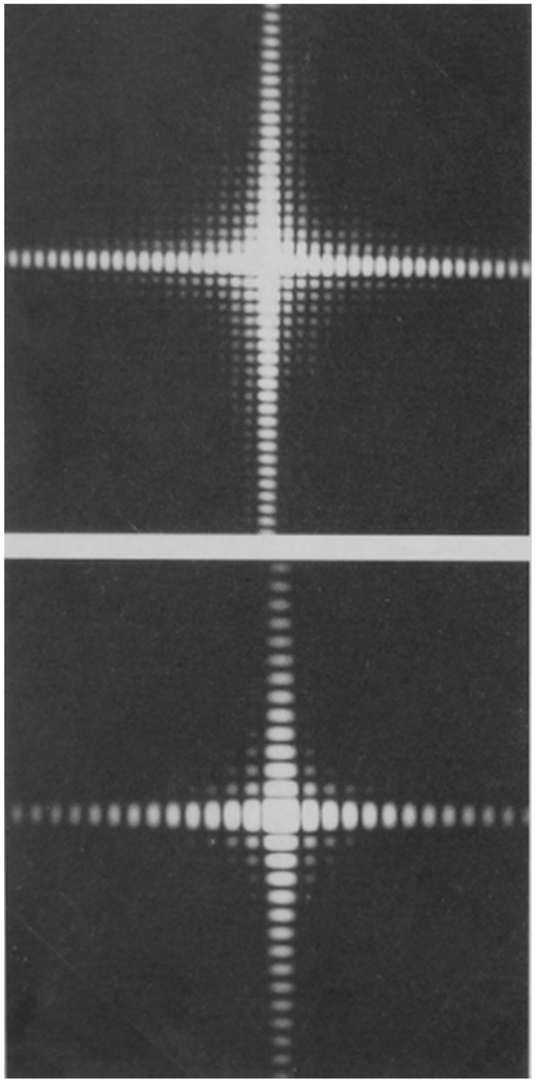
$$dE \approx \frac{\mathcal{E}_L e^{ikyY/R} e^{ikzZ/R} dydz}{R}$$

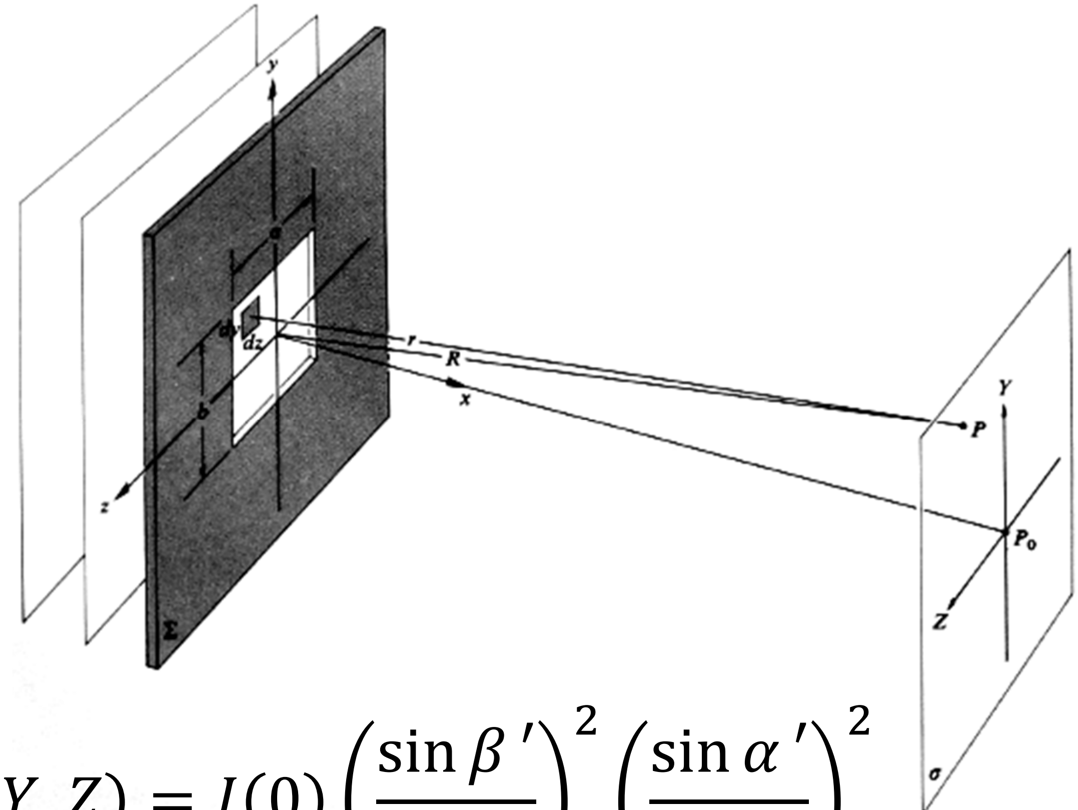
$$E = \frac{\mathcal{E}_L}{R}\left(\int_{-b/2}^{+b/2} e^{ikyY/R} dy\right)\left(\int_{-a/2}^{+a/2} e^{ikzZ/R} dz\right)$$



$$I(Y,Z) = I(0)\left(\frac{\sin\beta'}{\beta'}\right)^2\left(\frac{\sin\alpha'}{\alpha'}\right)^2$$

Rectangular Aperture



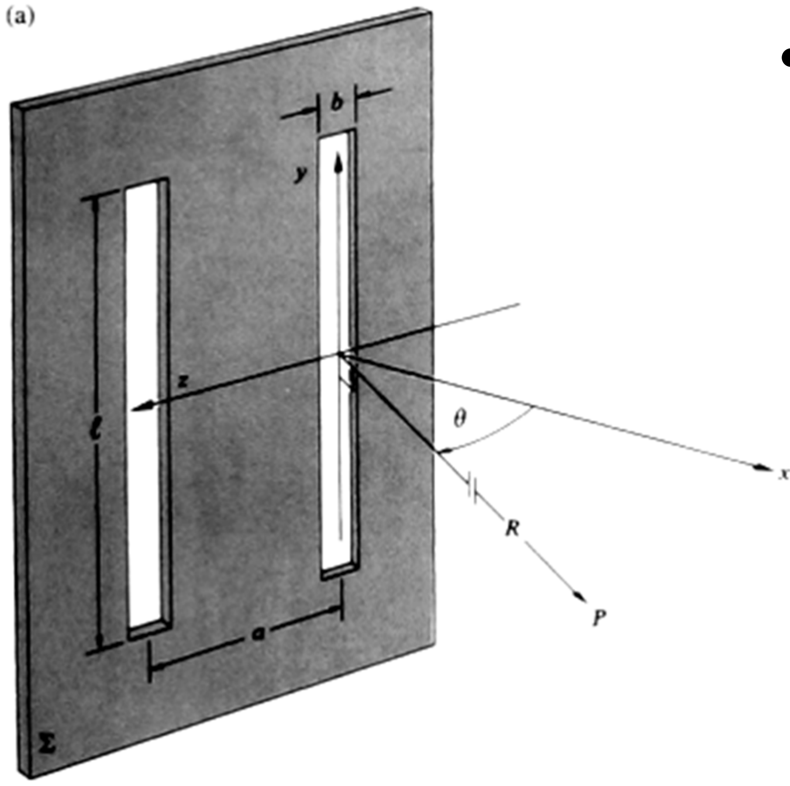


$$I(Y,Z) = I(0)\left(\frac{\sin\beta'}{\beta'}\right)^2\left(\frac{\sin\alpha'}{\alpha'}\right)^2$$

$$\beta' = \frac{1}{2}kbY/R$$

$$\alpha' = \frac{1}{2}kaZ/R$$

Double-Slit Fraunhofer Diffraction

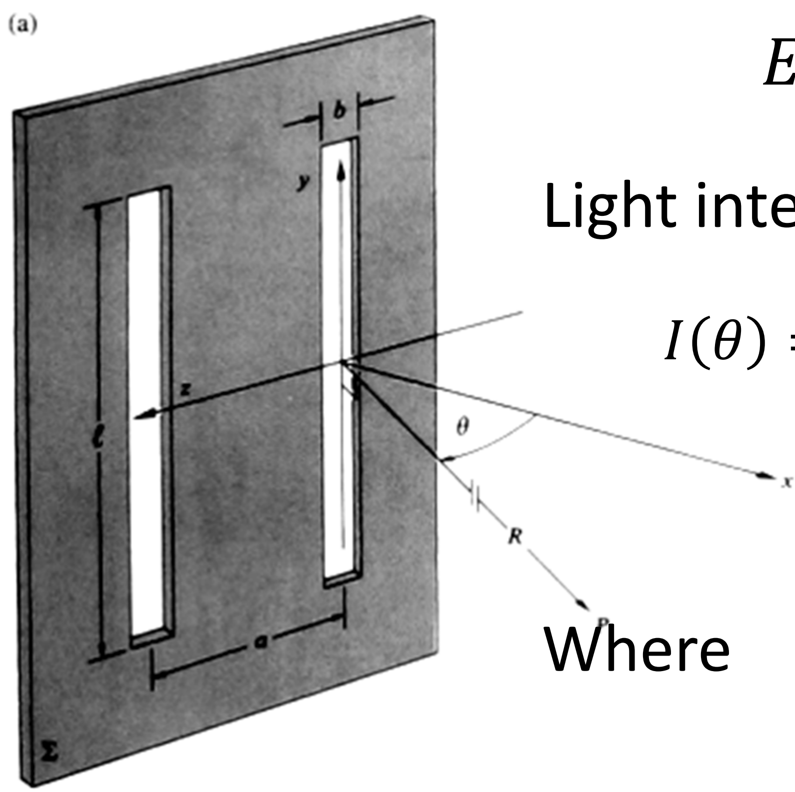


- Same idea, but this time we integrate over two slits:

$$E = \frac{\mathcal{E}_L}{R}\int_{-b/2}^{+b/2} e^{iky\sin\theta}\,dy + \frac{\mathcal{E}_L}{R}\int_{a-b/2}^{a+b/2} e^{iky\sin\theta}\,dy$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}\left(1 + e^{ika\,sin\,\theta}\right)$$

Double-Slit Fraunhofer Diffraction



$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta}\left(1 + e^{ika\sin\theta}\right)$$

Light intensity:

$$I(\theta) = 2I(0)\left(\frac{\sin\beta}{\beta}\right)^2 (1 + \cos(ka\sin\theta))$$

$$= 4I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha$$

Where

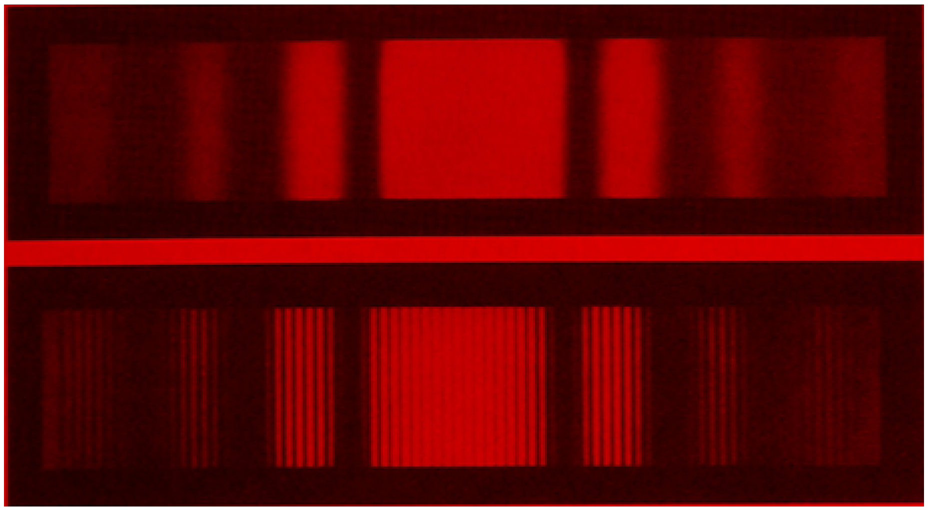
$$\alpha = \frac{1}{2} ka \sin\theta$$

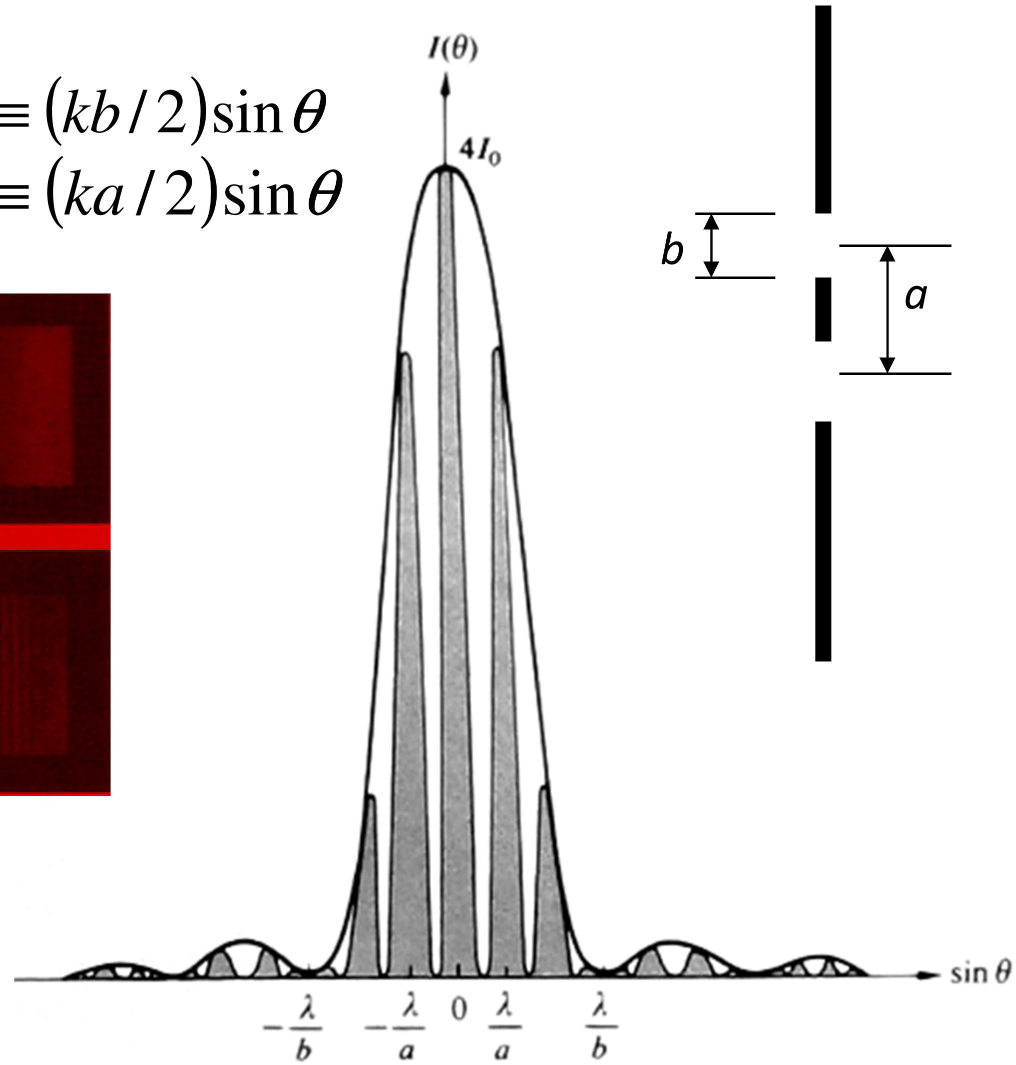
Since $a > b$, $\cos\alpha$ oscillates more rapidly than $\sin\beta$

Double Slit: Fraunhofer Diffraction

$$I(\theta) = 4I_0\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\alpha$$

$$\beta \equiv (kb/2)\sin\theta$$
$$\alpha \equiv (ka/2)\sin\theta$$





Minima: $\alpha = \pm\pi/2, \pm 3\pi/2$

$$a\sin\theta = (m + 1/2)\lambda$$

m=0, ±1, ±2...

or: β=$m\pi$, where m=±1, ±2...

$$b\sin\theta = m\lambda$$

Three-Slit Fraunhofer Diffraction

$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} \frac{e^{3i\delta} - 1}{e^{i\delta} - 1}$$

$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} \frac{e^{3i\delta/2}}{e^{i\delta/2}} \frac{e^{3i\delta/2} - e^{-3i\delta/2}}{e^{i\delta/2} - e^{-i\delta/2}}$$

$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} e^{i\delta} \frac{\sin 3\delta/2}{\sin\delta/2}$$

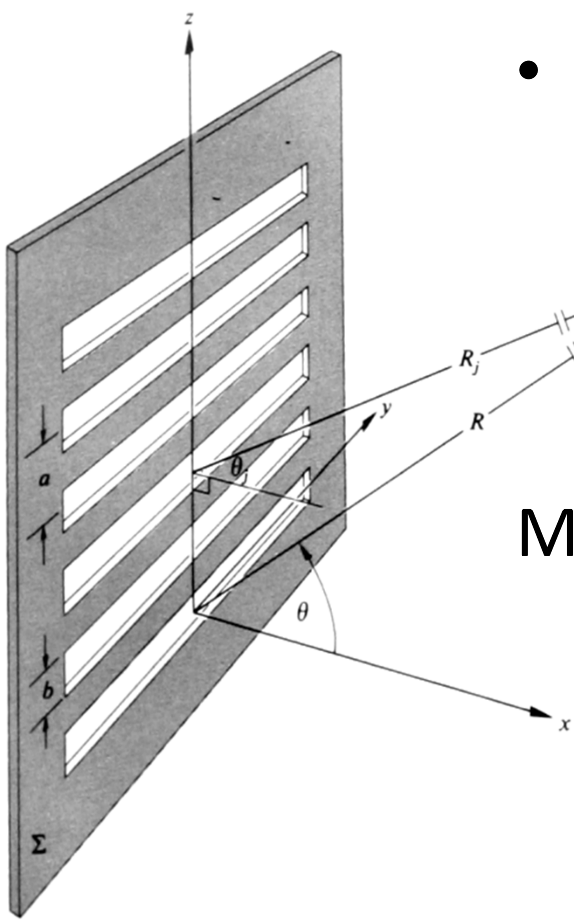
$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} e^{ika\sin\theta} \frac{\sin\left(\frac{3}{2} ka \sin\theta\right)}{\sin\left(\frac{1}{2} ka \sin\theta\right)}$$

$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} e^{2i\alpha} \frac{\sin 3\alpha}{\sin\alpha}$$

Light intensity: $I(\theta) = I(0) \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin 3\alpha}{\sin\alpha}\right)^2$

Three-Slit Fraunhofer Diffraction

- Light intensity: $I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin 3\alpha}{\alpha}\right)^2$

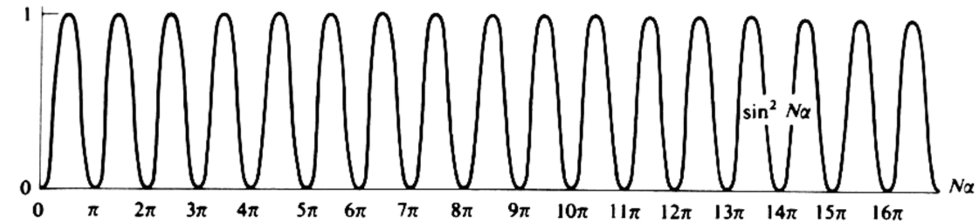


- In general, when there are N slits:

$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\alpha}\right)^2$$

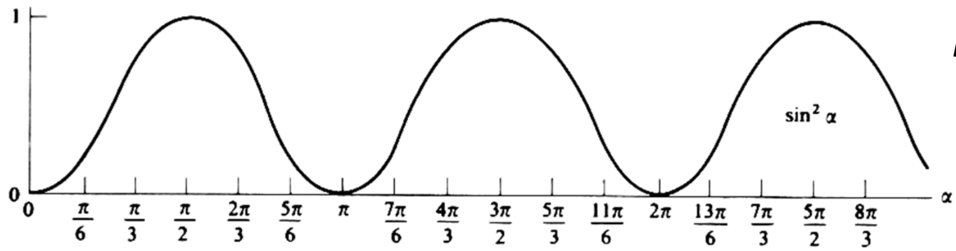
Maxima occur when $\alpha = \frac{1}{2}ka\sin\theta = m\pi$

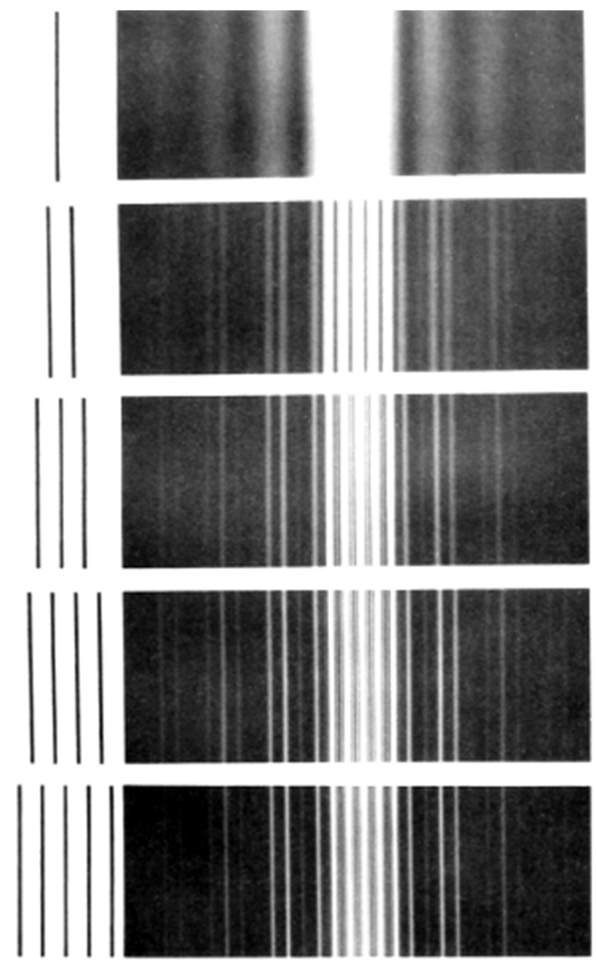
$$a\sin\theta = m\lambda$$

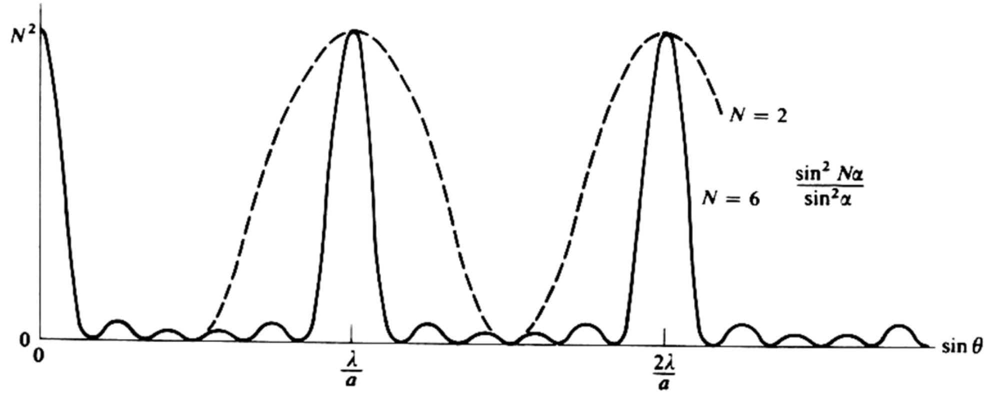


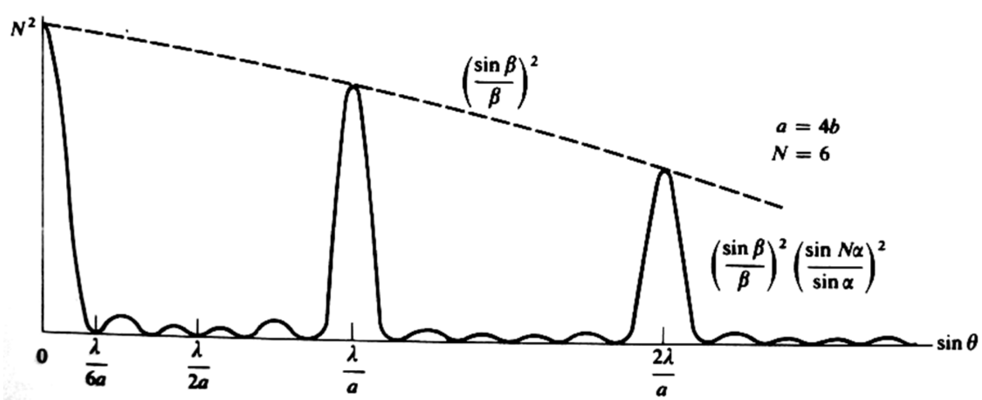
$$I(\theta) = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

a = 4b
N=6







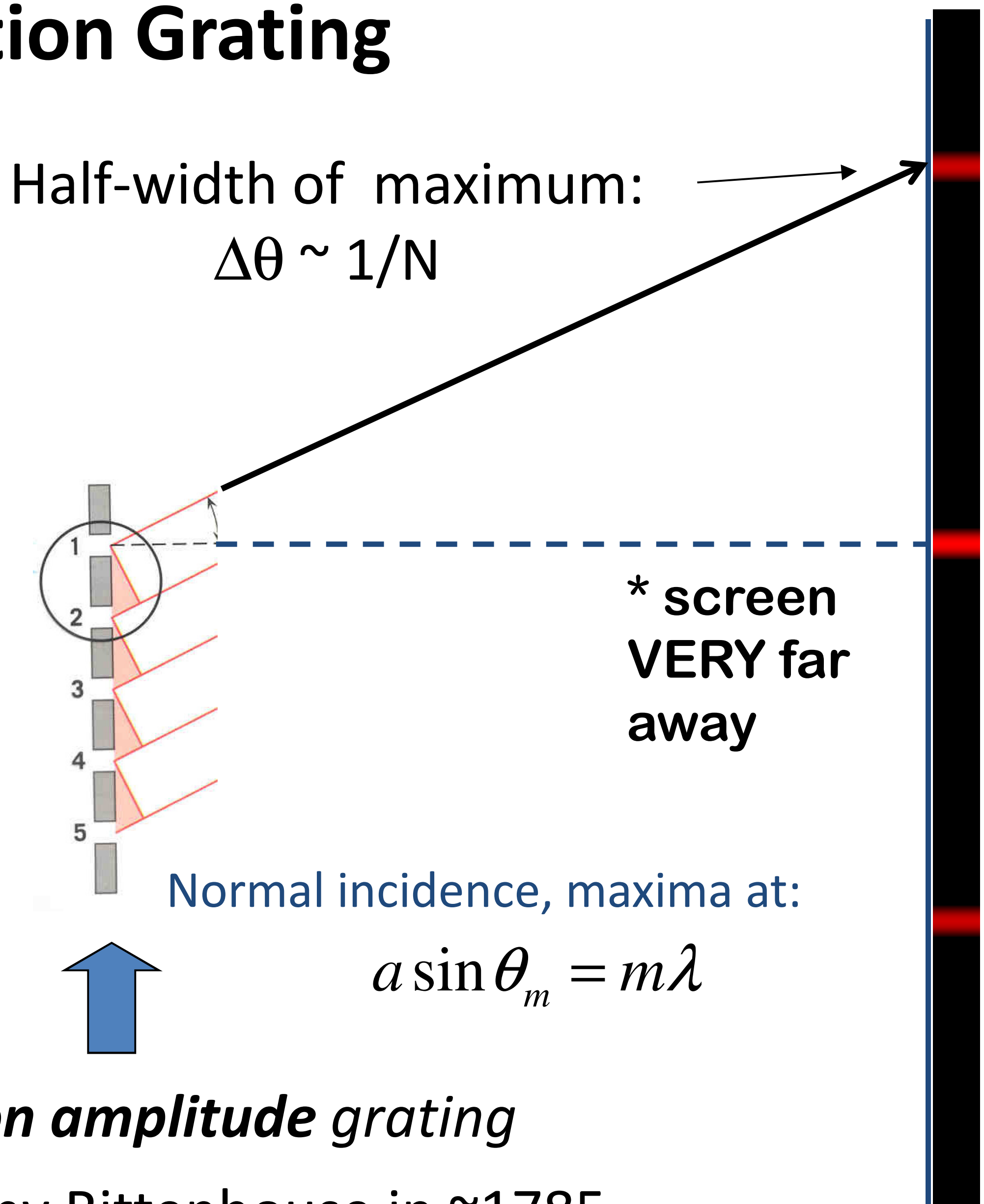


Diffraction Grating

Usually gratings have thousands of slits and are characterized by the number of slits per cm (for example: 6000 cm^{-1})

Half-width of maximum:

$\Delta\theta \sim 1/N$

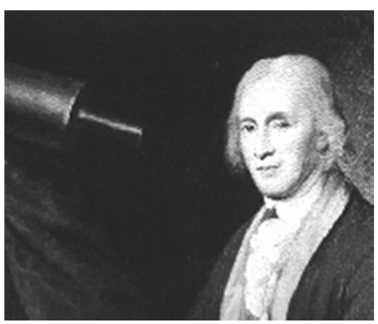


Normal incidence, maxima at:

$$a \sin\theta_m = m\lambda$$

Transmission amplitude *grating*

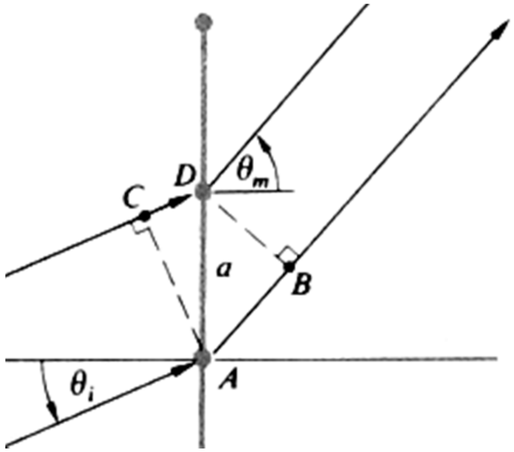
Introduced by Rittenhouse in ~1785

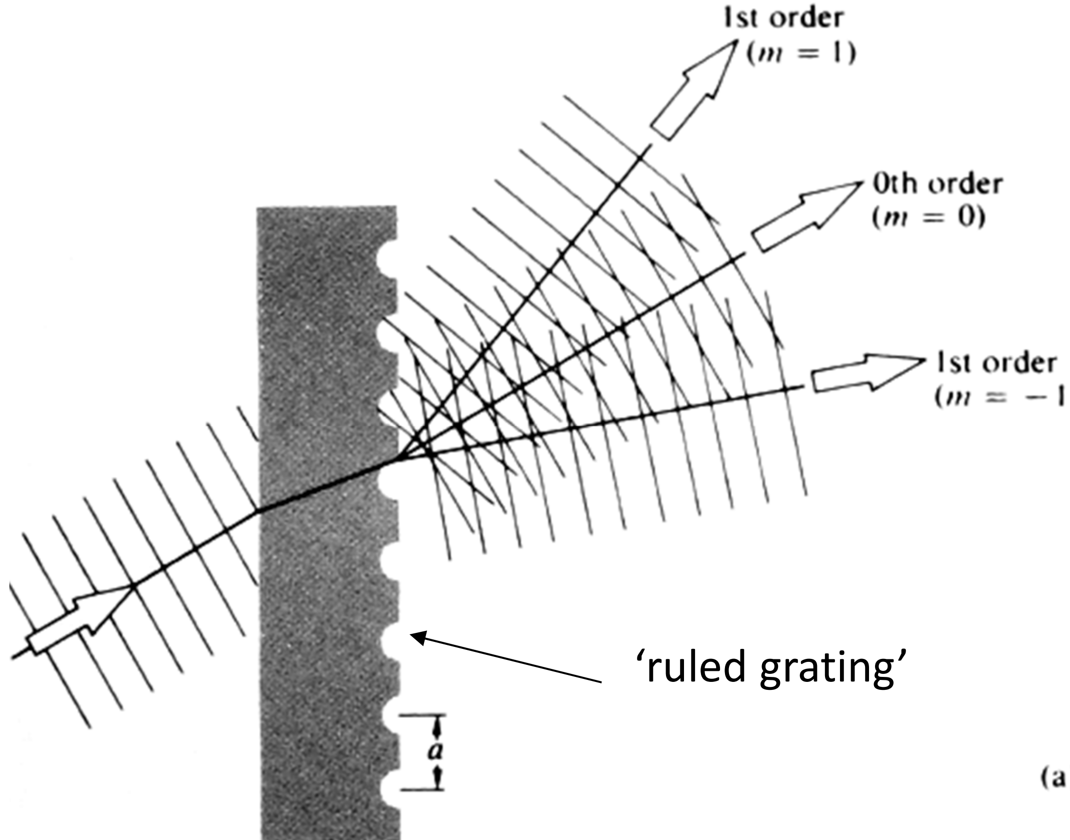


David Rittenhouse
1732 - 1796

Transmission Phase Grating

General case:
Incidence angle $\theta_i \neq 0$

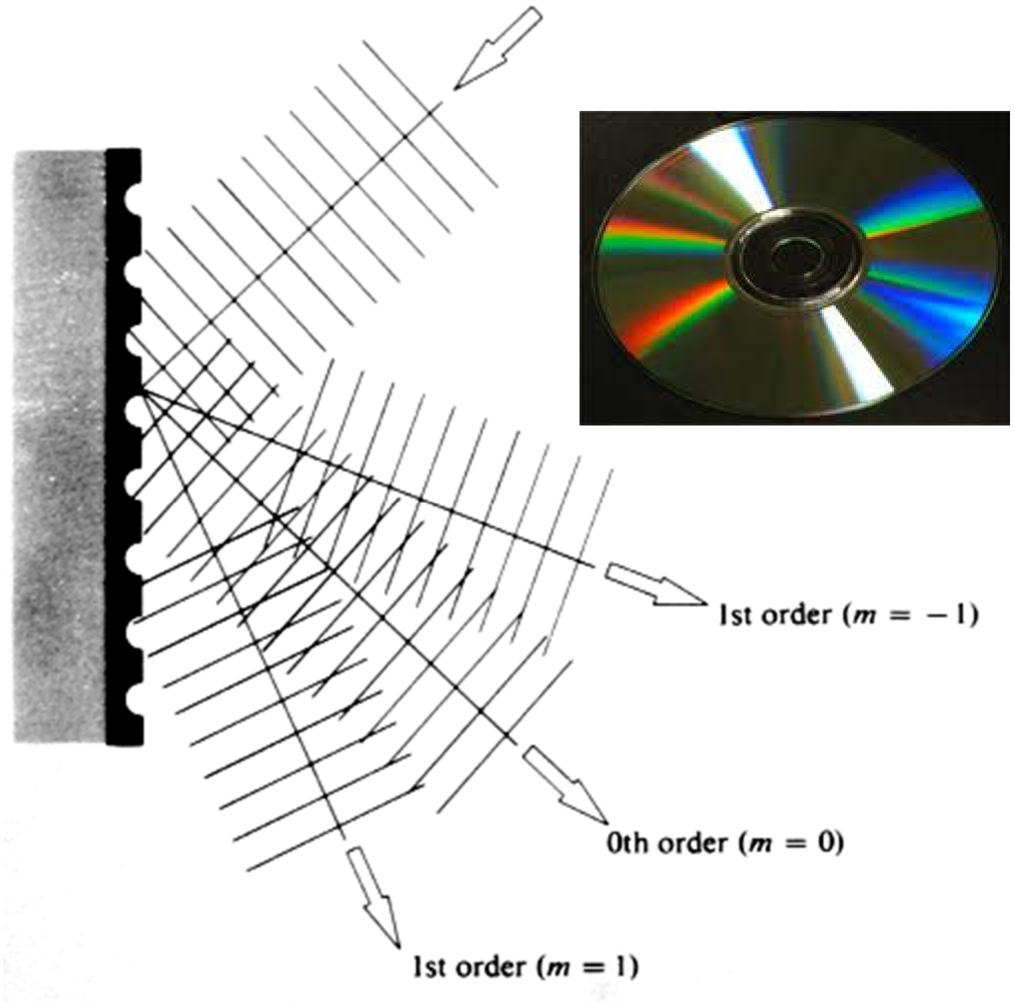




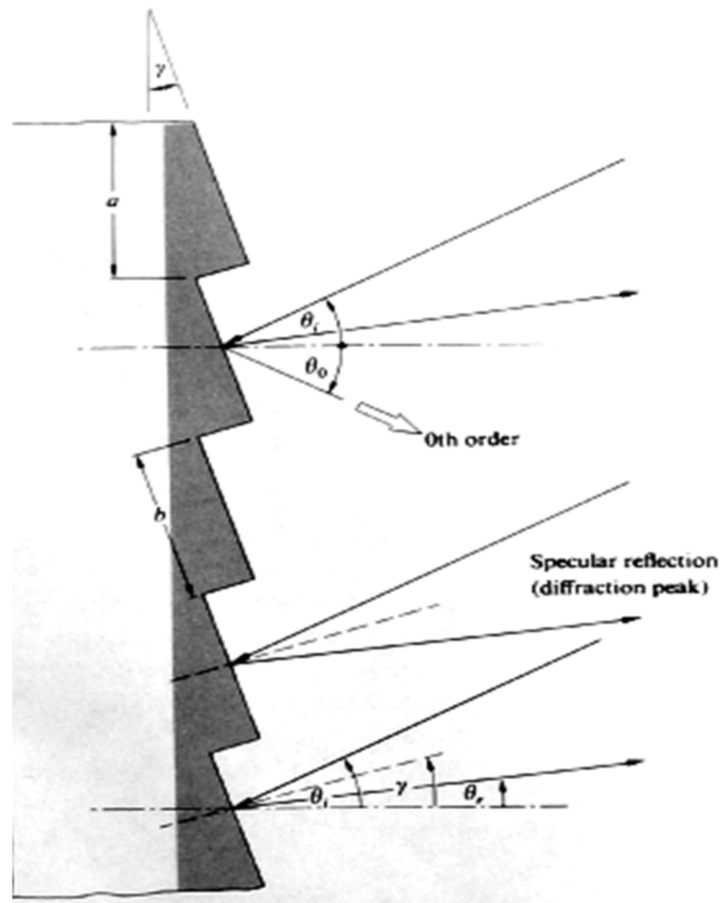
$$\overline{AB} - \overline{CD} = a\sin\theta_m - a\sin\theta_i$$

Maxima: $a(\sin\theta_m - \sin\theta_i) = m\lambda$ for arbitrary incidence angle

Reflection Phase Grating



Examples: CD disk
Finely machined surfaces

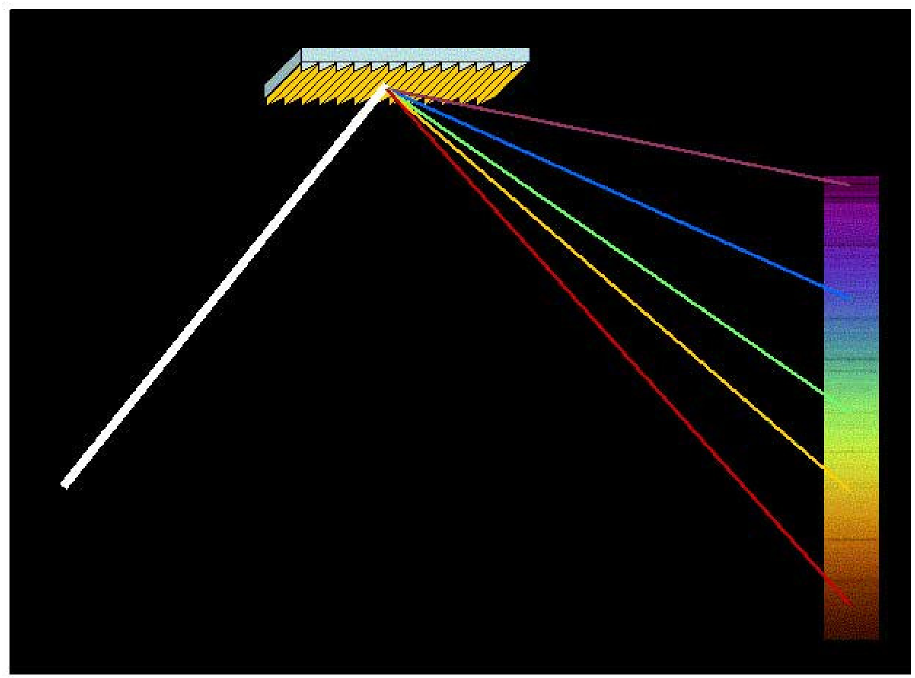


Diffraction Grating Spectrometers

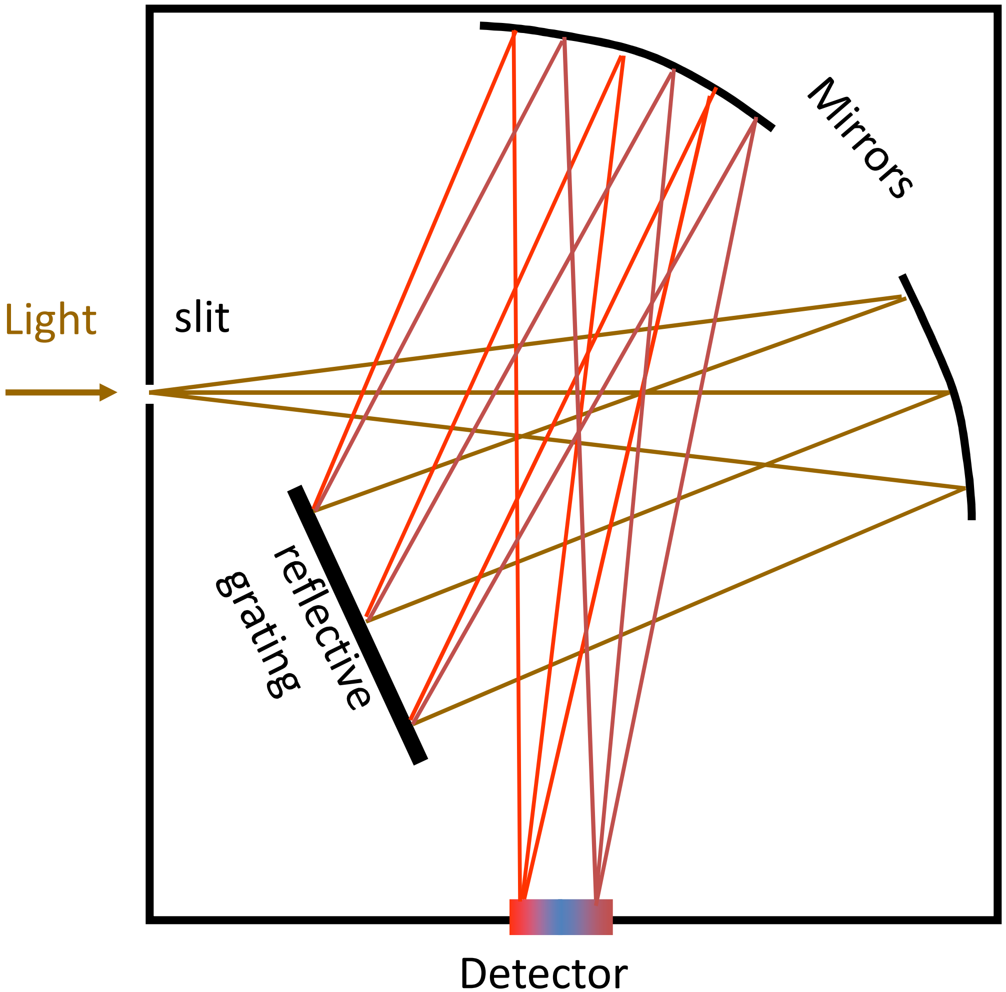
- Angle of maximum intensity depends on wavelength:

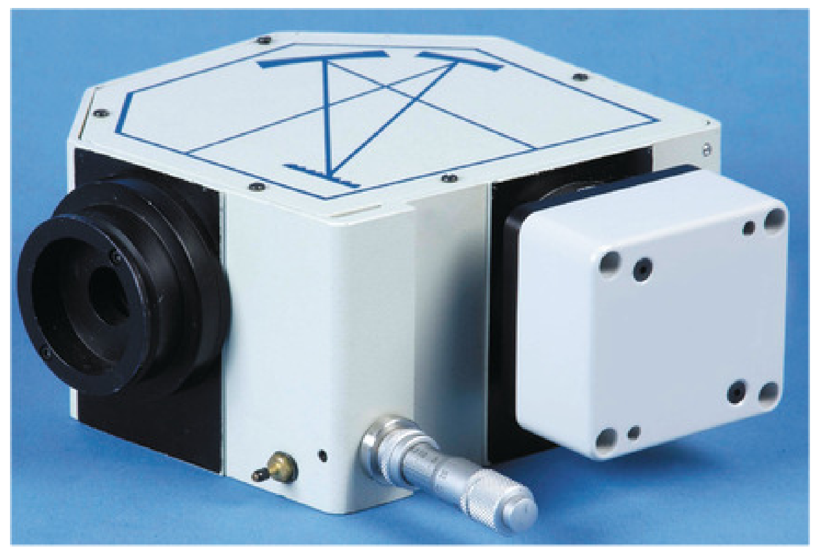
$$\sin\theta = \frac{m\lambda}{a}$$

- Diffraction gratings are used to separate and analyze the spectrum of light:

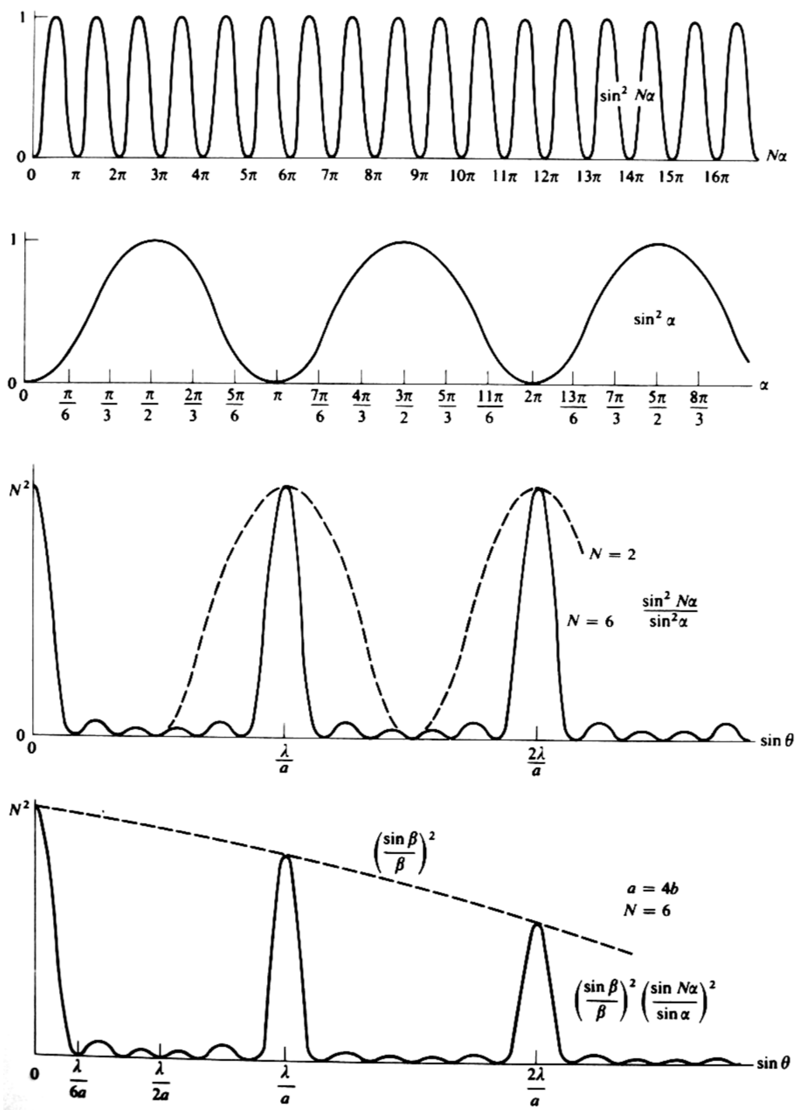


Diffraction Grating Spectrograph





Width of Spectral Lines



$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

- Maxima occur when

$$\alpha = m\pi$$

- Otherwise, zeros occur when

$$N\alpha = m'\pi$$

- Zeros on either side of a peak

$$N\alpha_{\pm} = (Nm \pm 1)\pi$$

- Width of peak:

$$\Delta\alpha = \alpha_{+} - \alpha_{-} = \frac{2\pi}{N}$$

Width of Spectral Lines

$$\Delta\alpha = \frac{2\pi}{N}$$

$$\alpha = \frac{1}{2} ak \sin\theta = \frac{\pi a}{\lambda} \sin\theta$$

$$\Delta\alpha = \frac{\pi a}{\lambda} \cos\theta \, \Delta\theta$$

- Angular resolution:

$$\Delta\theta = \frac{2\lambda}{Na \cos\theta}$$

$$(\Delta\theta)_{min} = \frac{1}{2} \Delta\theta = \frac{\lambda}{Na \cos\theta}$$

Angular Dispersion

- The angle depends on the wavelength:

$$a \sin\theta = m\lambda$$

$$a \cos\theta \, \Delta\theta = m\Delta\lambda$$

$$\frac{d\theta}{d\lambda} = \frac{m}{a \cos\theta}$$

- Chromatic resolving power is defined:

$$\mathcal{R} \equiv \frac{\lambda}{(\Delta\lambda)_{min}}$$

$$(\Delta\lambda)_{min} = \frac{a\cos\theta}{m}(\Delta\theta)_{min} = \frac{a\cos\theta}{m}\,\frac{2\lambda}{Na\cos\theta} = \frac{\lambda}{Nm}$$

$$\mathcal{R} = Nm = \frac{Na\sin\theta}{\lambda}$$

Resolving Power

- The chromatic resolving power is proportional to Na
- Example: 6000 lines per cm, 15 cm width

$$N = (6000\, lines/cm) \times (15\, cm) = 90{,}000$$

$$a = 1/(6000\, lines/cm) = 1.667\ \mu m$$

$$\left.\begin{aligned} \lambda &= 588.991\, nm \\ \lambda' &= 589.595\, nm \end{aligned}\right\} \Delta\lambda = 0.604\, nm$$

$m = 2$ (second order)

$$\sin\theta = \frac{m\lambda}{a} =$$

$$2 \times (589 \times 10^{-7} cm) \times (6000\, lines/cm)$$

$$= 0.707$$

$$\mathcal{R} = mN = 2 \times 90{,}000 = 180{,}000$$

$$(\Delta\lambda)_{min} = \frac{\lambda}{\mathcal{R}} = \frac{(589\, nm)}{180{,}000} = 0.00327\, nm$$

Overlapping Orders

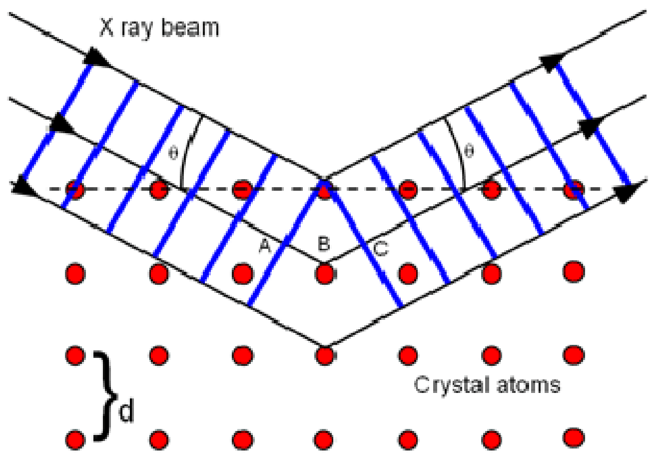
- Confusion can arise when a spectral line at one order overlaps with a different spectral line at a different order:

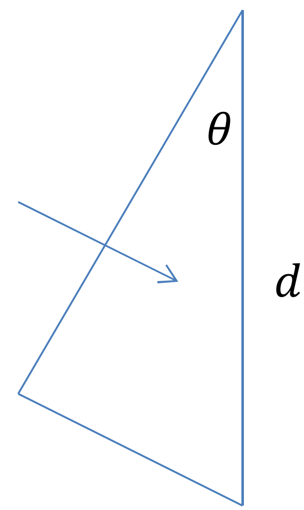
$$\sin\theta = \frac{(m+1)\lambda}{a} = \frac{m\lambda'}{a} = \frac{m(\lambda + \Delta\lambda)}{a}$$

$$\Delta\lambda = \frac{\lambda}{m} \equiv (\Delta\lambda)_{fsr} \text{ (free spectral range)}$$

X-ray Diffraction

- With short enough wavelengths, the atoms in a crystal lattice form the diffraction grating:



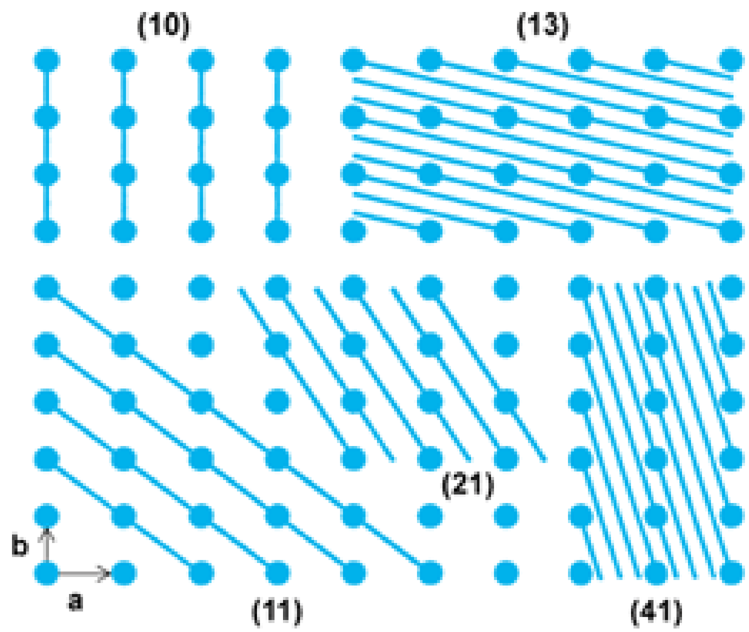


$$2d\sin\theta = n\lambda$$

(Bragg's Law)

X-ray Diffraction

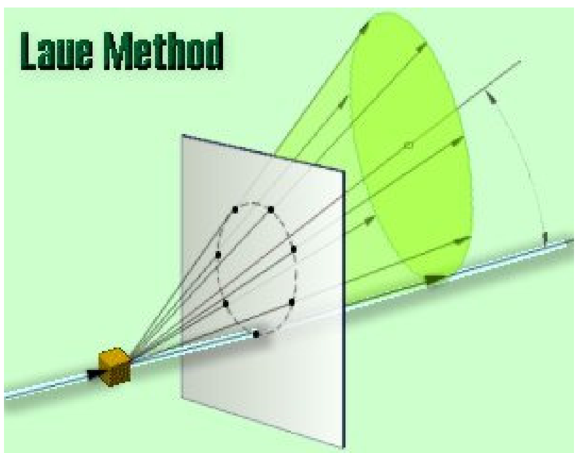
- Regular crystal lattices have many “planes”:

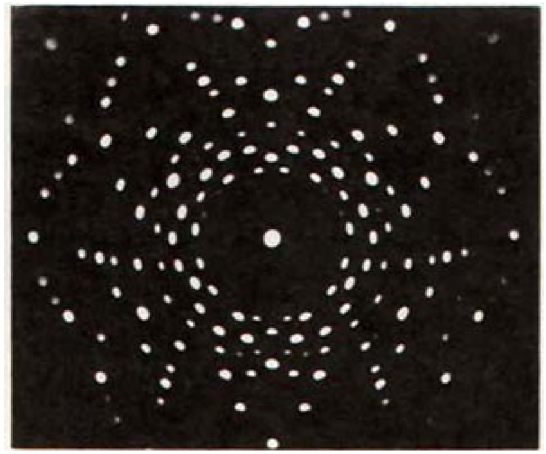


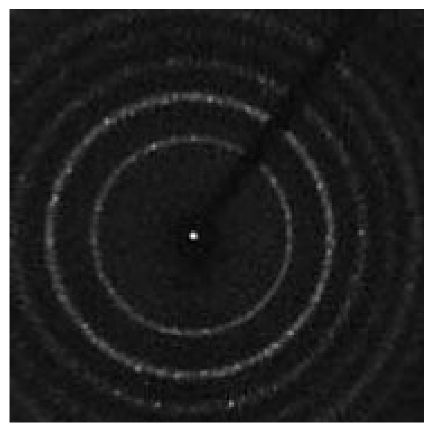
$$2d_{hkl} \sin \theta_{hkl} = n\lambda$$

X-ray Diffraction

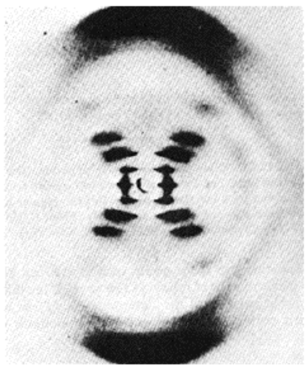
- Max von Laue exposed crystals to a continuous x-ray spectrum:





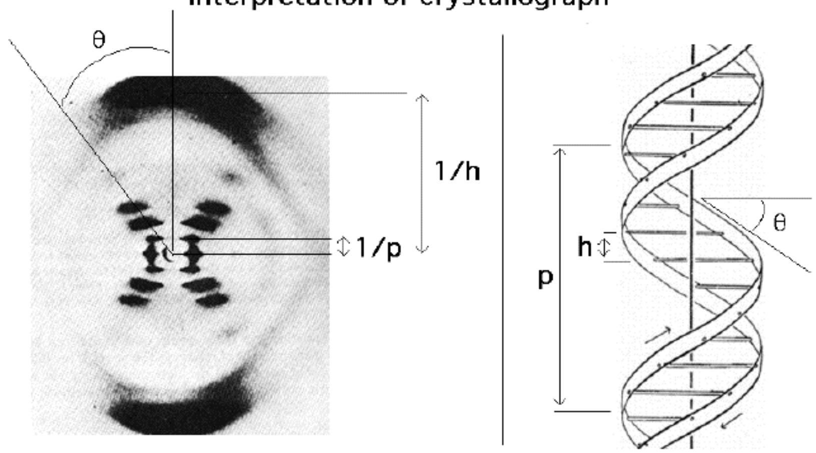


X-ray Diffraction



X-ray diffraction pattern from B form of DNA

Interpretation of crystallograph



θ - tilt of helix (angle from perpendicular to long axis)

h = 3.4 Å (Distance between bases)

p = 34 Å (Distance for one complete turn of helix; Repeat unit of the helix)