

Physics 42200  
**Waves & Oscillations**

Lecture 31 – Electromagnetic Waves

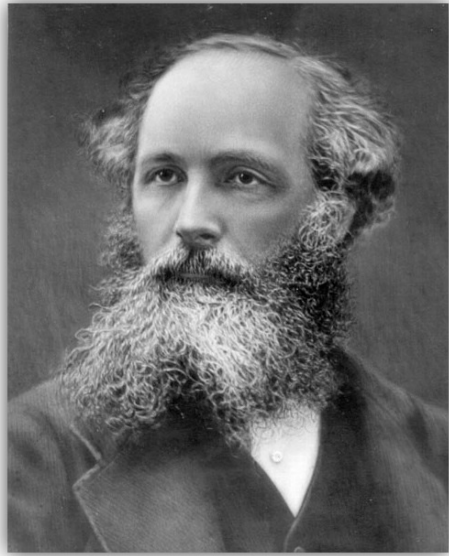
Spring 2015 Semester

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# Electromagnetism

- Geometric optics overlooks the wave nature of light.
  - Light inconsistent with longitudinal waves in an ethereal medium
  - Still an excellent approximation when feature sizes are large compared with the wavelength of light
- But geometric optics could not explain
  - Polarization
  - Diffraction
  - Interference
- A unified picture was provided by Maxwell c. 1864

# Maxwell's Equations (1864)



$$\oint_S \hat{n} \cdot \vec{E} dA = \frac{Q_{inside}}{\epsilon_0} \quad (1)$$

$$\oint_S \hat{n} \cdot \vec{B} dA = 0 \quad (2)$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt} \quad (3)$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\phi_e}{dt} \quad (4)$$

# Maxwell's Equations in Free Space

In “free space” where there are no electric charges or sources of current, Maxwell's equations are quite symmetric:

$$\oint_S \hat{n} \cdot \vec{E} dA = 0$$

$$\oint_S \hat{n} \cdot \vec{B} dA = 0$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\phi_e}{dt}$$

# Velocity of Electromagnetic Waves

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$
$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

Speed of wave propagation is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$
$$= \frac{1}{\sqrt{(4\pi \times 10^{-7} \text{ N/A}^2)(8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m})}}$$
$$= \mathbf{2.998 \times 10^8 \text{ m/s}}$$

(speed of light)

# Electromagnetic Waves

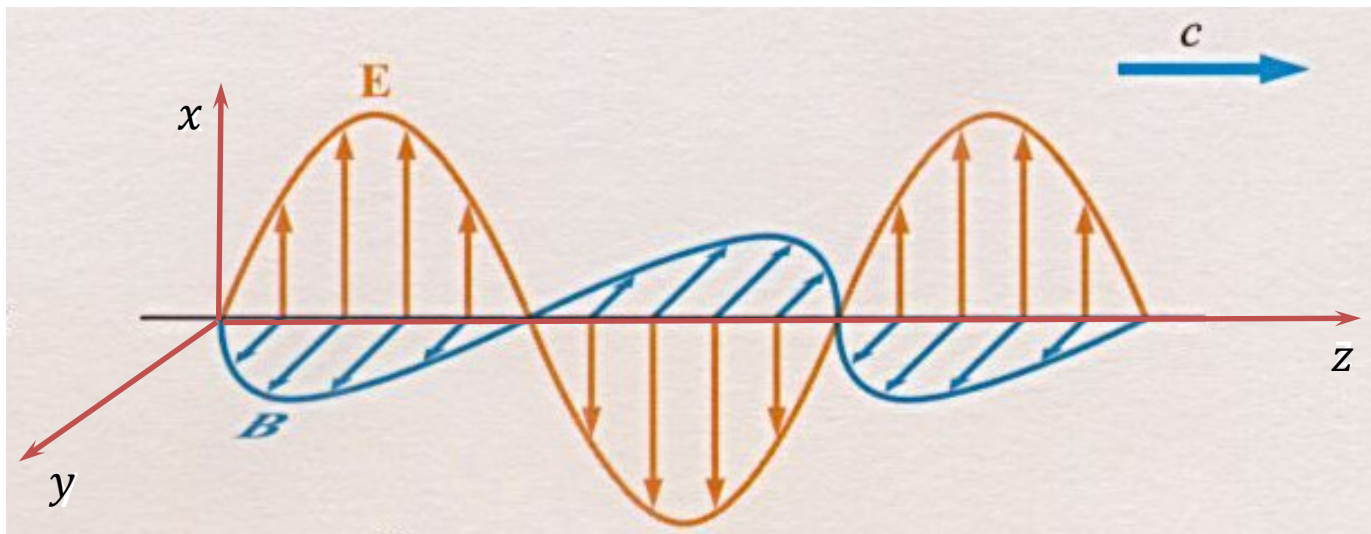
$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

- A solution is  $E_x(z, t) = E_0 \sin(kz - \omega t)$   
where  $\omega = kc = 2\pi c/\lambda$
- What is the magnetic field?

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} = -kE_0 \cos(kz - \omega t)$$

$$B_y(x, t) = \frac{k}{\omega} E_0 \sin(kx - \omega t)$$

# Electromagnetic Waves



- $\vec{E}$ ,  $\vec{B}$  and  $\vec{v}$  are mutually perpendicular.
- In general, the direction is

$$\hat{s} = \hat{E} \times \hat{B}$$

# Energy in Electromagnetic Waves

Energy density of electric and magnetic fields:

$$u_e = \frac{1}{2} \epsilon_0 E^2 \qquad u_m = \frac{1}{2\mu_0} B^2$$

For an electromagnetic wave,  $B = E/c = E\sqrt{\mu_0\epsilon_0}$

$$u_m = \frac{1}{2\mu_0} B^2 = \frac{1}{2} \epsilon_0 E^2 = u_e$$

The total energy density is

$$u = u_m + u_e = \epsilon_0 E^2$$



# Intensity of Electromagnetic Waves

- Intensity is defined as the average power transmitted per unit area.

Intensity = Energy density  $\times$  wave velocity

$$I = \epsilon_0 c \langle E^2 \rangle = \frac{\langle E^2 \rangle}{\mu_0 c}$$

$$\mu_0 c = 377 \, \Omega \equiv Z_0$$

(Impedance of free space)

# Polarization

- Light is an oscillating electromagnetic field
- The electric field has a direction

$$\vec{E}(x, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$$

- No need to specify the magnetic field direction:

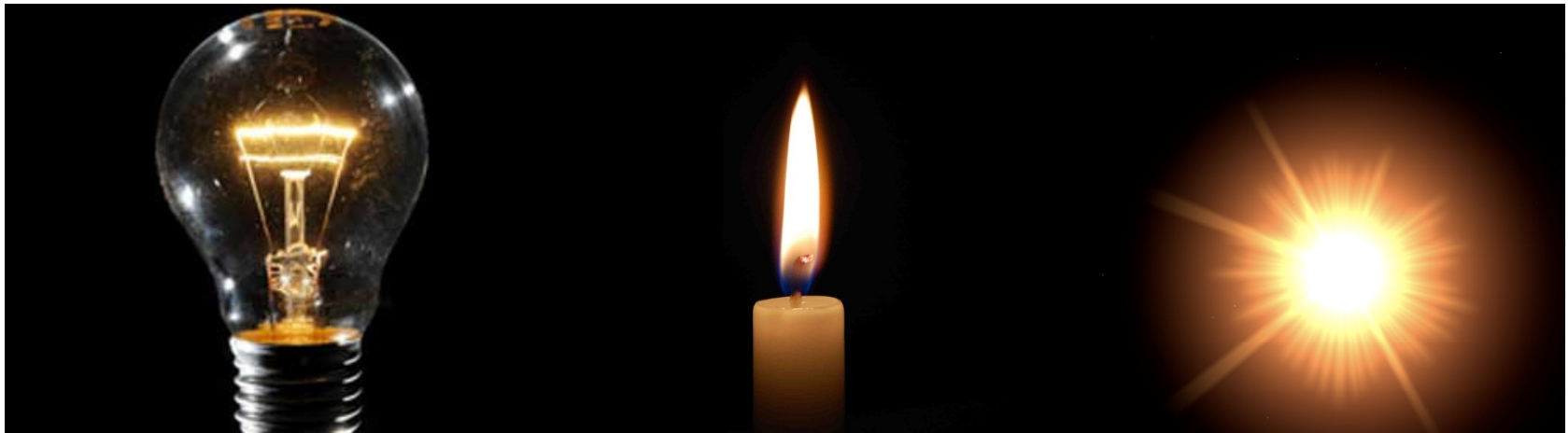
$$\vec{S} = \vec{E} \times \vec{H}$$

$$\vec{H} = (\hat{k} \times \vec{E})/Z \quad \text{where } Z = \sqrt{\mu/\epsilon}$$

- $\vec{H}$  refers to the magnetic field due to the light, not including any induced magnetic fields in the presence of matter.
- *Coherent* light has the same phase over macroscopic distances and time
- *Polarized* light has the electric field aligned over macroscopic distances and time

# Polarization

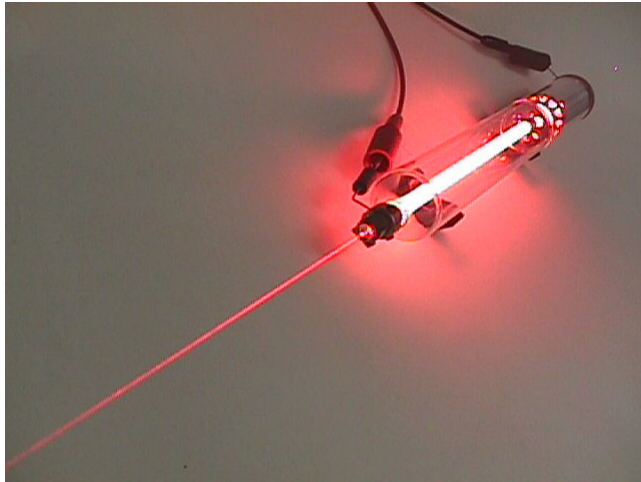
## Sources of un-polarized light



- Hot atoms transfer kinetic energy to electrons randomly
- Electrons randomly de-excite, emitting incoherent light – uncorrelated in phase and polarization

# Polarization

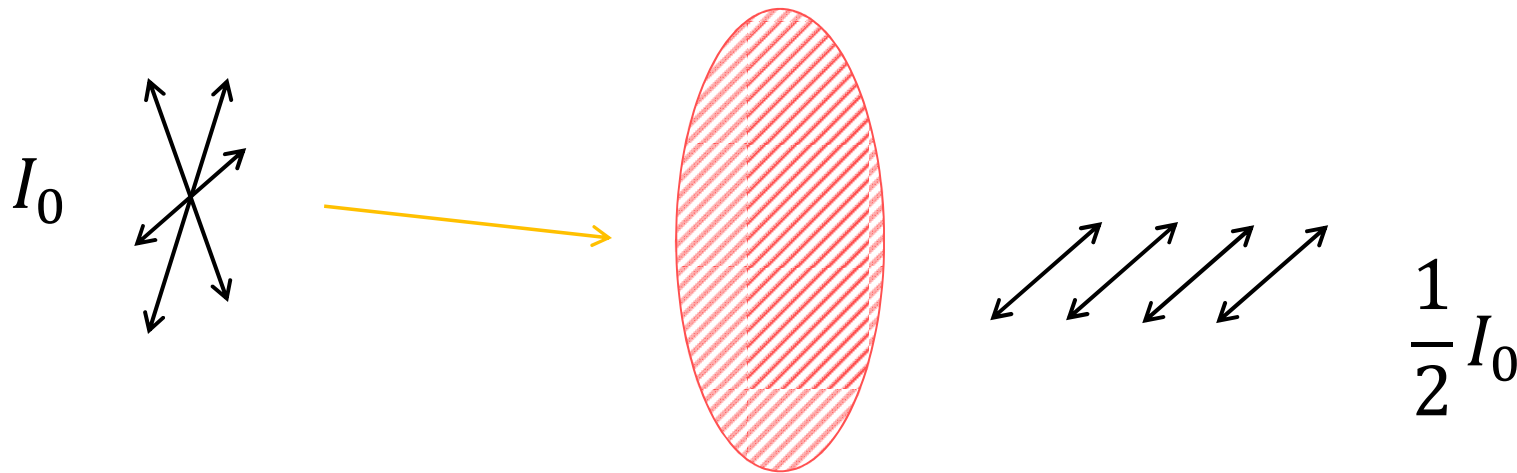
## Sources of polarized light



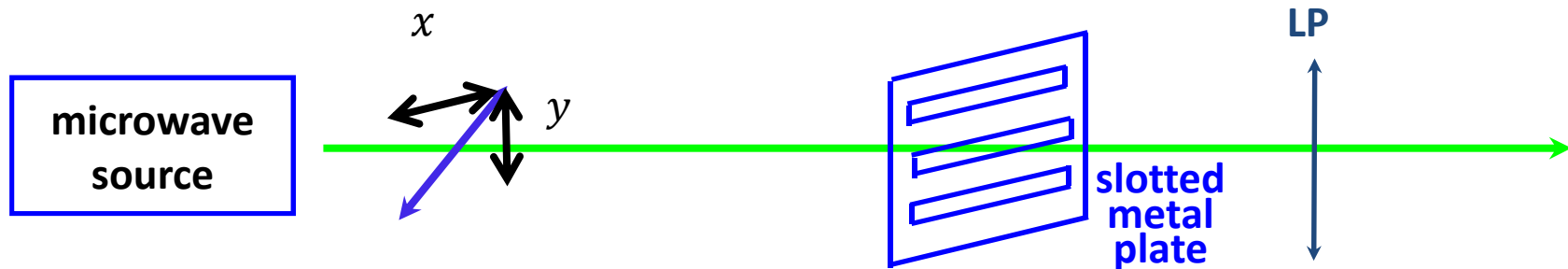
- Lasers produce light by stimulated emission
  - A photon causes an excited atom to emit another photon
  - The photon is emitted in phase and with the same polarization
- The resulting beam is highly coherent and polarized

# Polarization by Absorption

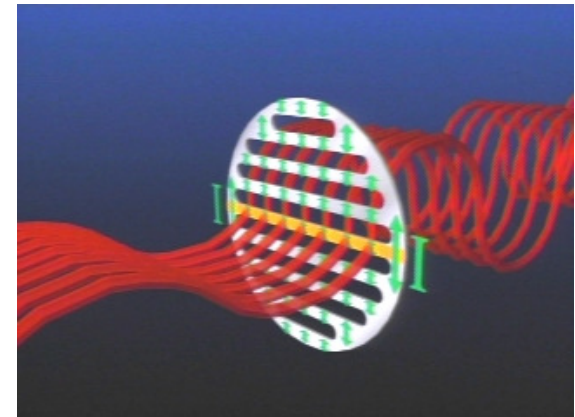
- A polarizer absorbs the component with  $\vec{E}$  oriented along a particular axis.
- The light that emerges is linearly polarized along the perpendicular axis.
- If the light is initially un-polarized, half the light is absorbed.



# Example with Microwaves



- The electric field in the  $x$ -direction induces currents in the metal plate and loses energy:
  - Horizontally polarized microwaves are absorbed
- No current can flow in the  $y$ -direction because of the slots
  - Vertically polarized microwaves are transmitted



# Polarization by Reflection



- Reflected light is preferentially polarized
- The other component must be transmitted
- Transmission and reflection coefficients must depend on the polarization

# Boundary Conditions

- In the presence of matter, the components electric and magnetic fields perpendicular to a surface change abruptly
- The component parallel to a surface is the same on both sides.
- Summary:

- Perpendicular to surface

$$\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$$

$$\mu_1 H_{1\perp} = \mu_2 H_{2\perp}$$

- Parallel to surface

$$E_{1\parallel} = E_{2\parallel}$$

$$H_{1\parallel} = H_{2\parallel}$$



# Reflection From a Surface

First case:

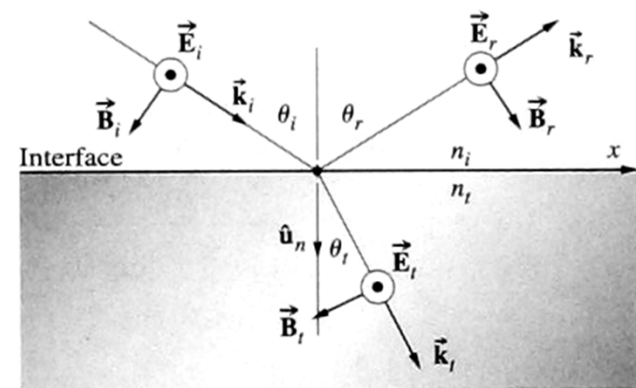
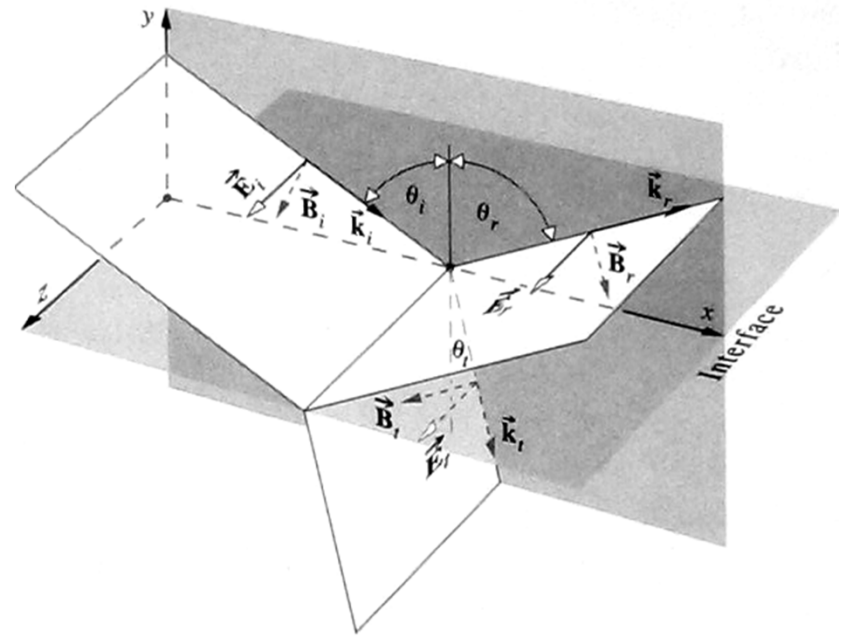
$\vec{E}$  is parallel to the surface...

$$E_i + E_r = E_t$$

$\vec{H}$  has components parallel and perpendicular to the surface

$$H_{\parallel i} + H_{\parallel r} = H_{\parallel t}$$

But  $H_{\parallel} = \vec{H} \cdot \hat{\tau}...$



# Reflection From a Surface

$\vec{E}$  is perpendicular to  $\hat{n}$

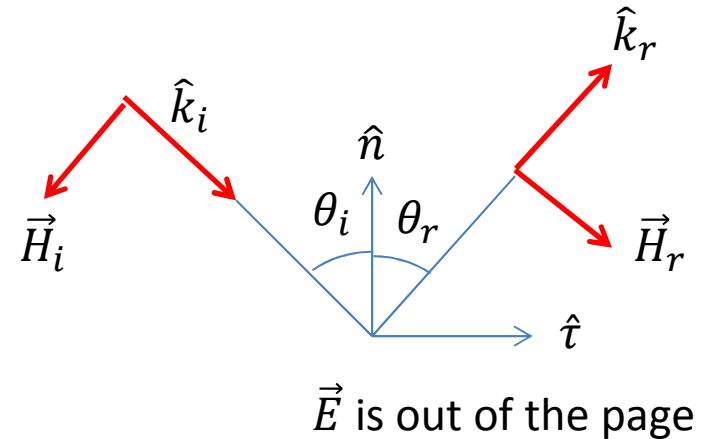
$$\hat{t} = \frac{\vec{n} \times \vec{E}}{|\vec{E}|}$$

$\vec{H}$  can be written

$$\vec{H} = \frac{\hat{k} \times \vec{E}}{Z}$$

So we can write

$$\begin{aligned} \vec{H}_i \cdot \hat{t} &= \frac{1}{ZE_i} (\hat{k} \times \vec{E}_i) \cdot (\vec{n} \times \vec{E}_i) \\ &= \frac{E_i}{Z} (\hat{k} \cdot \hat{n}) = -\frac{E_i}{Z} \cos \theta_i \end{aligned}$$



Likewise,

$$\vec{H}_r \cdot \hat{t} = \frac{E_r}{Z} \cos \theta_r$$

# $\vec{E}$ perpendicular to $\hat{n}$

- Boundary condition for  $\vec{H}$ :

$$H_{\parallel i} + H_{\parallel r} = H_{\parallel t}$$

- Previous results:

$$H_{i\parallel} = -\frac{E_i}{Z_1} \cos \theta_i \quad H_{r\parallel} = \frac{E_r}{Z_1} \cos \theta_r = \frac{E_r}{Z_1} \cos \theta_i$$

$$\frac{-E_i \cos \theta_i + E_r \cos \theta_i}{Z_1} = \frac{-E_t \cos \theta_t}{Z_2}$$
$$E_i + E_r = E_t$$

- Two equations in two unknowns...

**$\vec{E}$  perpendicular to  $\hat{n}$**

$$\frac{-E_i \cos \theta_i + E_r \cos \theta_i}{Z_1} = \frac{-E_t \cos \theta_t}{Z_2}$$
$$\frac{E_i \cos \theta_t + E_r \cos \theta_t}{Z_2} = \frac{E_t \cos \theta_t}{Z_2}$$

- Solve for  $E_r/E_i$ :

$$\frac{E_r}{E_i} = \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$$

- Solve for  $E_t/E_i$ :

$$\frac{E_t}{E_i} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$$

# Reflection From A Surface

$\vec{H}$  parallel to surface

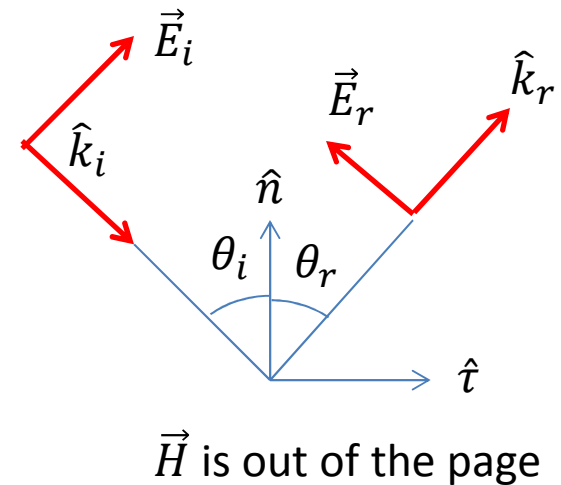
$$H_i + H_r = H_t$$

$$E_{\parallel i} + E_{\parallel r} = E_{\parallel t}$$

$$E_i \cos \theta_i - E_r \cos \theta_r = E_t \cos \theta_t$$

$$\frac{E_i}{Z_1} + \frac{E_r}{Z_1} = \frac{E_t}{Z_2}$$

- Two equations in two unknowns...



**$\vec{H}$  perpendicular to  $\hat{n}$**

$$\frac{E_i \cos \theta_i - E_r \cos \theta_i}{Z_2} = \frac{E_t \cos \theta_t}{Z_2}$$
$$\frac{E_i \cos \theta_t + E_r \cos \theta_t}{Z_1} = \frac{E_t \cos \theta_t}{Z_2}$$

- Solve for  $E_r/E_i$ :

$$\frac{E_r}{E_i} = \frac{Z_1 \cos \theta_i - Z_2 \cos \theta_t}{Z_1 \cos \theta_i + Z_2 \cos \theta_t}$$

- Solve for  $E_t/E_i$ :

$$\frac{E_t}{E_i} = \frac{2Z_1 \cos \theta_i}{Z_1 \cos \theta_i + Z_2 \cos \theta_t}$$

# Fresnel's Equations

- In most dielectric media,  $\mu_1 = \mu_2$  and therefore

$$\frac{Z_1}{Z_2} = \sqrt{\frac{\mu_1 \epsilon_2}{\mu_2 \epsilon_1}} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{n_2}{n_1} = \frac{\sin \theta_i}{\sin \theta_t}$$

- After some trigonometry...

$$\left(\frac{E_r}{E_i}\right)_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \quad \left(\frac{E_r}{E_i}\right)_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$\left(\frac{E_t}{E_i}\right)_{\perp} = \frac{(n_1/n_2)\sin(2\theta_i)}{\sin(\theta_i + \theta_t)} \quad \left(\frac{E_t}{E_i}\right)_{\parallel} = \frac{2 \cos(\theta_i) \sin(\theta_t)}{\sin(\theta_i + \theta_t) \cos(\theta_i - \theta_t)}$$

For  $\vec{E}$  perpendicular and parallel to **plane of incidence**.  
(not the same as perpendicular or parallel to the surface)

# Normal Incidence

- At normal incidence,  $\cos \theta = 1$  there really is no component parallel to the plane of incidence.
- In this case

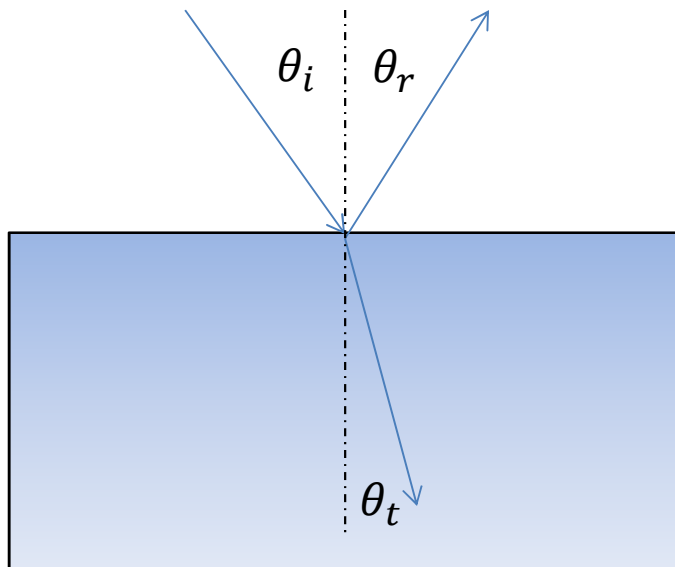
$$\left(\frac{E_r}{E_i}\right)_{\perp} = \frac{n_1 - n_2}{n_1 + n_2}$$
$$\left(\frac{E_t}{E_i}\right)_{\perp} = \frac{2n_2}{n_1 + n_2}$$

- This what we arrived at previously using only the difference in the speed of light in the two materials.



# Application of Fresnel's Equations

- Unpolarized light in air ( $n = 1$ ) is incident on a surface with index of refraction  $n' = 1.5$  at an angle  $\theta_i = 30^\circ$
- What are the magnitudes of the electric field components of the reflected light?



First calculate the angles of reflection and refraction...

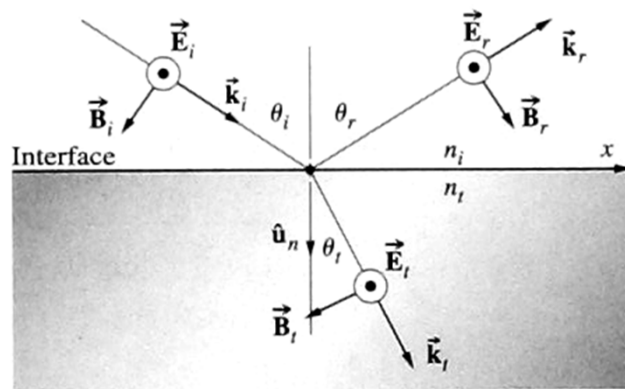
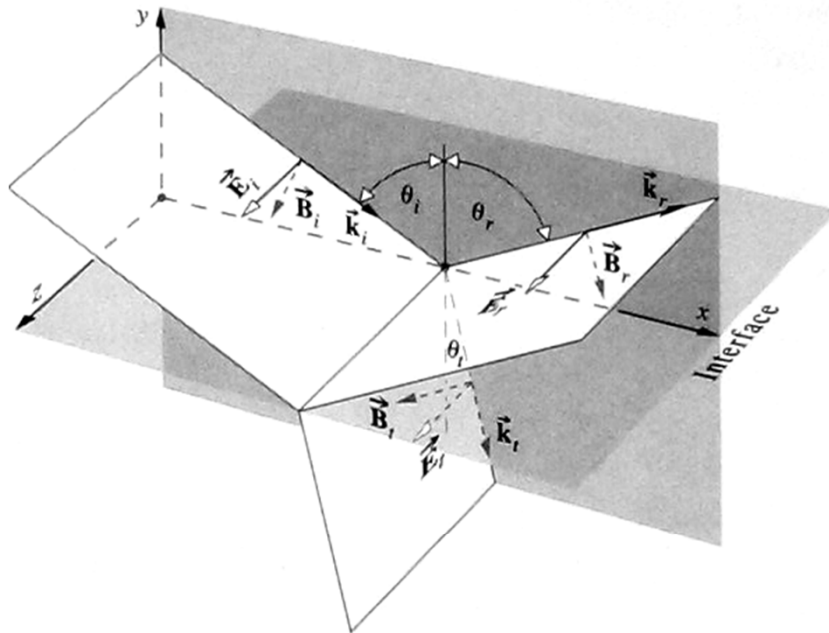
$$\theta_r = \theta_i = 30^\circ$$

$$\sin \theta_i = n' \sin \theta_t$$

$$\sin \theta_t = \frac{\sin 30^\circ}{1.5} = 0.333$$

$$\theta_t = 19.5^\circ$$

# Application of Fresnel's Equations



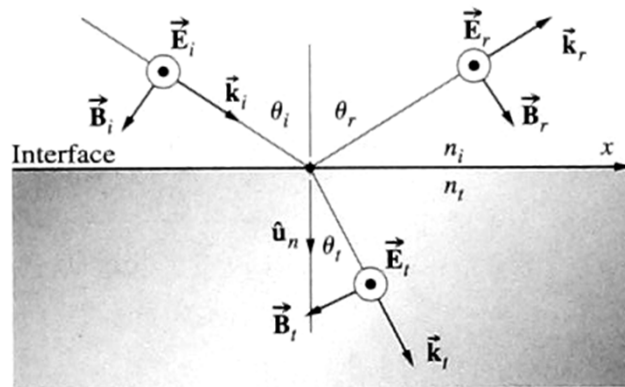
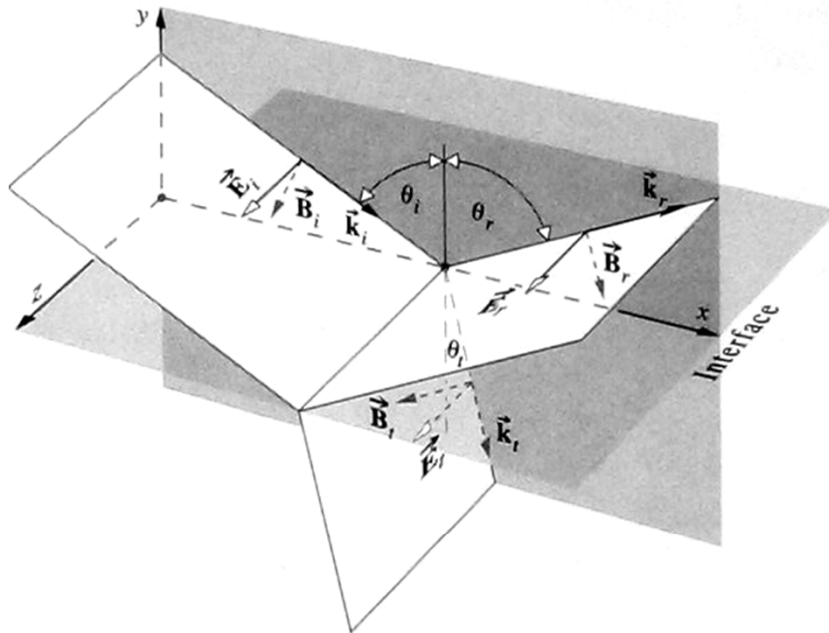
- Component of  $\vec{E}$  parallel to the surface is perpendicular to the plane of incidence

$$\left(\frac{E_r}{E_i}\right)_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$= -\frac{\sin(30^\circ - 19.5^\circ)}{\sin(30^\circ + 19.5^\circ)}$$

$$= -0.240$$

# Application of Fresnel's Equations



- Component of  $\vec{E}$  perpendicular to the surface is parallel to the plane of incidence

$$\left(\frac{E_r}{E_i}\right)_{\perp} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$= \frac{\tan(30^\circ - 19.5^\circ)}{\tan(30^\circ + 19.5^\circ)}$$

$$= 0.158$$

- Reflected light is preferentially polarized parallel to the surface.

# Reflected Components

- Since  $\theta_t < \theta_i$  the component perpendicular to the plane of incidence is always negative:

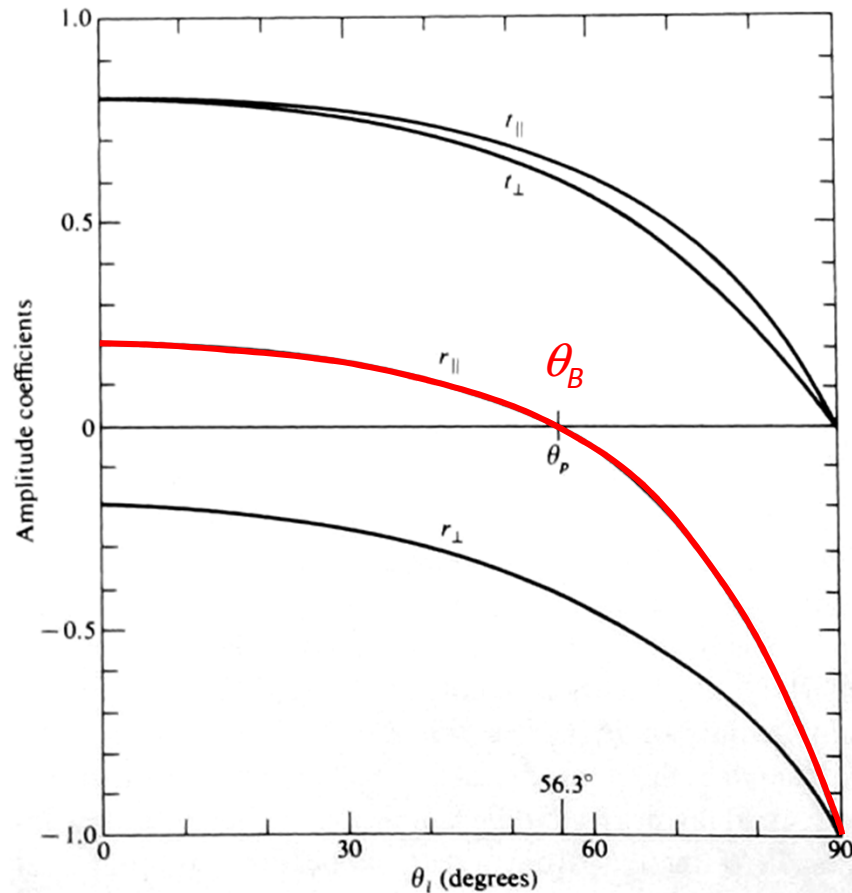
$$\left(\frac{E_r}{E_i}\right)_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

- The component parallel to the plane of incidence could be positive or negative:

$$\left(\frac{E_r}{E_i}\right)_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

- What happens when  $\theta_i + \theta_t = 90^\circ$ ?
  - Can this happen? Sure... check when  $\theta_i \rightarrow 90^\circ$ .

# Brewster's Angle



$\tan \theta \rightarrow \infty$  as  $\theta \rightarrow 90^\circ$   
while  $\tan(\theta_i - \theta_t)$   
remains finite.

Therefore,

$$\left( \frac{E_r}{E_i} \right)_{\parallel} \rightarrow 0$$

Brewster's angle,  $\theta_B$  is  
the angle of incidence  
for which  $r_{\parallel} \rightarrow 0$ .

# Brewster's Angle

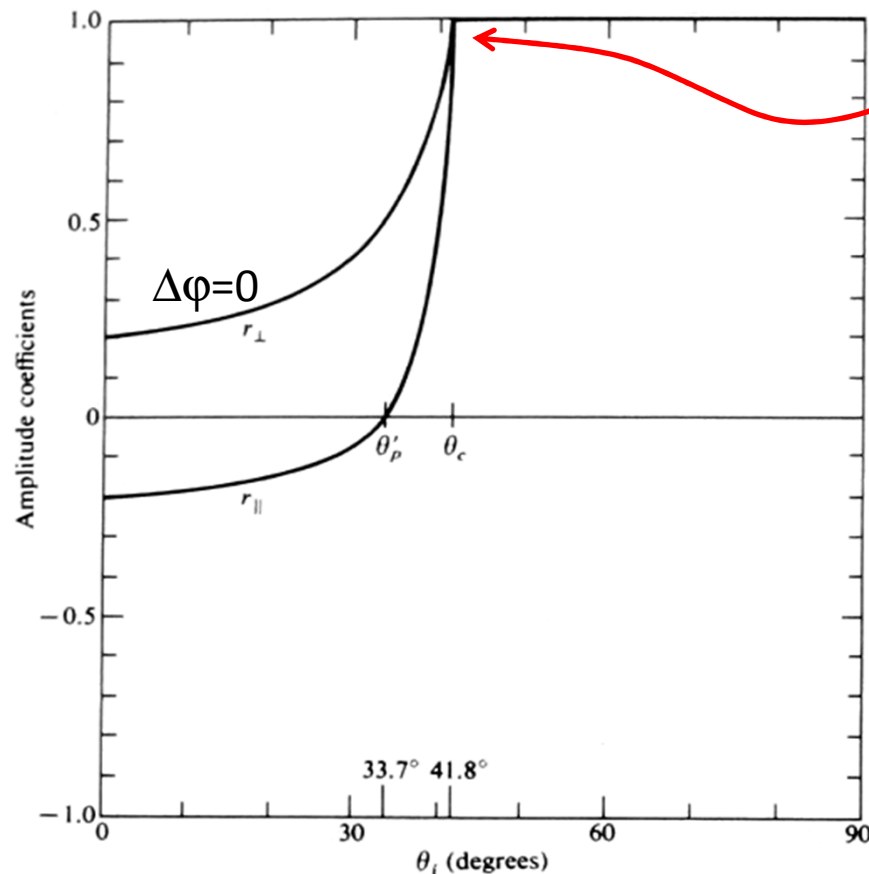
- Brewster's angle can be calculated from the relation  $\theta_i + \theta_t = 90^\circ$
- We can always calculate  $\theta_t$  using Snell's law
- Good assignment question:

$$\textit{Show that } \theta_B = \tan^{-1} \left( \frac{n_2}{n_1} \right)$$

- Light reflected from a surface at an angle  $\theta_B$  will be linearly polarized parallel to the surface (perpendicular to the plane of incidence)

# Total Internal Reflection

- Consider the other case when  $n_i > n_t$ , for example, glass to air:



At some incidence angle (critical angle  $\theta_c$ ) everything is reflected (and nothing transmitted).

It can be shown that for any angle larger than  $\theta_c$  no waves are transmitted into media: **total internal reflection.**

No phase shift upon reflection.

# Reflected Intensity

- Remember that the *intensity* (irradiance) is related to the energy carried by light:

$$I = \epsilon v \langle E^2 \rangle_T$$

(averaged over some time  $T \gg 1/f$ )

- Reflectance is defined as

$$R_{\perp} = (r_{\perp})^2 = \frac{\sin^2(\theta_i - \theta_t)}{\sin^2(\theta_i + \theta_t)}$$

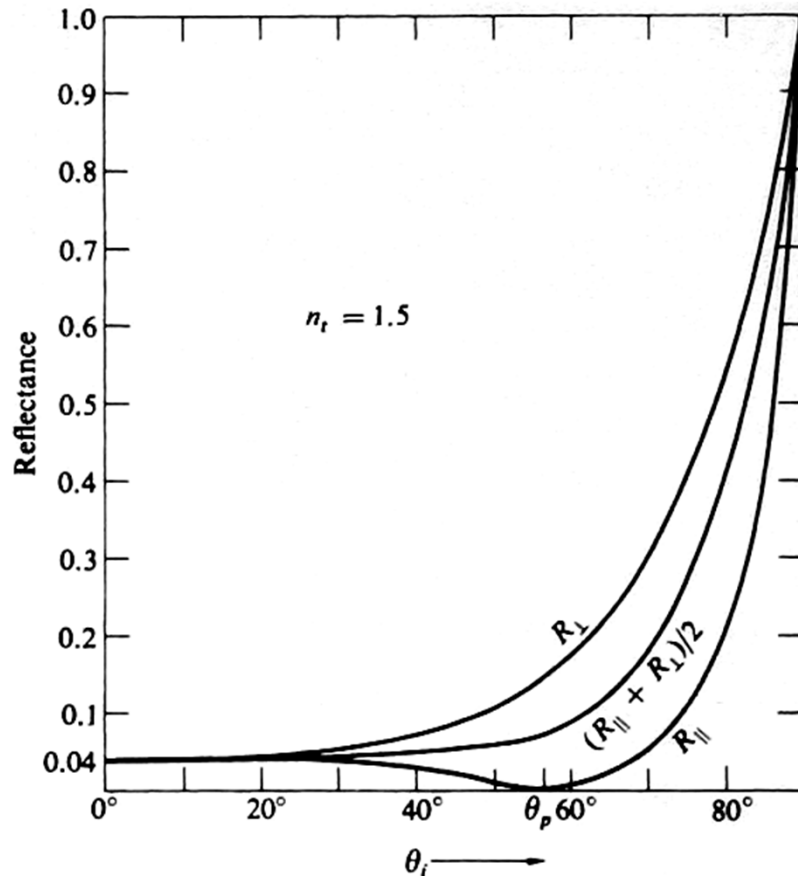
$$R_{\parallel} = (r_{\parallel})^2 = \frac{\tan^2(\theta_i - \theta_t)}{\tan^2(\theta_i + \theta_t)}$$

- Unpolarized reflectance:

$$R = \frac{1}{2} (R_{\perp} + R_{\parallel})$$



# Reflected Intensity



- How polarized is the reflected light?
- Degree of polarization:  

$$V = I_p / I_{total}$$
- Measured using an analyzing polarizer

