

Physics 42200  
**Waves & Oscillations**

Lecture 24 – Review

Spring 2015 Semester

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# Midterm Exam:

Date: Thursday, March 12<sup>th</sup>

Time: 8:00 – 10:00 pm

Room: PHYS 114

Material: French, chapters 1-8

You can bring one double sided page  
of notes, formulas, examples, etc.

# Review

1. Simple harmonic motion (one degree of freedom)
  - mass/spring, pendulum, floating objects, RLC circuits
  - damped harmonic motion
2. Forced harmonic oscillators
  - amplitude/phase of steady state oscillations
  - transient phenomena
3. Coupled harmonic oscillators
  - masses/springs, coupled pendula, RLC circuits
  - forced oscillations
4. Uniformly distributed discrete systems
  - masses on string fixed at both ends
  - lots of masses/springs

# Review

5. Continuously distributed systems (standing waves)
  - string fixed at both ends
  - sound waves in pipes (open end/closed end)
  - transmission lines
  - Fourier analysis
6. Progressive waves in continuous systems
  - reflection/transmission coefficients

# Simple Harmonic Motion

- Any system in which the force is opposite the displacement will oscillate about a point of stable equilibrium
- If the force is proportional to the displacement it will undergo simple harmonic motion
- Examples:
  - Mass/massless spring
  - Elastic rod (characterized by Young's modulus)
  - Floating objects
  - Torsion pendulum (shear modulus)
  - Simple pendulum
  - Physical pendulum
  - LC circuit

# Simple Harmonic Motion

- You should be able to draw a free-body diagram and express the force in terms of the displacement.
- Use Newton's law:  $m\ddot{x} = F$  or  $I\ddot{\theta} = N$
- Write it in standard form:

$$\ddot{x} + \omega_0^2 x = 0$$

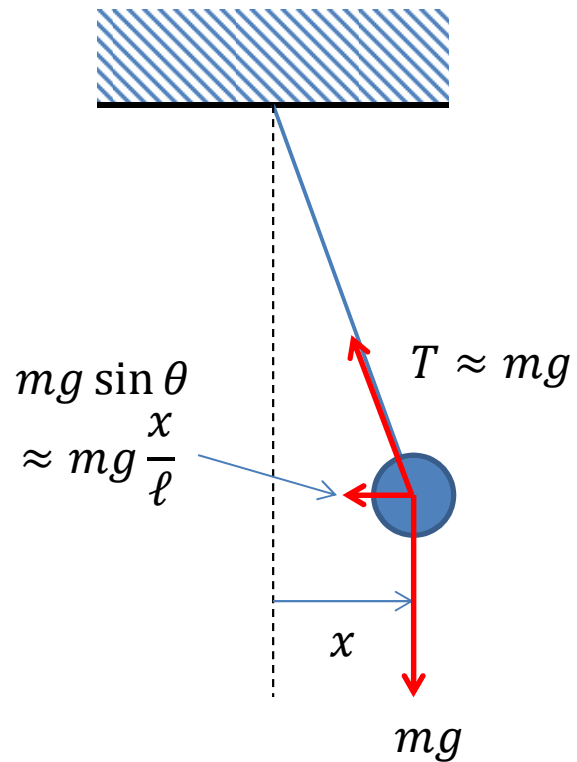
- Solutions are of the form:

$$x(t) = A \cos(\omega_0 t - \delta)$$

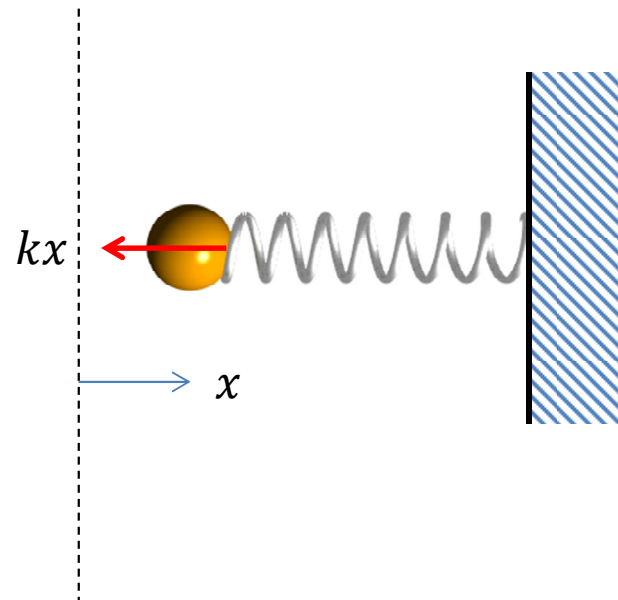
$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

- ***You must be able to use the initial conditions to solve for the constants of integration***

# Examples

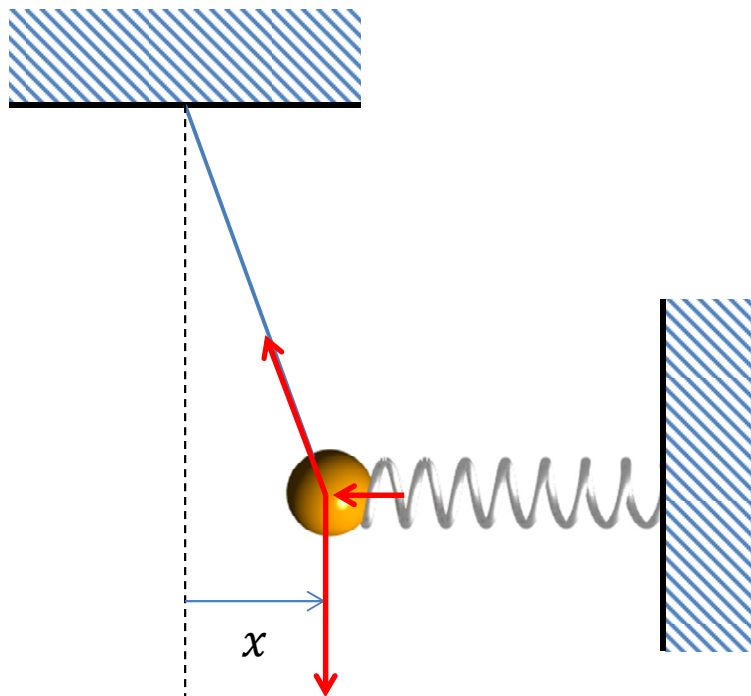


$$m\ddot{x} = -mgx/\ell$$



$$m\ddot{x} = -kx$$

# Examples



$$m\ddot{x} = ?$$



# Damped Harmonic Motion

- Damping forces remove energy from the system
- We will only consider cases where the force is proportional to the velocity:  $F = -bv$
- You should be able to construct a free-body diagram and write the resulting equation of motion:

$$m\ddot{x} + b\dot{x} + kx = 0$$

- You should be able to write it in the standard form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

- ***You must be able to solve this differential equation!***

# Damped Harmonic Motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

$$\text{Let } x(t) = Ae^{\alpha t}$$

- Characteristic polynomial:

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

- Roots (use the quadratic formula):

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

- Classification of solutions:

- Over-damped:  $\gamma^2/4 - (\omega_0)^2 > 0$  (distinct real roots)
- Critically damped:  $\gamma^2/4 = (\omega_0)^2$  (one root)
- Under-damped:  $\gamma^2/4 - (\omega_0)^2 < 0$  (complex roots)

# Damped Harmonic Motion

- Over-damped motion:  $\gamma^2/4 - (\omega_0)^2 > 0$

$$x(t) = Ae^{-\frac{\gamma}{2}t} e^{t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}} + Be^{-\frac{\gamma}{2}t} e^{-t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}}$$

- Under-damped motion:  $\gamma^2/4 - (\omega_0)^2 < 0$

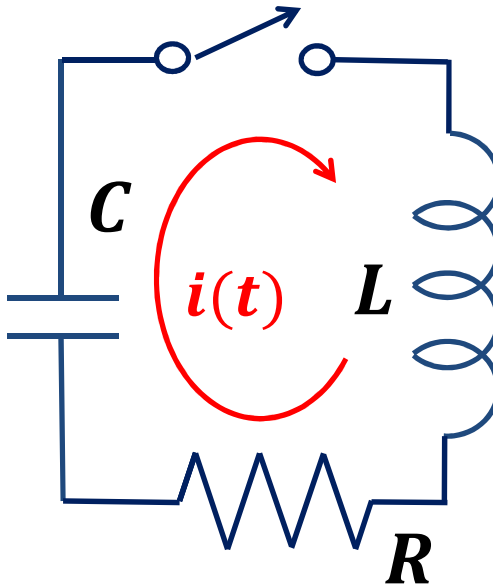
$$x(t) = Ae^{-\frac{\gamma}{2}t} e^{it\sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}} + Be^{-\frac{\gamma}{2}t} e^{-it\sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}}$$

- Critically damped motion:

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

- ***You must be able to use the initial conditions to solve for the constants of integration***

# Example



Sum of potential differences:

$$-L \frac{di}{dt} - i(t)R - \frac{1}{C} \left( Q_0 + \int_0^t i(t) dt \right) = 0$$

Initial charge,  $Q_0$ , defines the initial conditions.

# Example

$$L \frac{di}{dt} + i(t)R + \frac{1}{C} \left( Q_0 + \int_0^t i(t) dt \right) = 0$$

Differentiate once with respect to time:

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0$$

$$\frac{d^2 i}{dt^2} + \gamma \frac{di}{dt} + \omega_0^2 i(t) = 0$$

***Remember, the solution is  $i(t)$  but the initial conditions might be in terms of  $Q(t) = Q_0 + \int i(t) dt$***

(See examples from the lecture notes...)

# Forced Harmonic Motion

- Now the differential equation is

$$m\ddot{x} + b\dot{x} + kx = F(\omega) = F_0 \cos \omega t$$

- Driving function is not always given in terms of a real force... (think about non-inertial reference frames):

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = -\frac{d^2\eta}{dt^2} = C\omega^2 \cos \omega t$$

- General properties:
  - Steady state properties:  $t \gg 1/\gamma$
  - Solution is  $y(t) = A \cos(\omega t - \delta)$
  - Amplitude,  $A$ , and phase,  $\delta$ , depend on  $\omega$

# Forced Harmonic Motion

“Q” quantifies the amount of damping:

$$Q = \frac{\omega_0}{\gamma}$$

(large Q means small damping force)

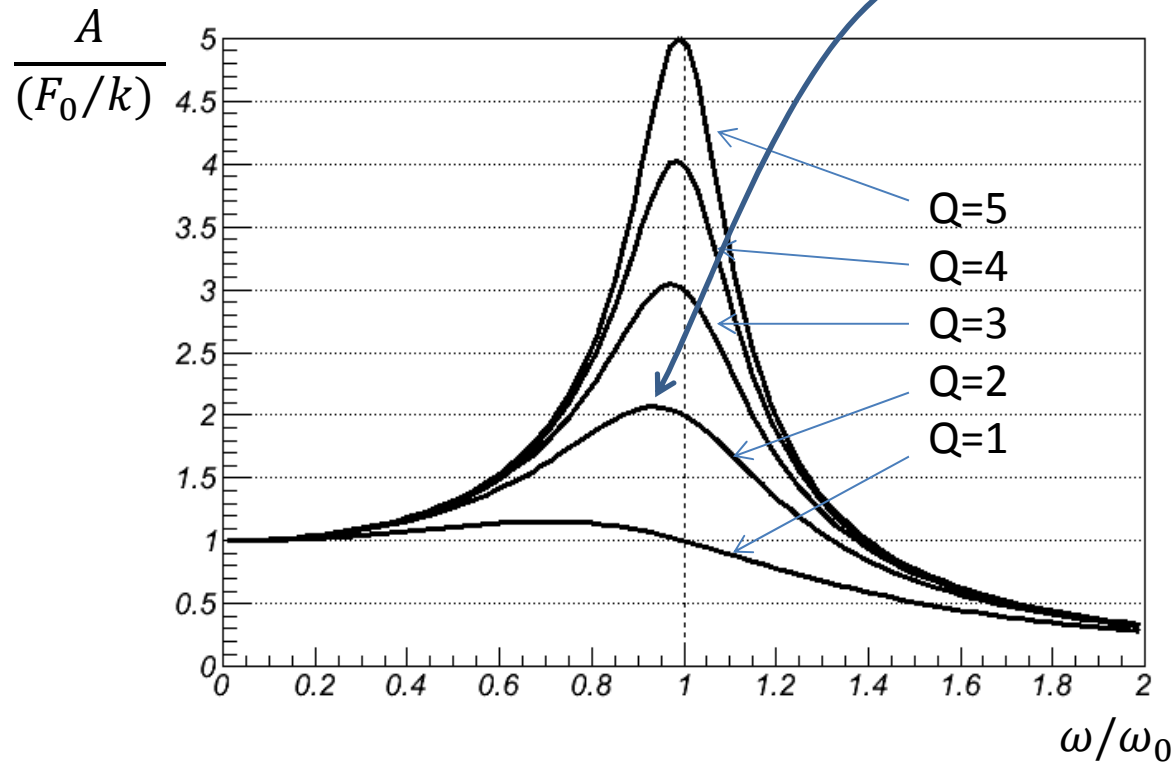
$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[ \left( \frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \right)^2 + \frac{1}{Q^2} \right]^{1/2}}$$

$$\delta = \tan^{-1} \left( \frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}} \right)$$

But watch out when  $F_0 = C\omega^2$

# Resonance

- Qualitative features: amplitude



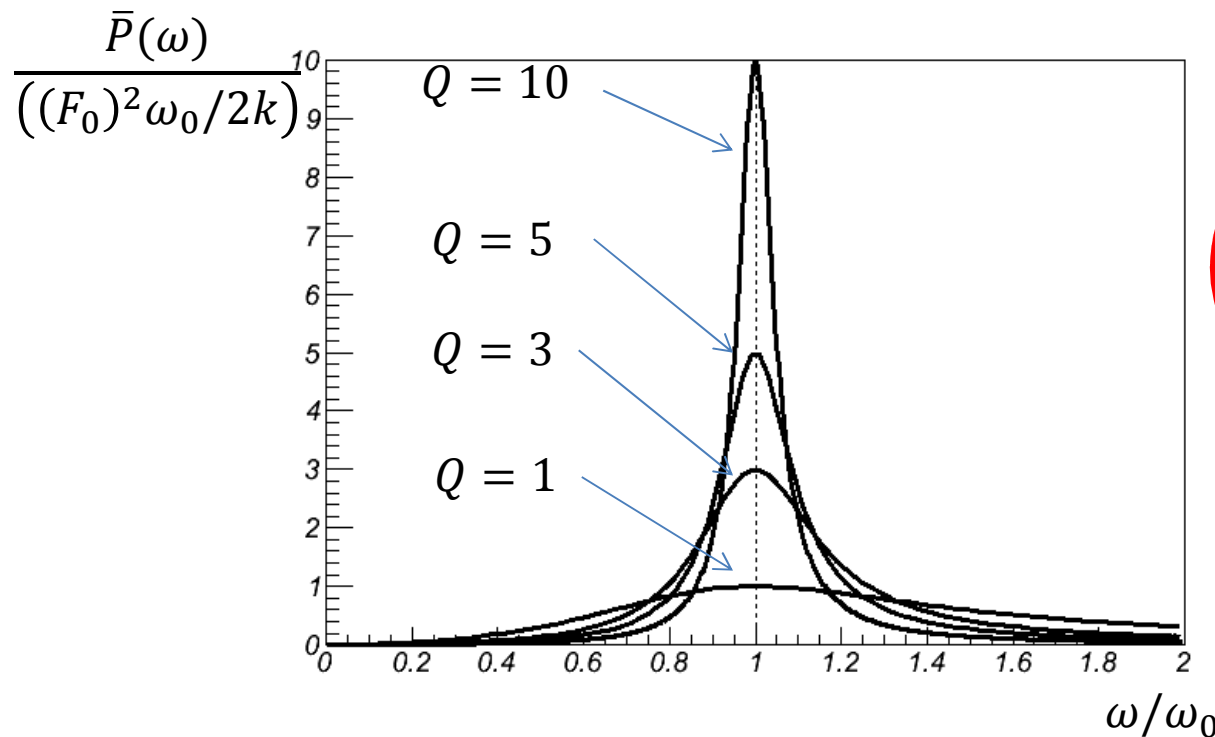
$$\frac{\omega_{free}}{\omega_0} = 1$$
$$\omega_{free} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$



# Average Power

- The rate at which the oscillator absorbs energy is:

$$\bar{P}(\omega) = \frac{(F_0)^2 \omega_0}{2kQ} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$



Full-Width-at-Half-Max:

$$FWHM = \frac{\omega_0}{Q} = \gamma$$

# Resonance

- Qualitative features: phase shift

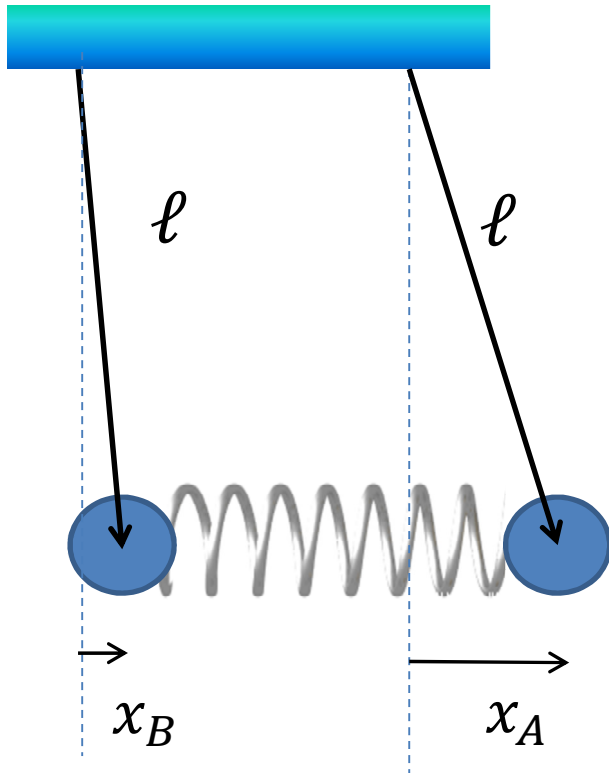
$$\delta = \tan^{-1} \left( \frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}} \right)$$

$\delta \rightarrow 0$  at low frequencies

$\delta \rightarrow \pi$  at high frequencies

$$\delta = \frac{\pi}{2} \text{ when } \omega = \omega_0$$

# Coupled Oscillators



- Restoring force on pendulum A:

$$F_A = -k(x_A - x_B)$$

- Restoring force on pendulum B:

$$F_B = k(x_A - x_B)$$

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$
$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

# Coupled Oscillators

- You must be able to draw the free-body diagram and set up the system of equations.*

$$\begin{aligned} m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) &= 0 \\ m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) &= 0 \end{aligned}$$

- You must be able to write this system as a matrix equation.*

$$\begin{pmatrix} \ddot{x}_A \\ \ddot{x}_B \end{pmatrix} + \begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = 0$$

# Coupled Oscillators

- Assume solutions are of the form

$$\begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \cos(\omega t - \delta)$$

- Then,

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

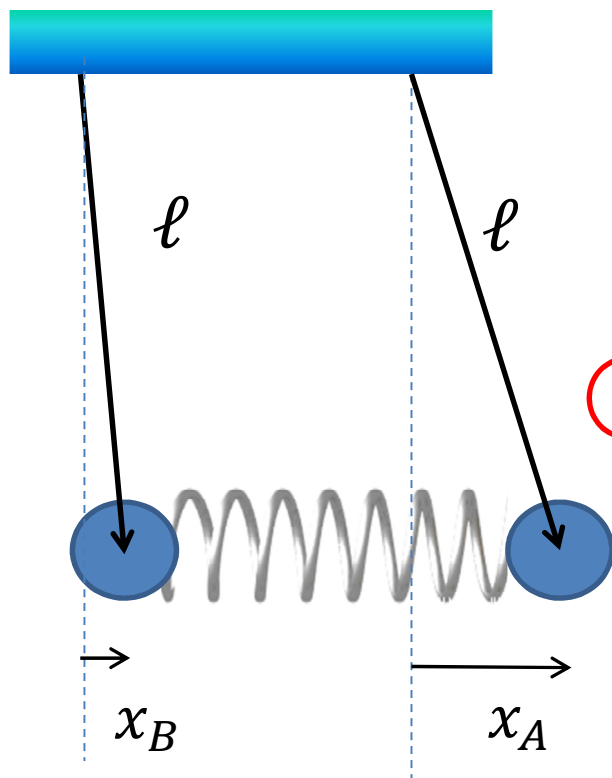
- ***You must be able to calculate the eigenvalues of a 2x2 or 3x3 matrix.***
  - ***Calculate the determinant***
  - ***Calculate the roots by factoring the determinant or using the quadratic formula.***
- These are the frequencies of the normal modes of oscillation.

# Coupled Oscillators

- *You must be able to calculate the eigenvectors of a 2x2 or 3x3 matrix*
- General solution:  
$$\vec{x}(t) = \mathbf{A}\vec{x}_1 \cos(\omega_1 t - \alpha) + \mathbf{B}\vec{x}_2 \cos(\omega_2 t - \beta) + \dots$$
- *You must be able to solve for the constants of integration using the initial conditions.*

# Coupled Discrete Systems

- The general method of calculating eigenvalues will always work, but for simple systems you should be able to decouple the equations by a change of variables.*



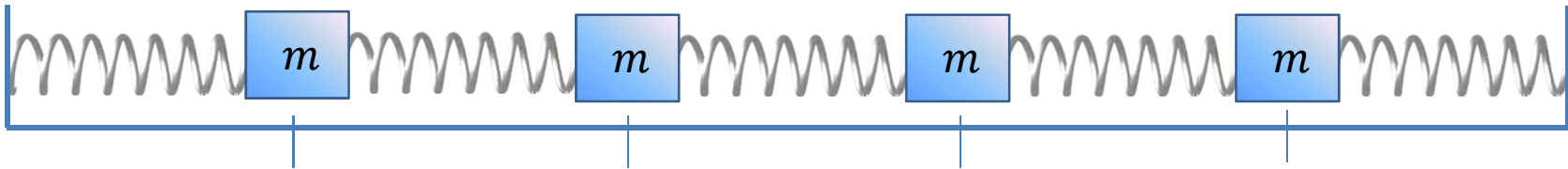
$$\begin{aligned}
 & \textcircled{1} \left\{ \begin{aligned} m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) &= 0 \\ m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) &= 0 \end{aligned} \right. \\
 & \textcircled{2} \left\{ \begin{aligned} \ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2x_B &= 0 \\ \ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2x_A &= 0 \end{aligned} \right. \\
 & \qquad \qquad \omega_0 = \sqrt{g/\ell}, \quad \omega_c = \sqrt{k/m} \\
 & \textcircled{3} \left\{ \begin{aligned} q_1 &= x_A + x_B \\ q_2 &= x_A - x_B \end{aligned} \right. \\
 & \textcircled{4} \left\{ \begin{aligned} \ddot{q}_1 + (\omega_0)^2q_1 &= 0 \\ \ddot{q}_2 + (\omega')^2q_2 &= 0 \end{aligned} \right.
 \end{aligned}$$

# Forced Oscillations

- We mainly considered the qualitative aspects
  - We did not analyze the behavior when damping forces were significant
- Main features:
  - Resonance occurs at each normal mode frequency
  - Phase difference is  $\delta = \pi/2$  at resonance
- Example:  $x_A$  driven by the force  $F(\omega) = F_0 \cos \omega t$ 
  - Calculate force term applied to normal coordinates
$$F_1(\omega) = F_2(\omega) = F_0 \cos \omega t$$
  - Reduced to two one-dimensional forced oscillators:
$$\ddot{q}_1 + (\omega_0)^2 q_1 = F_0/m \cos \omega t$$
$$\ddot{q}_2 + (\omega')^2 q_2 = F_0/m \cos \omega t$$

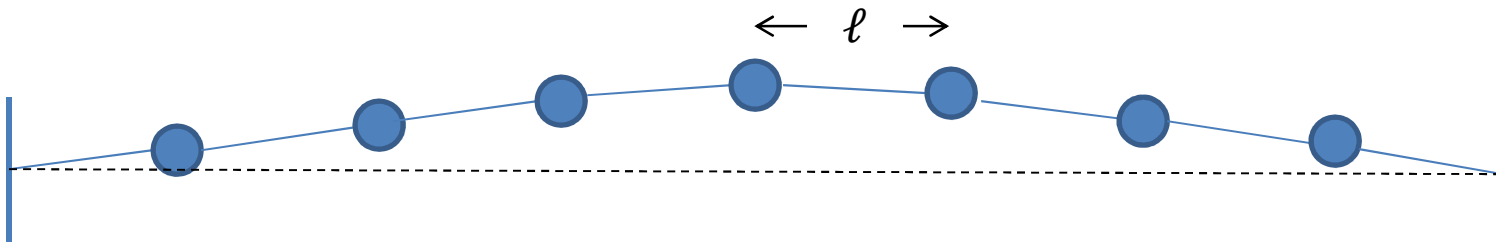


# Uniformly Distributed Discrete Systems



Equations of motion for masses in the middle:

$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0$$
$$(\omega_0)^2 = k/m$$



$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$
$$(\omega_0)^2 = T/m\ell$$

# Uniformly Distributed Discrete Masses

- Proposed solution:

$$x_n(t) = A_n \cos \omega t$$

$$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- We solved this to determine  $A_n$  and  $\omega_k$ :

$$A_{n,k} = C \sin \left( \frac{nk\pi}{N+1} \right)$$

$$\omega_k = 2\omega_0 \sin \left( \frac{k\pi}{2(N+1)} \right)$$

Amplitude of mass  $n$   
oscillating in normal  
mode  $k$

Frequency of normal  
mode  $k$

- General solution:

$$x_n(t) = \sum_{k=1}^N a_k \sin \left( \frac{nk\pi}{N+1} \right) \cos(\omega_k t - \delta_k)$$

# Vibrations of Continuous Systems

- General solution for mass  $n$ :

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

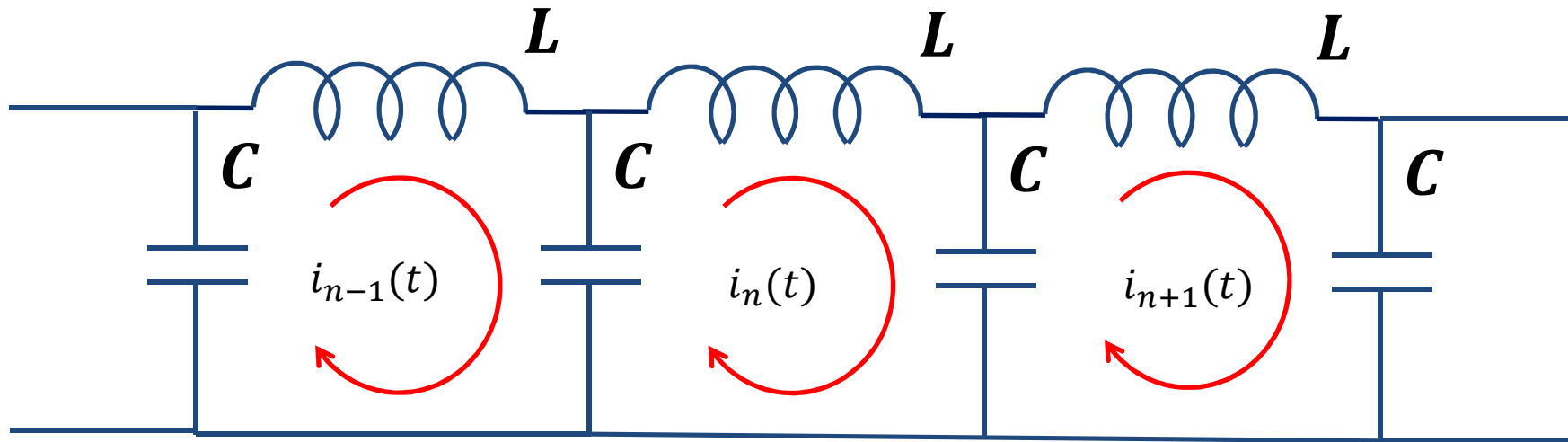
- Orthogonality relation:

$$\sum_{k=1}^N \sin\left(\frac{mk\pi}{N+1}\right) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} \delta_{mn}$$

- Solution to initial value problem:

$$\sum_{n=1}^N x_n(0) \sin\left(\frac{nk\pi}{N+1}\right) = \frac{N}{2} a_k \cos \delta_k$$

# Lumped LC Circuit



$$-L \frac{di_n}{dt} - \frac{1}{C} \int (i_n - i_{n+1}) dt - \frac{1}{C} \int (i_n - i_{n-1}) dt = 0$$

$$\frac{d^2 i_n}{dt^2} + 2\omega_0^2 i_n - \omega_0^2 (i_{n-1} + i_{n+1}) = 0$$

*This is the exact same problem as the previous two examples.*

# Forced Coupled Oscillators

- Qualitative features are the same:
  - Motion can be decoupled into a set of  $N$  independent oscillator equations (normal modes)
  - Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
  - Phase difference approaches  $\pi/2$  at resonance
- *You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.*

# Continuous Distributions

Limit as  $N \rightarrow \infty$  and  $m/\ell \rightarrow \mu$ :

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Boundary conditions specified at  $x = 0$  and  $x = L$ :

- Fixed ends:  $y(0) = y(L) = 0$
- Maximal motion at ends:  $\dot{y}(0) = \dot{y}(L) = 0$
- Mixed boundary conditions





Normal modes will be of the form

$$y_n(x, t) = a_n \sin(k_n x) \cos(\omega_n t - \alpha_n)$$

or 
$$y_n(x, t) = a_n \cos(k_n x) \cos(\omega_n t - \alpha_n)$$

# Properties of the Solutions

$$y(L, t) \sim \sin k_n L = 0 \quad \Rightarrow \quad k_n L = n\pi$$

	mode	wavelength	frequency
	first	$2L$	$\frac{v}{2L}$
	second	$L$	$\frac{v}{L}$
	third	$\frac{2L}{3}$	$\frac{3v}{2L}$
	fourth	$\frac{L}{2}$	$\frac{2v}{L}$

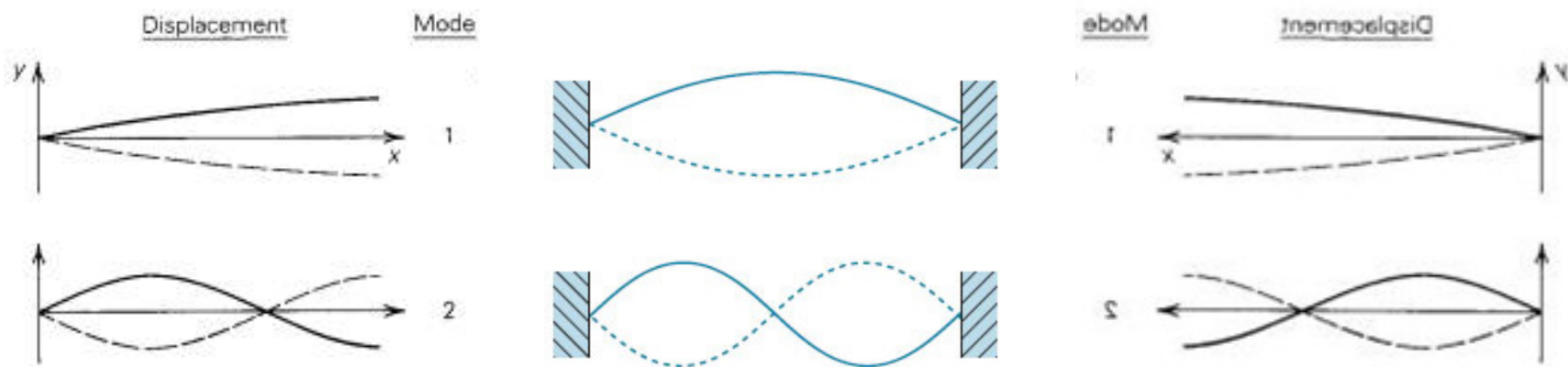
$$\lambda_n = \frac{2L}{n}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

# Boundary Conditions

- Examples:
  - String fixed at both ends:  $y(0) = y(L) = 0$
  - Organ pipe open at one end:  $\dot{y}(0) = \dot{y}(L) = 0$ 
    - Driving end has maximal pressure amplitude
  - Organ pipe closed at one end:  $\dot{y}(0) = 0, y(L) = 0$
  - Transmission line open at one end:  $i(L) = 0$
  - Transmission line shorted at one end:  $v(L) \propto \frac{di(L)}{dt} = 0$





# Fourier Analysis

- Normal modes satisfying  $y(0) = y(L) = 0$ :

$$y_n(x, t) = a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- General solution:

$$y(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- Initial conditions:

$$y(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\alpha_n) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}(x, 0) = - \sum_{n=1}^{\infty} a_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \sin(\alpha_n) = \sum_{n=1}^{\infty} b'_n \sin\left(\frac{n\pi x}{L}\right)$$

# Fourier Analysis

- Fourier sine transform:

$$u(x) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$
$$a'_n = \frac{2}{L} \int_0^L u(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Fourier cosine transform:

$$b'_n = \frac{2}{L} \int_0^L v(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

# Fourier Analysis

$$a'_n = a_n \cos \alpha_n$$
$$b'_n = a_n \omega_n \sin \alpha_n$$

Solve for amplitudes:

$$a_n = \sqrt{a'^2_n + \frac{b'^2_n}{\omega_n^2}}$$

Solve for phase:

$$\tan \alpha_n = \frac{b'_n}{a'_n \omega_n}$$

# Fourier Analysis

- *Suggestion: don't simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.*

- Example:

- If you are given  $\dot{y}(x, 0) = 0$  and  $y(0) = y(L) = 0$  then you know that solutions are of the form

$$y(x, t) = \sum a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

- If you are given  $y(x, 0) = 0$  and  $\dot{y}(0) = \dot{y}(L) = 0$  then solutions are of the form

$$y(x, t) = \sum_{\text{odd } n} a_n \sin\left(\frac{n\pi x}{L}\right) \sin \omega_n t$$

# Progressive Waves

- Far from the boundaries, other descriptions are more transparent:

$$y(x, t) = f(x \pm vt)$$

- The Fourier transform gives the frequency components:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx) dx$$

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx) dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx) dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx) dx \end{aligned}$$

- Narrow pulse in space  $\rightarrow$  wide range of frequencies
- Pulse spread out in space  $\rightarrow$  narrow range of frequencies

# Properties of Progressive Waves

- Power carried by a wave:
  - String with tension  $T$  and mass per unit length  $\mu$

$$P = \frac{1}{2} \mu \omega^2 A^2 v = \frac{1}{2} Z \omega^2 A^2$$

- Impedance of the medium:

$$Z = \mu v = T/v$$

- Important properties:
  - *Impedance is a property of the medium, not the wave*
  - *Energy and power are proportional to the square of the amplitude*

# Reflections

- Wave energy is reflected by discontinuities in the impedance of a system
- Reflection and transmission coefficients:
  - The wave is incident and reflected in medium 1
  - The wave is transmitted into medium 2

$$\rho = \frac{Z_2 - Z_1}{Z_1 + Z_2}$$
$$\tau = \frac{2Z_2}{Z_1 + Z_2}$$

Important: when is  
this negative?

Always positive

- Wave amplitudes:

$$A_r = \rho A_i$$

$$A_t = \tau A_i$$

# Reflected and Transmitted Power

- Power is proportional to the square of the amplitude.
  - Reflected power:  $P_r = \rho^2 P_i$
  - Transmitted power:  $P_t = \tau^2 P_i$
- ***You should be able to demonstrate that energy is conserved:***  
***ie, show that  $P_i = P_r + P_t$***



# That's all for now...

- Study these topics – make sure you understand the examples and assignment questions.
- Next topics: ***waves applied to optics.***