

Physics 42200

Waves & Oscillations

Lecture 22 – French, Chapter 8

Spring 2015 Semester

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Midterm Exam:

Date: Thursday, March 12th

Time: 8:00 – 10:00 pm

Room: PHYS 112

Material: French, chapters 1-8

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

- The solution, $y(x, t)$, describes the shape of a string as a function of x and t .
- This is a transverse wave: the displacement is perpendicular to the direction of propagation.
- This would confuse the following discussion...
- Instead, let's now consider longitudinal waves, like the pressure waves due to the propagation of sound in a gas.

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- The solution, $p(x, t)$, describes the excess pressure in the gas as a function of x and t .
- What if the wave was propagating in the y -direction?

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- What if the wave was propagating in the z -direction?

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

Waves in Three Dimensions

- The excess pressure is now a function of \vec{x} and t .
- Wave equation in three dimensions:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- But we like to write it this way:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- Where ∇^2 is called the “Laplacian operator”, but you just need to think of it as a bunch of derivatives:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Waves in Three Dimensions

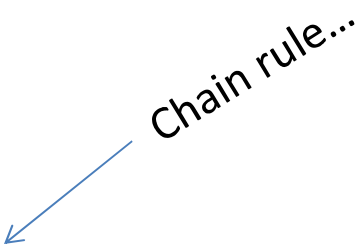
- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- How do we solve this? Here's how...

$$p(\vec{x}, t) = p_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

- One partial derivatives:

$$\begin{aligned} \frac{\partial p}{\partial x} &= ip_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \frac{\partial}{\partial x} (\vec{k} \cdot \vec{x} - \omega t) \\ &= ip_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \frac{\partial}{\partial x} (k_x x + k_y y + k_z z - \omega t) \\ &= ik_x p(\vec{x}, t) \end{aligned}$$


- Second derivative:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2 p(\vec{x}, t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

- Second derivatives:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2 p(\vec{x}, t)$$

$$\frac{\partial^2 p}{\partial y^2} = -k_y^2 p(\vec{x}, t)$$

$$\frac{\partial^2 p}{\partial z^2} = -k_z^2 p(\vec{x}, t)$$

$$\frac{\partial^2 p}{\partial t^2} = -\omega^2 p(\vec{x}, t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2} \frac{\partial^2 p}{\partial t^2}$$

$$-(k_x^2 + k_y^2 + k_z^2)p(\vec{x}, t) = -\frac{\omega^2}{v^2}p(\vec{x}, t)$$

- Any values of k_x, k_y, k_z satisfy the equation, provided that

$$\omega = v \sqrt{k_x^2 + k_y^2 + k_z^2} = v|\vec{k}|$$

- If $k_y = k_z = 0$ then $p(\vec{x}, t) = p_0 e^{i(k_x x - \omega t)}$ but this describes a wave propagating in the $+x$ direction.

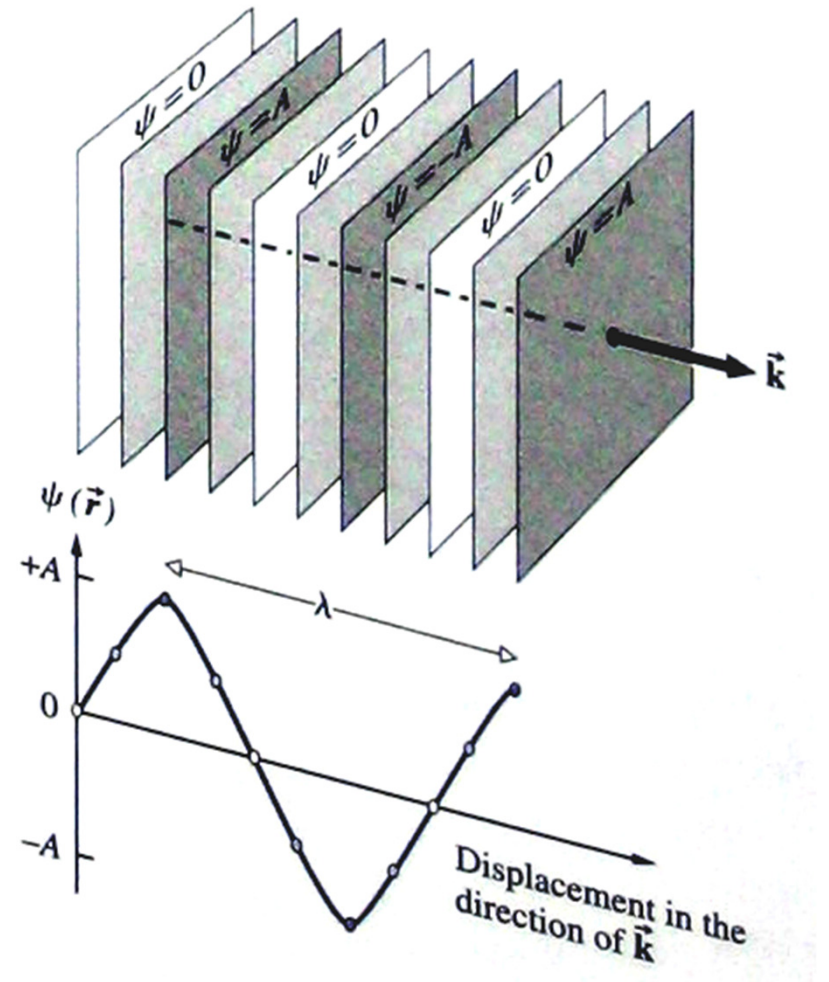
Waves in Three Dimensions

$$p(\vec{x}, t) = p_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

- The vector, \vec{k} , points in the direction of propagation
- The wavelength is $\lambda = 2\pi/|\vec{k}|$
- How do we visualize this solution?
 - Pressure is equal at all points \vec{x} such that $\vec{k} \cdot \vec{x} - \omega t = \phi$ where ϕ is some constant phase.
 - Let \vec{x}' be some other point such that $\vec{k} \cdot \vec{x}' - \omega t = \phi$
 - We can write $\vec{x}' = \vec{x} + \vec{u}$ and this tells us that $\vec{k} \cdot \vec{u} = 0$.
 - \vec{k} and \vec{u} are perpendicular.
 - *All points in the plane perpendicular to \vec{k} have the same phase.*

Waves in Three Dimensions

- As usual, we are mainly interested in the real component:
$$\psi(\vec{r}, t) = A \cos(\vec{k} \cdot \vec{x} - \omega t)$$
- A wave propagating in the opposite direction would be described by
$$\psi'(\vec{r}, t) = A' \cos(\vec{k} \cdot \vec{x} + \omega t)$$
- The points in a plane with a common phase is called the “wavefront”.



Waves in Three Dimensions

$$\psi(\vec{r}, t) = A \cos(\vec{k} \cdot \vec{x} \mp \omega t)$$

- Sometimes we are free to pick a coordinate system in which to describe the wave motion.
- If we choose the x -axis to be in the direction of propagation, we get back the one-dimensional solution we are familiar with:

$$\psi(\vec{r}, t) = A \cos(kx \mp \omega t)$$

- But in one-dimension we saw that any function that satisfied $f(x \pm vt)$ was a solution to the wave equation.
- What is the corresponding function in three dimensions?

Waves in Three Dimensions

$$\omega = v \sqrt{k_x^2 + k_y^2 + k_z^2} = v |\vec{k}|$$

- General solution to the wave equation are functions that are twice-differentiable of the form:

$$\psi(\vec{r}, t) = C_1 f(\hat{k} \cdot \vec{r} - vt) + C_2 g(\hat{k} \cdot \vec{r} + vt)$$

$$\text{where } \hat{k} = \vec{k} / |\vec{k}|$$

- Just like in the one-dimensional case, these do not have to be harmonic functions.

Example

- Is the function $\psi(\vec{x}, t) = (ax + bt + c)^2$ a solution to the wave equation?

- It should be because we can write it as

$$\psi(\vec{x}, t) = (a(\mathbf{x} + \mathbf{v}t) + c)^2$$

where $v = b/a$ which is of the form $g(x + vt)$

- We can check explicitly:

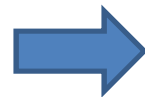
$$\frac{\partial \psi}{\partial x} = 2a(ax + bt + c)$$

$$\frac{\partial \psi}{\partial x} = 2b(ax + bt + c)$$

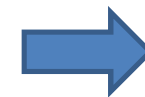
$$\frac{\partial^2 \psi}{\partial x^2} = 2a^2$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2b^2$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$



$$2a^2 = 2b^2/v^2$$



$$v = b/a$$

Example

- Is the function $\psi(\vec{x}, t) = ax^{-2} + bt$, where $a > 0, b > 0$, a solution to the wave equation?
- It is twice differentiable...

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{6a}{x^4} \qquad \frac{\partial^2 \psi}{\partial t^2} = 0$$

- But it is not a solution:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad \Rightarrow \quad \frac{6a}{x^4} = 0$$

- Only true if $a = 0$, which we already said was not the case.
- This is not a solution to the wave equation.

Waves in Two Dimensions

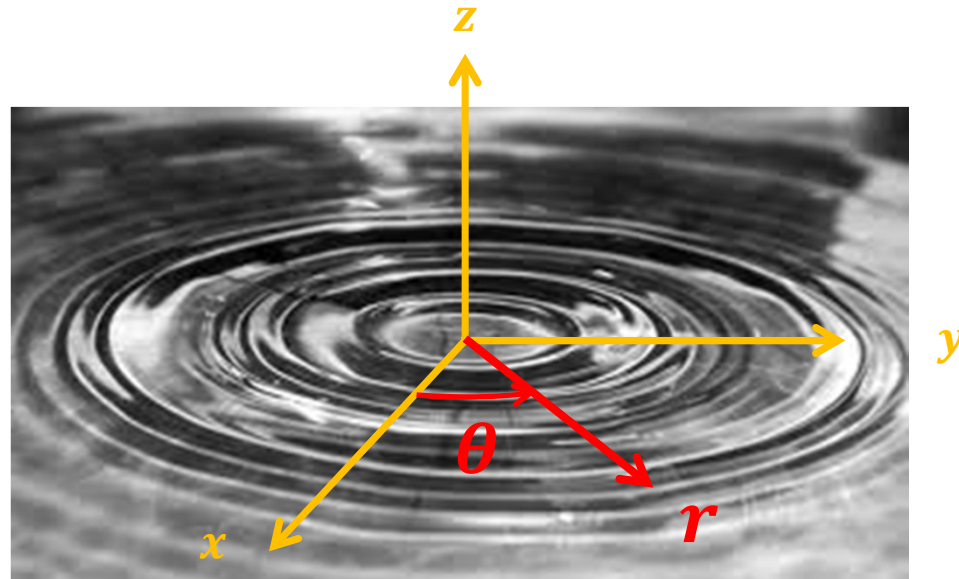
- Plane waves frequently provide a good description of physical phenomena, but this is usually an approximation:



- This looks like a wave... can the wave equation describe this?

Waves in Two Dimensions

- Rotational symmetry:
 - Cartesian coordinates are not well suited for describing this problem.
 - Use polar coordinates instead.
 - Motion should depend on r but should be independent of θ



Waves in Two Dimensions

- Wave equation: $\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$
- How do we write ∇^2 in polar coordinates?

$$\left. \begin{aligned} r &= \sqrt{x^2 + y^2} \\ x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \right\} \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

- Derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

Waves in Two Dimensions

$$\begin{aligned}
 &= \cos \theta \left[\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial r} \right] + \sin \theta \left[\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial r} \right] \\
 &= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2}
 \end{aligned}$$

Similarly,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial u}{\partial x} + r \cos \theta \frac{\partial u}{\partial y}.$$

Taking one more derivative, we see

$$\begin{aligned}
 \frac{\partial^2 u}{\partial \theta^2} &= r \frac{\partial}{\partial \theta} \left[-\sin \theta \frac{\partial u}{\partial x} + \cos \theta \frac{\partial u}{\partial y} \right] \\
 &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r \left[-\sin \theta \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \theta} \right) + \cos \theta \left(\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \theta} \right) \right] \\
 &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r \left[\sin^2 \theta \frac{\partial^2 u}{\partial x^2} + r \cos \theta \frac{\partial^2 u}{\partial x \partial y} \right] + \cos \theta \left(-r \sin \theta \frac{\partial^2 u}{\partial x \partial y} + r \cos \theta \frac{\partial^2 u}{\partial y^2} \right) \\
 &= -r \cos \theta \frac{\partial u}{\partial x} - r \sin \theta \frac{\partial u}{\partial y} + r^2 \sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \\
 &= -r \frac{\partial u}{\partial r} + r^2 \left[\sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \right]
 \end{aligned}$$

Now we're ready to put everything together:

$$\begin{aligned}
 \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= \cos^2 \theta \frac{\partial^2 u}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u}{\partial y^2} - \frac{1}{r} \frac{\partial u}{\partial r} + \sin^2 \theta \frac{\partial^2 u}{\partial x^2} - 2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 u}{\partial y^2} \\
 &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 u}{\partial x^2} + (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 u}{\partial y^2} - \frac{1}{r} \frac{\partial u}{\partial r}
 \end{aligned}$$

Waves in Two Dimensions

- Laplacian in polar coordinates:

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

- When the geometry does not depend on θ or z :

$$\begin{aligned} \nabla^2 \psi &= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) \end{aligned}$$

- Wave equation:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

Waves in Two Dimensions

- Wave equation:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

- If we assume that $\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$ then the equation is:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\omega^2}{v^2} \psi = 0$$

- Change of variables: Let $\rho = r\omega/v$

$$\frac{\omega^2}{v^2} \frac{\partial^2 \psi}{\partial \rho^2} + \frac{\omega^2}{v^2} \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\omega^2}{v^2} \psi = 0$$

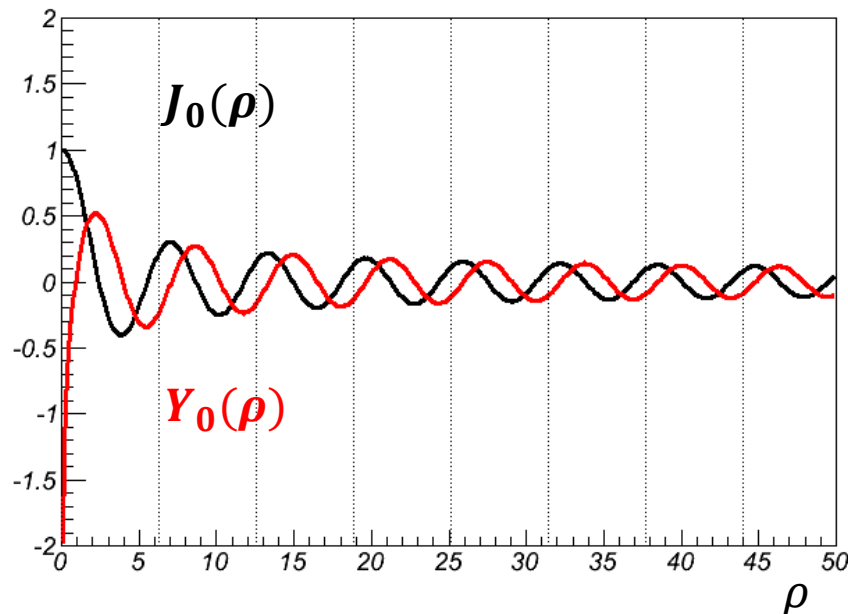
$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \psi(\rho) = 0$$

Waves in Two Dimensions

- Bessel's Equation:

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \psi(\rho) = 0$$

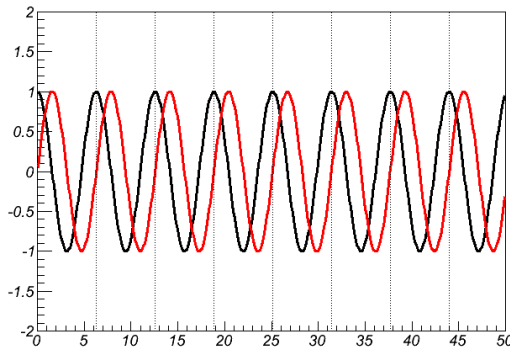
- Solutions are “Bessel functions”: $J_0(\rho)$, $Y_0(\rho)$



Bessel Functions?

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega^2}{v^2} \psi = 0$$

- Solutions: $\sin kx, \cos kx$
- Graphs:

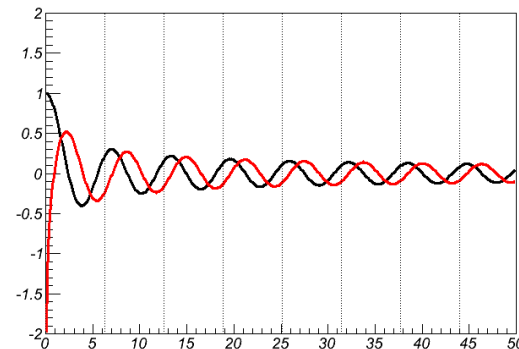


- Series representation:

$$\cos kx = \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^{2n}}{(2n)!}$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\omega^2}{v^2} \psi = 0$$

- Solutions: $J_0(kr), Y_0(kr)$
- Graphs:

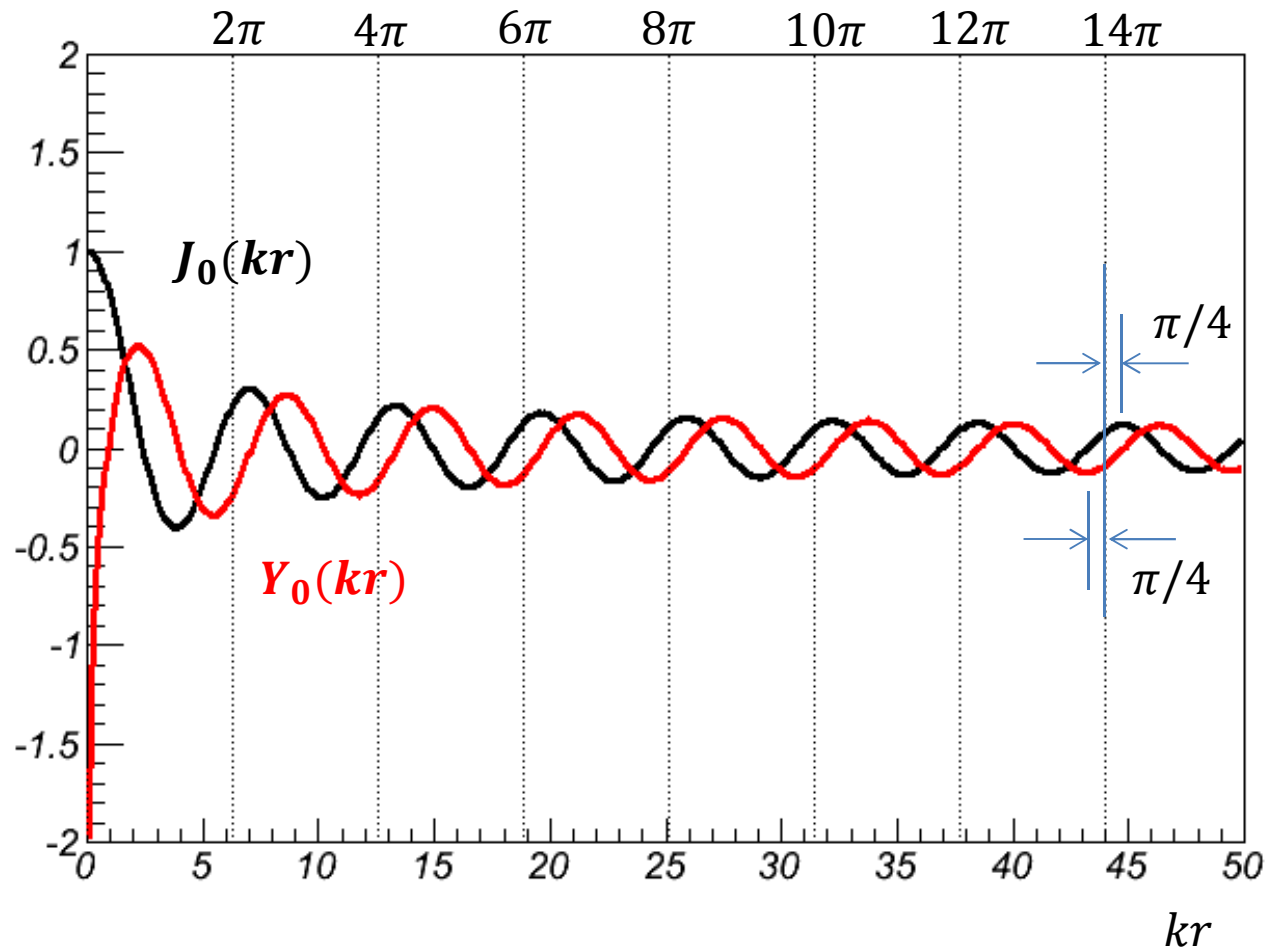


- Series representation:

$$J_0(kr) = \sum_{n=0}^{\infty} \frac{(-1)^n (kr)^{2n}}{2^{2n} (n!)^2}$$

Asymptotic Properties

- At large values of r ...

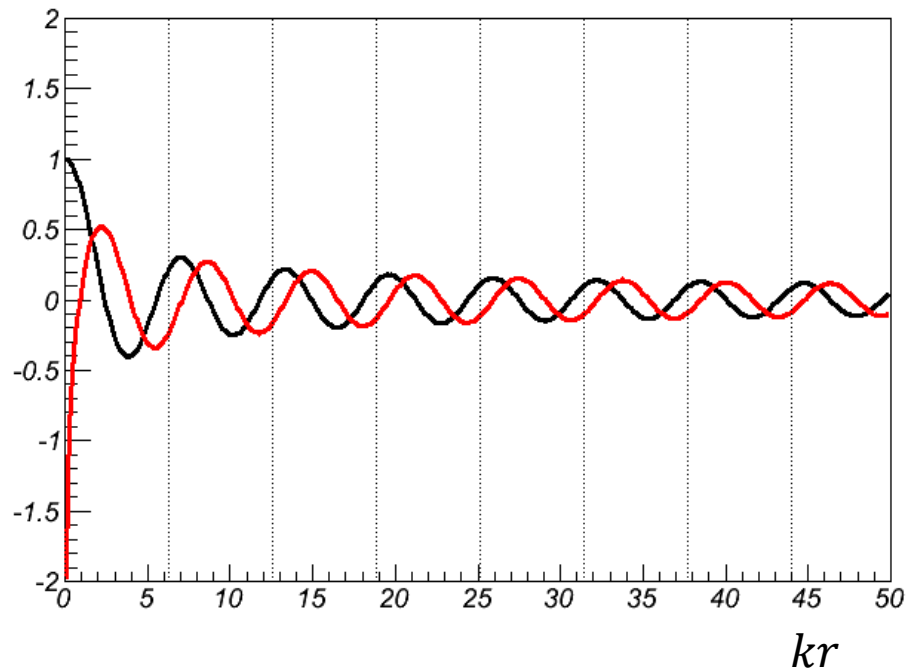


Asymptotic Properties

- When r is large, for example, $kr \gg 1$

$$J_0(kr) \approx \sqrt{2/\pi} \frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$

$$Y_0(kr) \approx \sqrt{2/\pi} \frac{\sin(kr - \pi/4)}{\sqrt{kr}}$$



Example

- What are the frequencies of the rotationally symmetric normal modes of oscillation for the surface of a circular drum of radius R for which the speed of wave propagation is v ?
 - The speed would depend on things like the surface tension and mass per unit area, but the solution only depends on the value of v .

Example

- How is this similar to the string with fixed ends?
 - Look for solutions to the wave equation that satisfy the boundary conditions.

- When $y(x, t) = 0$ for $x = 0$ and $x = L$, these were

$$y_n(x, t) = \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

- We substitute this back into the wave equation to find ω_n :

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \\ \left[\left(\frac{n\pi}{L}\right)^2 - \frac{\omega_n^2}{v^2} \right] y_n(x, t) &= 0 \\ \omega_n &= \frac{n\pi v}{L} \end{aligned}$$

Example

- In the case of a circular drum we have to pick the form of the solution that we expect:
 - It can't be $Y_0(kr)$ because this one diverges at $r = 0$
 - It must be $J_0(kr)$ but only when k makes it satisfy the boundary condition $J_0(kR) = 0$.
 - Asymptotic form of the solution:

$$J_0(kr) \approx \sqrt{2/\pi} \frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$

- The argument of the cosine function must be:

$$kR - \pi/4 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{etc...}$$

Example

$$kR - \pi/4 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

- In general, $k_n R - \pi/4 = (2n - 1)\pi/2$

$$\begin{aligned} k_n &= \frac{1}{R} \left(\frac{\pi}{2} (2n - 1) + \frac{\pi}{4} \right) \\ &= \frac{n\pi}{R} - \frac{\pi}{4R} \end{aligned}$$

- Frequencies are

$$\omega_n = v k_n = \frac{v}{R} \left(n\pi - \frac{\pi}{4} \right)$$

Example

- Various numerical methods are available to evaluate $J_0(kr)$ and to find its roots
 - Just like there are numerical methods at your disposal to evaluate $\sin(kx)$ and $\cos(kx)$.

n	Rk_n (approx)	z_n (exact)
1	$3\pi/4 = 2.3562$	2.4048
2	$7\pi/4 = 5.4978$	5.5201
3	$11\pi/4 = 8.6394$	8.6537
4	$15\pi/4 = 11.7810$	11.7915
5	$19\pi/4 = 14.9226$	14.9309

Energy

- The energy carried by a wave is proportional to the square of the amplitude.
- When $\psi(r, t) \sim A \frac{\cos kr}{\sqrt{r}}$ the energy density decreases as $1/r$
- But the wave is spread out on a circle of circumference $2\pi r$
- The total energy is constant, independent of r
- At large r they look like plane waves:

