Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 21 – French, Chapter 8

Spring 2015 Semester

Matthew Jones

Midterm Exam:

Date:	Thursday, March 12th
Time:	8:00 – 10:00 pm
Room:	PHYS 112
Material:	French, chapters 1-8

Electrical Impedance

- Reflection coefficient:

$$\rho = \frac{Z' - Z}{Z' + Z}$$

- Transmission coefficient:

$$\tau = \frac{2Z'}{Z' + Z}$$

- Limiting cases to remember:
  - Open circuit: $\rho = 1, \tau = 2$
  - Short circuit: $\rho = -1, \tau = 0$
  - Matched, $Z' = Z$: $\rho = 0, \tau = 1$.

Drivers/Receivers

- Now we can model the entire cable:

- Current from the source:

$$I = \frac{V}{Z_s + Z}$$

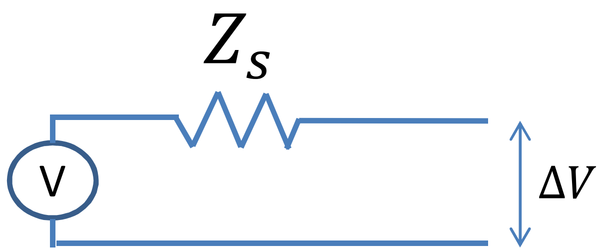
$$\rho = \frac{Z_r - Z}{Z_r + Z}$$

- Voltage at the left end of the cable:

$$V_i = V - I\, Z_s = V\, \frac{Z}{Z_s + Z}$$

Low Frequency Limiting Cases

- What if there was no cable?



- No current flows through the open circuit so we measure $\Delta V = V$ for any voltage source.
- What if a short cable was attached?

Limiting Cases

- Matched source and cable impedance:

$$\rho = \frac{Z_r - Z}{Z_r + Z} = 1$$

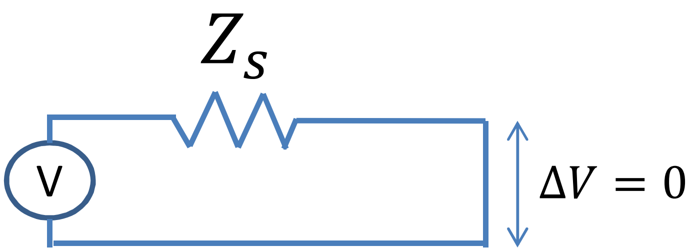
- Voltage from the source:

$$V_i = V - I\, Z_s = V\,\frac{Z}{Z_s + Z} = \frac{V}{2}$$

- Reflected signal is $V_r = V_i$ because $\rho = 1$
- Measured voltage is $\Delta V = V_r + V_i = V$ as before.
- Assumes that the pulse is much longer than the electrical length of the cable.

Low Frequency Limiting Cases

- What if the source was shorted:



- The electric potential is the same everywhere in a conductor.
- The electric potential difference across a wire is zero.
- What if a short cable was attached?

Limiting Cases

- Matched source and cable impedance:

$$\rho = \frac{Z_r - Z}{Z_r + Z} = -1$$

- Voltage from the source:

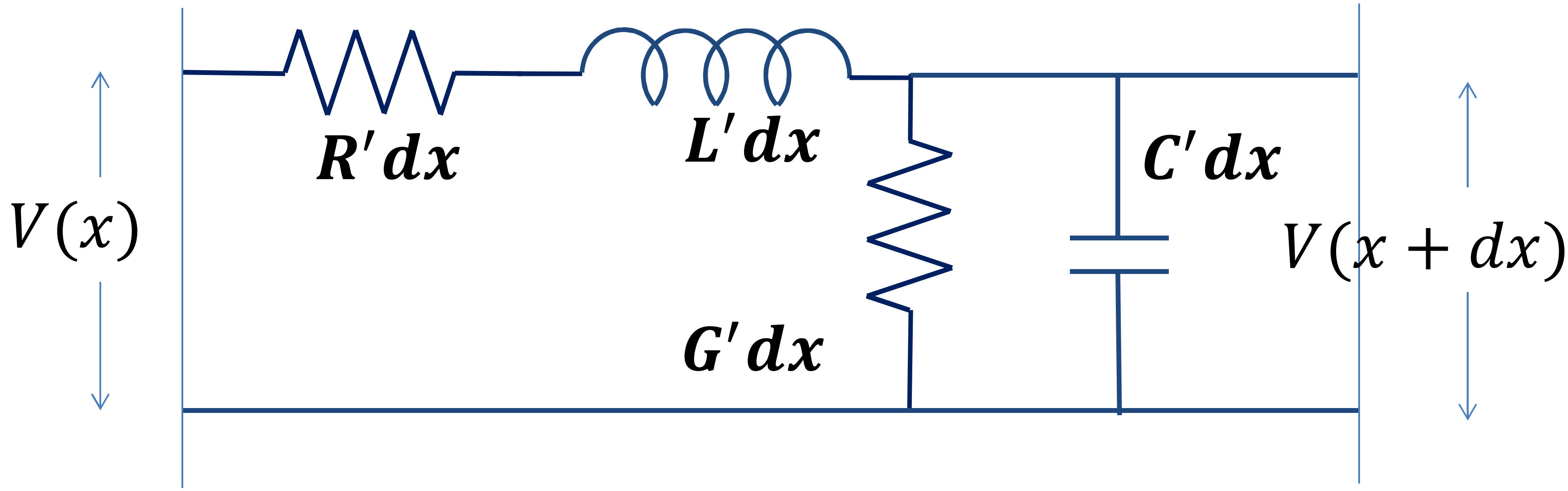
$$V_i = V - I\,Z_s = V\,\frac{Z}{Z_s + Z} = \frac{V}{2}$$

- Reflected signal is $V_r = -V_i$ because $\rho = -1$
- Measured voltage is $\Delta V = V_r + V_i = 0$ as before.
- Assuming that the pulse is much longer than the electrical length of the cable.

Real Transmission Lines

- In addition to reflections from mismatched impedance, real transmission lines also attenuate signals over large distances.
- What properties of the transmission line determine how energy is lost as the wave propagates?

Electrical Circuits



$$\frac{\partial^2 V}{\partial x^2} = XY\,V(x)$$

$$V(x,t) = V(x)e^{i\omega t}$$

$$XY = (R' + i\omega L')(G' + i\omega C')$$

Suppose we try a solution of the form $V(x) = e^{-\gamma x}$*?*

Propagation Constant

- Assume that a solution is of the form

$$V(x,t) = e^{i\omega t - \gamma x}$$

$$\frac{\partial^2 V}{\partial x^2} = \gamma^2 V = (R' + i\omega L')(G' + i\omega C')V$$

- The propagation constant is

$$\gamma = \pm\sqrt{(R' + i\omega L')(G' + i\omega C')}$$

- How do we take the square root of a complex number?
  - $z = r\, e^{i\theta} \rightarrow \sqrt{z} = \sqrt{r} e^{i\theta/2}$
  - $\sqrt{z} = \alpha + i\beta \rightarrow z = (\alpha^2 - \beta^2) + 2i\alpha\beta$

Propagation Constant

- In general, $G' = 0$ is a good approximation.

$$\gamma^2 = (R' + i\omega L')(i\omega C')$$
$$= -\omega^2 L'C' + i\omega R'C'$$
$$= (\alpha^2 - \beta^2) + 2i\alpha\beta$$
$$\approx -\beta^2 + 2i\alpha\beta$$

- When $\alpha \ll \beta$,

$$\beta = \omega\sqrt{L'C'} = \omega/v$$
$$\alpha = \frac{\omega R'C'}{2\beta} = \frac{1}{2}R'\sqrt{\frac{C'}{L'}} = \frac{R'}{2Z}$$

- Where as before we are using $v = \frac{1}{\sqrt{L'C'}}$ and $Z = \sqrt{\frac{L'}{C'}}$.

Attenuation in Transmission Lines

$$V(x,t) = Ae^{i\omega t - \gamma x}$$

$$\gamma = \alpha + i\beta = \frac{R'}{2Z} \pm i\omega/v$$

- Wave propagating in the +x direction:

$$V(x,t) = Ae^{i(\omega t - kx)}e^{-\alpha x}$$

- Wave propagating in the –x direction:

$$V(x,t) = Ae^{i(\omega t + kx)}e^{+\alpha x}$$

- Example:

$$Z = 50\ \Omega, R' = 0.015\ \Omega/\text{ft}\,, L = 300\ \text{ft}$$

$$\mathrm{e}^{-\alpha L} = 0.95$$

Examples

Waves in Three Dimensions

- Wave equation in one dimension:
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$
- The solution, $y(x, t)$, describes the shape of a string as a function of x and t.
- This is a transverse wave: the displacement is perpendicular to the direction of propagation.
- This would confuse the following discussion…
- Instead, let's now consider longitudinal waves, like the pressure waves due to the propagation of sound in a gas.

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- The solution, $p(x, t)$, describes the excess pressure in the gas as a function of x and t.
- What if the wave was propagating in the y-direction?

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- What if the wave was propagating in the z-direction?

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

Waves in Three Dimensions

- The excess pressure is now a function of $\vec{x}$ and t.
- Wave equation in three dimensions:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- But we like to write it this way:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- Where ∇^2 is called the "Laplacian operator", but you just need to think of it as a bunch of derivatives:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Waves in Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- How do we solve this? Here's how…

$$p(\vec{x},t) = p_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

- One partial derivatives:

Chain rule…

$$\frac{\partial p}{\partial x} = i p_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\frac{\partial}{\partial x}\left(\vec{k}\cdot\vec{x}-\omega t\right)$$

$$= i p_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\frac{\partial}{\partial x}\left(k_x x + k_y y + k_z z - \omega t\right)$$

$$= i k_x p(\vec{x},t)$$

- Second derivative:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2\, p(\vec{x},t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- Second derivatives:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2\, p(\vec{x}, t)$$
$$\frac{\partial^2 p}{\partial y^2} = -k_y^2\, p(\vec{x}, t)$$
$$\frac{\partial^2 p}{\partial z^2} = -k_z^2\, p(\vec{x}, t)$$
$$\frac{\partial^2 p}{\partial t^2} = -\omega^2\, p(\vec{x}, t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

$$-\left(k_x^2 + k_y^2 + k_z^2\right)p(\vec{x},t) = -\frac{\omega^2}{v^2}p(\vec{x},t)$$

- Any values of k_x, k_y, k_z satisfy the equation, provided that

$$\omega = v\sqrt{k_x^2 + k_y^2 + k_z^2} = v\left|\vec{k}\right|$$

- If $k_y = k_z = 0$ then $p(\vec{x},t) = p_0 e^{i(k_x x - \omega t)}$ but this described a wave propagating in the $+x$ direction.

Waves in Three Dimensions

$$p(\vec{x}, t) = p_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

- The vector, $\vec{k}$, points in the direction of propagation
- The wavelength is $\lambda = 2\pi/|\vec{k}|$
- How do we visualize this solution?
  - Pressure is equal at all points $\vec{x}$ such that $\vec{k} \cdot \vec{x} - \omega t = \phi$ where ϕ is some constant phase.
  - Let $\vec{x}'$ be some other point such that $\vec{k} \cdot \vec{x}' - \omega t = \phi$
  - We can write $\vec{x}' = \vec{x} + \vec{u}$ and this tells us that $\vec{k} \cdot \vec{u} = 0$.
  - $\vec{k}$ and $\vec{u}$ are perpendicular.
  - *All points in the plane perpendicular to $\vec{k}$ have the same phase.*

Waves in Three Dimensions

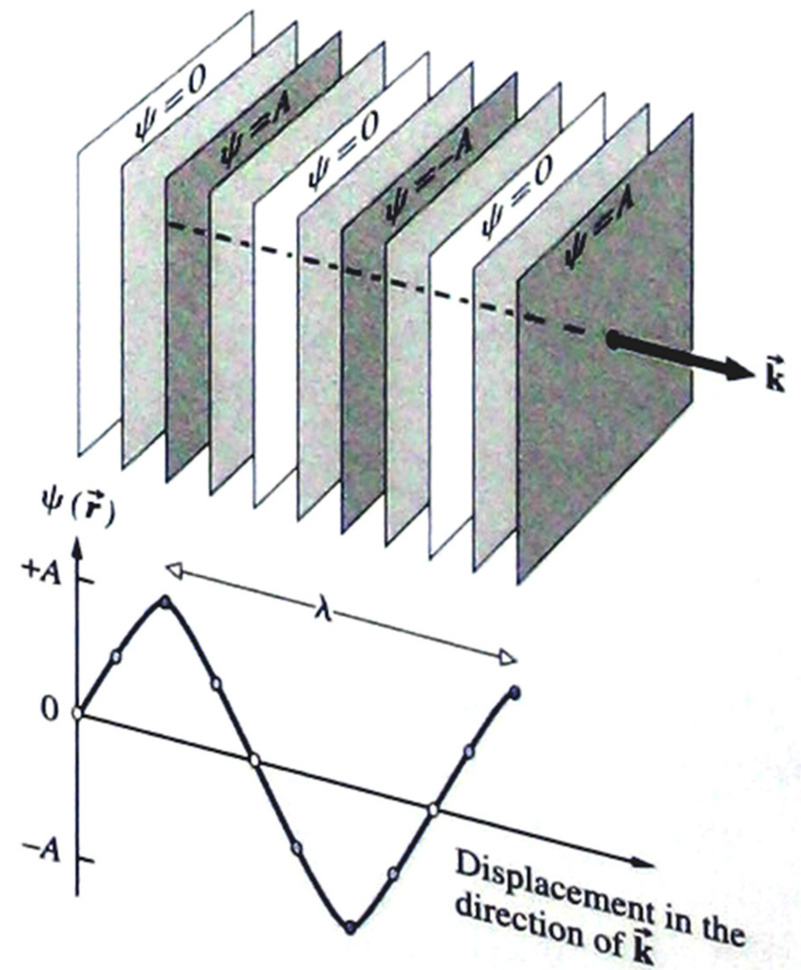
- As usual, we are mainly interested in the real component:

$$\psi(\vec{r},t) = A\cos(\vec{k}\cdot\vec{x} - \omega t)$$

- A wave propagating in the opposite direction would be described by

$$\psi'(\vec{r},t) = A'\cos(\vec{k}\cdot\vec{x} + \omega t)$$

- The points in a plane with a common phase is called the "wavefront".



Waves in Three Dimensions

$$\psi(\vec{r},t) = A\cos\left(\vec{k}\cdot\vec{x} \mp \omega t\right)$$

- Sometimes we are free to pick a coordinate system in which to describe the wave motion.
- If we choose the x-axis to be in the direction of propagation, we get back the one-dimensional solution we are familiar with:

$$\psi(\vec{r},t) = A\cos(kx \mp \omega t)$$

- But in one-dimension we saw that any function that satisfied $f(x \pm vt)$ was a solution to the wave equation.
- What is the corresponding function in three dimensions?

Waves in Three Dimensions

$$\omega = v\sqrt{k_x^2 + k_y^2 + k_z^2} = v\left|\vec{k}\right|$$

- General solution to the wave equation are functions that are twice-differentiable of the form:

$$\psi(\vec{r},t) = C_1 f\left(\hat{k}\cdot\vec{r} - vt\right) + C_2 g\left(\hat{k}\cdot\vec{r} + vt\right)$$

where $\hat{k} = \vec{k}/\left|\vec{k}\right|$

- Just like in the one-dimensional case, these do not have to be harmonic functions.

Example

- Is the function $\psi(\vec{x},t) = (ax + bt + c)^2$ a solution to the wave equation?
- It should be because we can write it as

$$\psi(\vec{x},t) = (a(\boldsymbol{x} + \boldsymbol{vt}) + c)^2$$

where $v = b/a$ which is of the form $g(x + vt)$

- We can check explicitly:

$$\frac{\partial\psi}{\partial x} = 2a(ax + bt + c) \qquad \frac{\partial\psi}{\partial x} = 2b(ax + bt + c)$$

$$\frac{\partial^2\psi}{\partial x^2} = 2a^2 \qquad \frac{\partial^2\psi}{\partial x^2} = 2b^2$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \Rightarrow 2a^2 = 2b^2/v^2 \Rightarrow v = b/a$$

Example

- Is the function $\psi(\vec{x}, t) = ax^{-2} + bt$, where $a > 0, b > 0$, a solution to the wave equation?
- It is twice differentiable…

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{6a}{x^4} \qquad \frac{\partial^2 \psi}{\partial t^2} = 0$$

- But it is not a solution:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \Rightarrow \frac{6a}{x^4} = 0$$

  - Only true if $a = 0$, which we already said was not the case.
- This is not a solution to the wave equation.