PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 2 – French, Chapter 1

Spring 2015 Semester

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Simple Harmonic Motion

- Mass-spring system:
  - Force given by Hooke's law:

$$F = -kx$$

  - Newton's second law:

$$F = ma = m\ddot{x} = m\frac{d^2x}{dt^2}$$

  - Equation of motion:

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0$$

where $\omega = \sqrt{k/m}$

Simple Harmonic Motion

- Differential equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

- Solutions can be written in various ways:

$$x(t) = \boldsymbol{A}\ \cos(\omega t + \boldsymbol{\varphi})$$

$$x(t) = \boldsymbol{A} \sin \omega t + \boldsymbol{B}\ \cos \omega t$$

(and many others…)

- Two *constants of integration* need to be determined from initial conditions or other information.

Simple Harmonic Motion

- How do we know that these are solutions?
- Compute the derivatives:

$$x(t) = A\ \cos(\omega t + \varphi)$$
$$\dot{x}(t) = -A\omega \sin(\omega t + \varphi)$$
$$\ddot{x}(t) = -A\omega^2 \cos(\omega t + \varphi) = -\omega^2 x(t)$$

- Substitute into the differential equation:

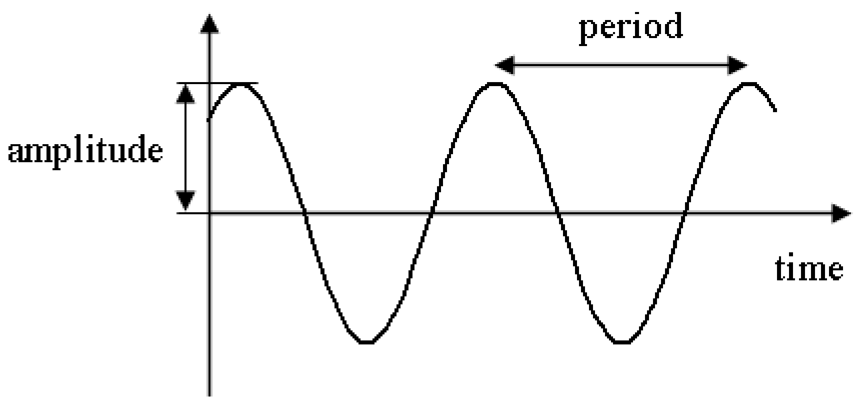
$$\ddot{x} + \omega^2 x = \left(-\omega^2 x(t)\right) + \omega^2 x(t) = 0$$

- Mathematical details:
  - How could we deduce that this was a solution if we didn't already know it was?

Simple Harmonic Motion

- Properties of the solution:

$$x(t) = \boldsymbol{A}\ \cos(\boldsymbol{\omega} t + \boldsymbol{\varphi})$$



- Notation:

Amplitude: $\boldsymbol{A}$

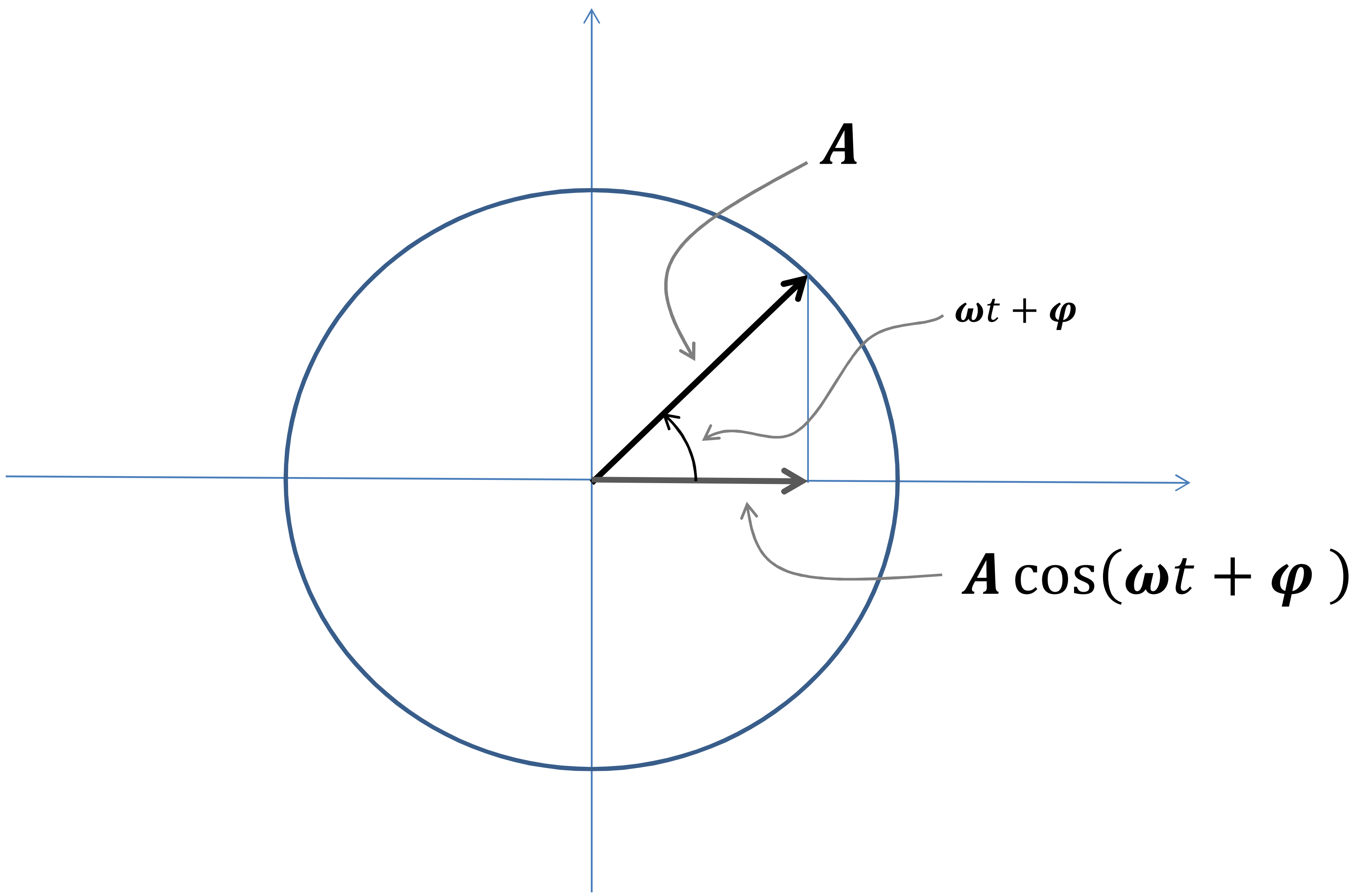
Initial phase: $\boldsymbol{\varphi}$

Angular frequency: $\boldsymbol{\omega}$

Frequency: $\boldsymbol{f = \omega/2\pi}$

Period: $\boldsymbol{T = 1/f = 2\pi/\omega}$

Descriptions of Harmonic Motion

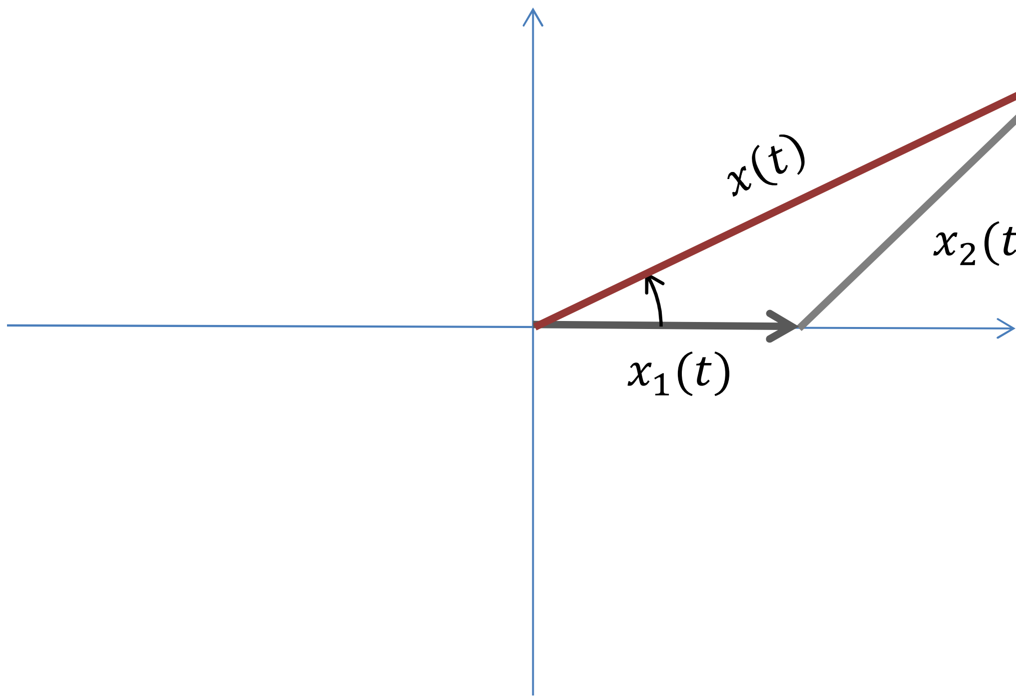


Uniform Circular Motion

- The graph of $x(t) = A\ \cos(\omega t + \varphi)$ is the same as the projection onto the x-axis of a vector of length A, rotating with angular frequency ω.
- This is a useful geometric description of the motion.
  - Two-component vectors are introduced only for convenience (we call them “phasors”)
  - The solution we are interested in is just the projection onto the x-axis.
- Example…

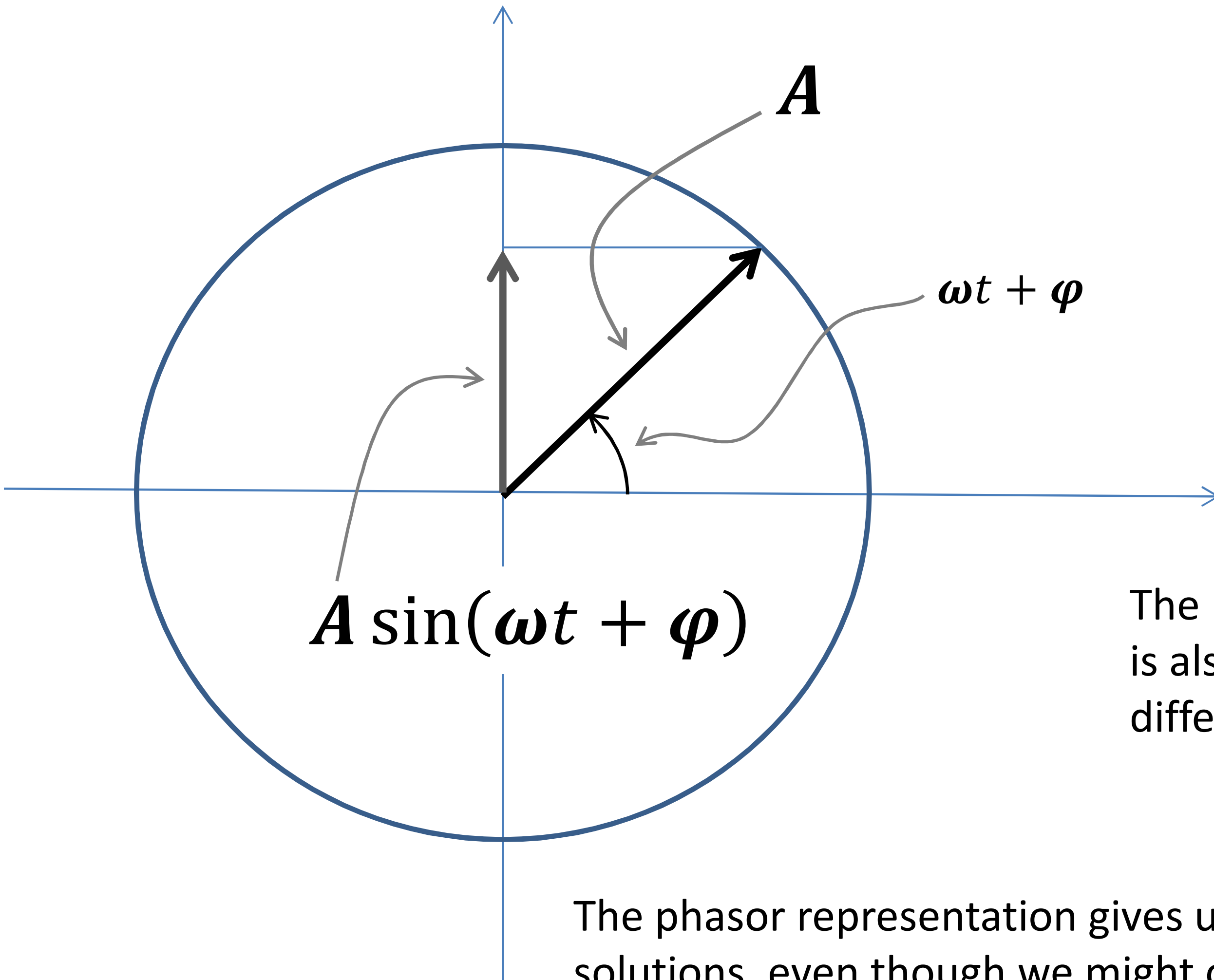
Uniform Circular Motion

- The differential equation $\ddot{x} + \omega^2 x = 0$ is linear:
  - Suppose $x_1(t)$ and $x_2(t)$ are both solutions
  - Then the function $x(t) = a\, x_1(t) + b\, x_2(t)$ is also a solution for any real numbers a and b.



Actually, the *functions* $x_1(t)$, $x_2(t)$ and $x(t)$ are the projections of these vectors onto the x-axis.

Uniform Circular Motion



The projection onto the y-axis is also a solution to the differential equation.

The phasor representation gives us two independent solutions, even though we might only want to use only one of them to describe the motion.

Phasor Representation

- The phasor provides all the information we need to describe the motion
  - If we just knew the value of x at one time t, we still don't know what A and φ are.
  - But if we know x and y at time t then we have enough information to calculate both A and φ.
- The more general description of the motion can be useful for analyzing problems even if the "physical" solution to the equations of motion is just one of its projections.

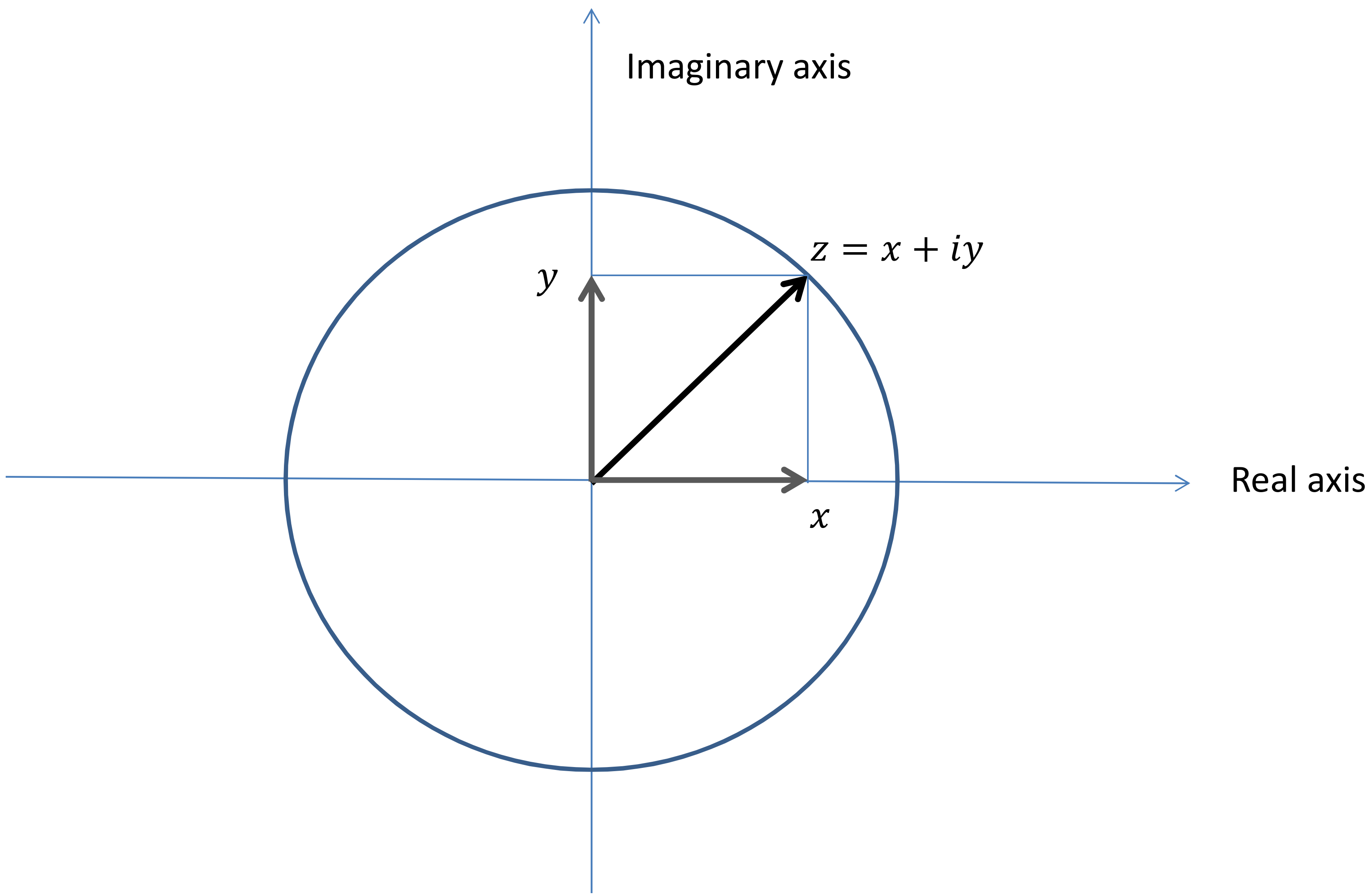
Complex Representation

- Basic definitions:

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$z = x + iy$$
$$z^* = x - iy$$
$$|z| = \sqrt{z^* z} = \sqrt{x^2 + y^2}$$
$$Re(z) = x = (z + z^*)/2$$
$$Im(z) = y = (z - z^*)/2i$$

(where x and y are real numbers)

Complex Representation



Complex Representation

- But complex numbers are way better…
- Euler's identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- Complex numbers in this form satisfy:

$$\left(e^{i\theta}\right)^* = e^{-i\theta}$$

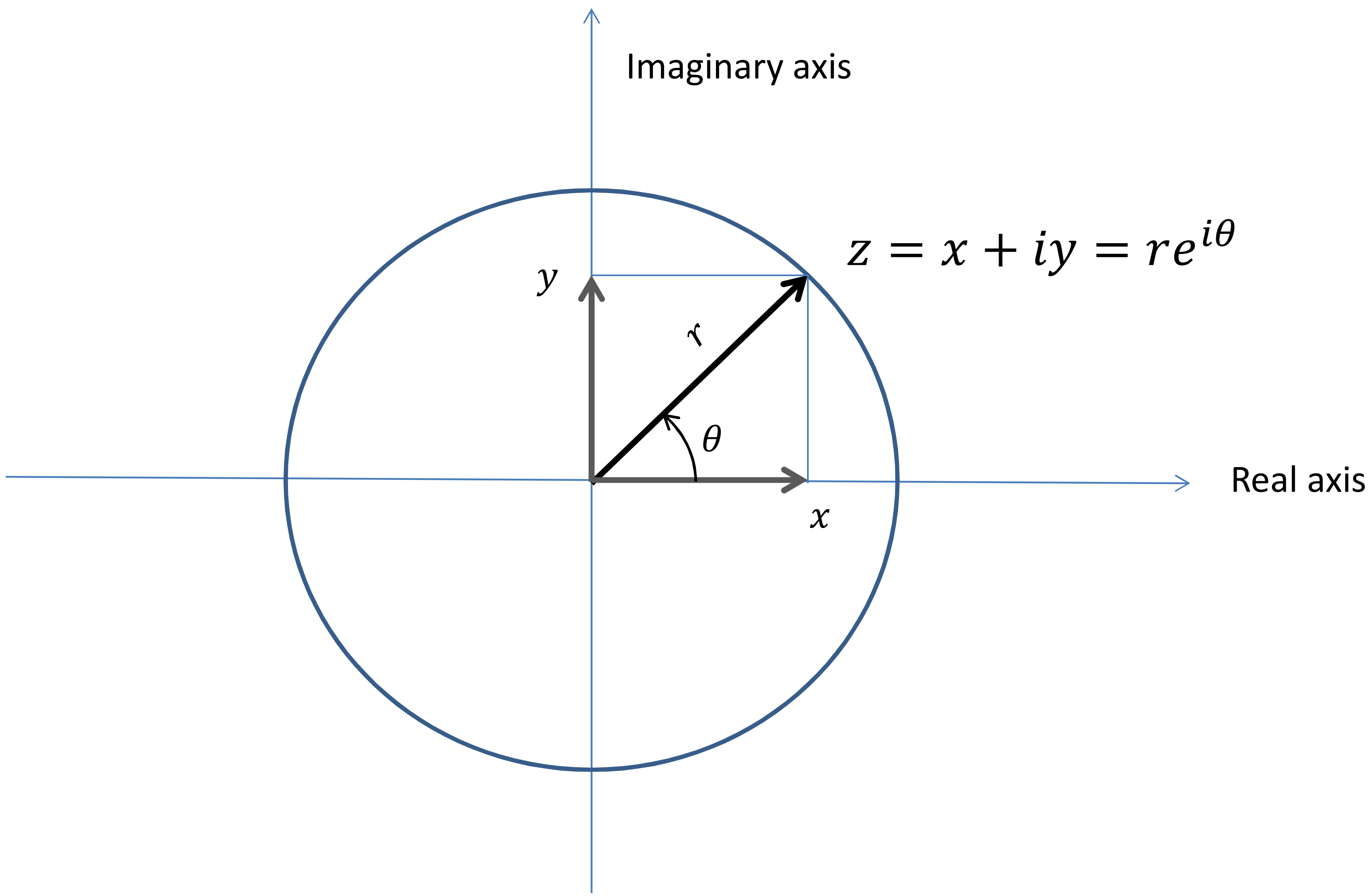
$$e^{i\theta}e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$$

$$= cos^2\theta + sin^2\theta = 1$$

- In general, we can always write

$$z = x + iy = re^{i\theta}$$

Complex Representation



Simple Harmonic Motion

- The other way we will describe solutions to

$$\ddot{x} + \omega^2 x = 0$$

will be using complex numbers…

  – Let $x(t) = \boldsymbol{r}e^{i(\omega t+\boldsymbol{\varphi})} = \underbrace{\left(\boldsymbol{r}e^{i\boldsymbol{\varphi}}\right)}e^{i\omega t} = \boldsymbol{c}e^{i\omega t}$

This part is just a constant.

  – Derivatives are:

$$\dot{x}(t) = i\omega\, \boldsymbol{c}e^{i\omega t}$$

$$\ddot{x}(t) = (i\omega)^2\boldsymbol{c}e^{i\omega t} = -\omega^2 x(t)$$

- It is a solution:

$$\ddot{x} + \omega^2 x = \left(-\omega^2 x(t)\right) + \omega^2 x(t) = 0$$

Simple Harmonic Motion

- The *physical* displacement of the mass must be a real number.
- The displacement as a function of time is given by the real component, $Re[x(t)]$.
- The complex representation contains more information than is present in just the function describing the physical displacement.
  - It provides *both* amplitude *and* phase information

Initial Value Problems

- A solution to the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

can be written

$$x(t) = \boldsymbol{A}\ \cos(\omega t + \boldsymbol{\varphi})$$

- We need more information if have to also determine the constants of integration $\boldsymbol{A}$ and $\boldsymbol{\varphi}$.
- This information is usually given as *initial conditions* (or *boundary conditions*).

Initial Value Problems

Example:

"A mass of 100 g is attached to a spring with spring constant of 1 N/m that is initially stretched to a length of 2 cm. What is the solution to the equation of motion if it is released from rest at time t=0?"

Initial Value Problems

1. Analyze the problem in general – don't use the numbers yet. *If you make a mistake you will never find it in a big mess of numbers.*

Let $m = 100\ g$ *be the mass,* $k = 1\ N/m$ *be the spring constant and* x_0 *be the initial displacement.*

2. Introduce other symbols if it is convenient to do so:

Let $\omega = \sqrt{k/m}$.

Initial Value Problems

3. Now describe the system in terms of the relevant physical principles…

The force acting on the mass is described by Hooke's law:

$$F(x) = -k\,x$$

and the resulting motion is given by Newton's second law:

$$m\,\frac{d^2x}{dt^2} = -k\,x\ldots$$

Initial Value Problems

4. Write it in a "standard form" where you know what the solution is:

... which can be written

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

and has solutions of the form,

$$x(t) = A\ \cos(\omega t + \varphi)$$

where A and φ are constants of integration that remain to be determined.

Initial Value Problems

5. Calculate the initial conditions at t=0:

At $t = 0$, the initial displacement is

$$x(0) = A\ \cos\varphi = x_0$$

and the initial velocity is

$$\dot{x}(0) = \left.\frac{dx}{dt}\right|_{t=0} = -A\omega\sin\varphi = 0.$$

6. What values of A and φ will satisfy these two equations?

Initial Value Problems

$$x(0) = A\cos\varphi = x_0$$

$$\dot{x}(0) = \left.\frac{dx}{dt}\right|_{t=0} = -A\omega\sin\varphi = 0$$

- $A \neq 0$ and $\omega \neq 0$ so we must have $\varphi = 0$.
- Then $A = x_0$.

The initial conditions are satisfied by $A = x_0$ and $\varphi = 0$, so the solution to the initial value problem is

$$x(t) = x_0\cos\omega t$$

Initial Value Problems

7. Now you can substitute in the numbers if you want…

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\ N/m}{0.1\ kg}} = \sqrt{\frac{1\ [kg \cdot m/s^2]/m}{0.1\ kg}}$$

$$= 3.162\ s^{-1}$$

$$f = \omega/2\pi = 0.503\ s^{-1}$$

$$x(t) = (2\ cm)\cos[(3.162\ s^{-1}) \times t]$$

Make sure you write the units everywhere. That way it is clear that $x(t)$ is in cm and t is in seconds.

Initial Value Problems

- What if the initial conditions were more complicated?
  - Initial displacement: x_0
  - Initial velocity: v_0
- How can we determine the constants of integration?
  - You could use lots of trigonometric identities
  - You could look at the problem in a geometric way to see the solution…

Initial Value Problem

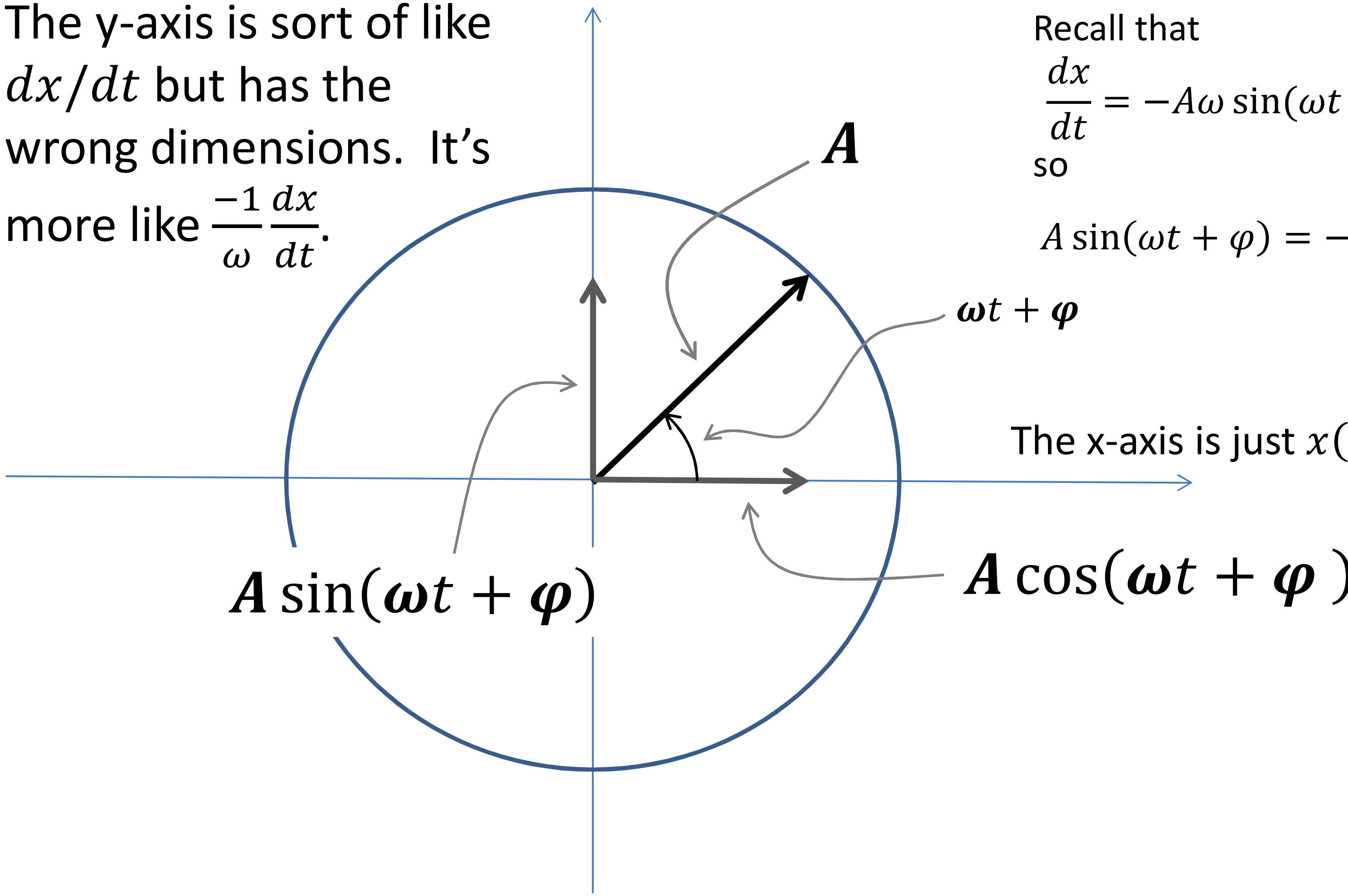
The y-axis is sort of like dx/dt but has the wrong dimensions. It's more like $\frac{-1}{\omega}\frac{dx}{dt}$.

Recall that

$$\frac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$$

so

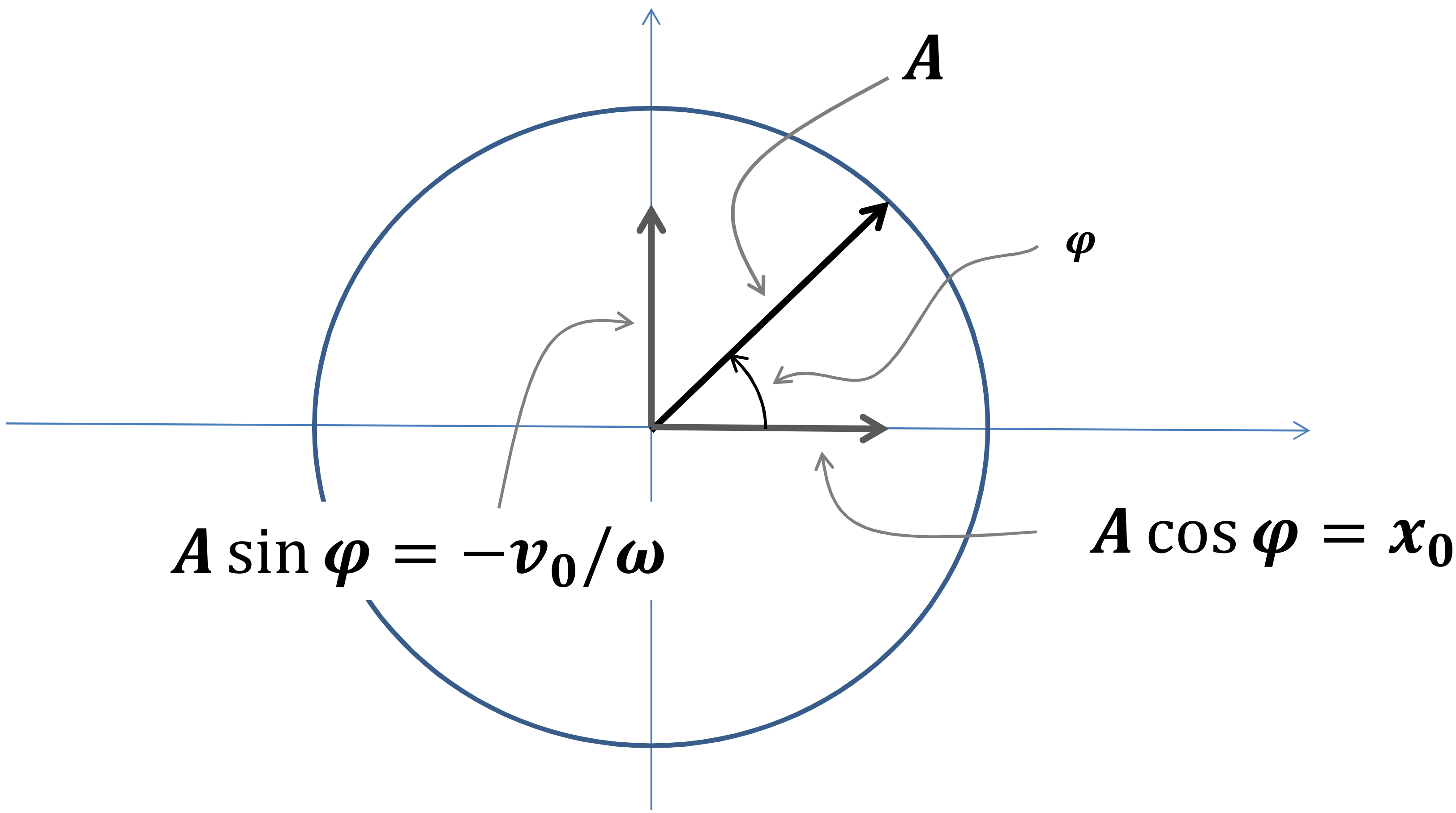
$$A\sin(\omega t + \varphi) = -\frac{1}{\omega}\frac{dx}{dt}$$



The x-axis is just $x(t)$.

Initial Value Problem

At t=0, the x-component is $x(0) = x_0$ and the y-component is $-\frac{\dot{x}(0)}{\omega} = -v_0/\omega$.



Initial Value Problem

- Solve for A and φ:

$$\frac{A\ \sin\varphi}{A\ \cos\varphi} = \tan\varphi = \frac{-v_0}{\omega\ x_0}$$

$$\varphi = \tan^{-1}\left(\frac{-v_0}{\omega x_0}\right)$$

$$A^2\sin^2\varphi + A^2\cos^2\varphi = A^2 = x_0^2 + v_0^2/\omega^2$$

$$A = \pm\sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$$

- You still need the initial conditions to get the right sign.