

Physics 42200
Waves & Oscillations

Lecture 18 – French, Chapter 6

Spring 2015 Semester

Matthew Jones

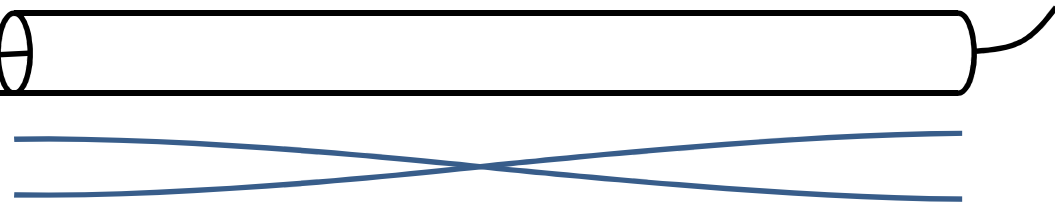
Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
 - **Open circuit: $I(L) = 0$**
 - Short circuit: $V(L) = 0$



$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi v \frac{n}{2L}$$

$$f_n = \frac{nv}{2L}$$



If $L = 1 \text{ m}$ and $v = 20 \text{ cm/ns}$...

$$n = 1$$

$$f = 100 \text{ MHz}$$

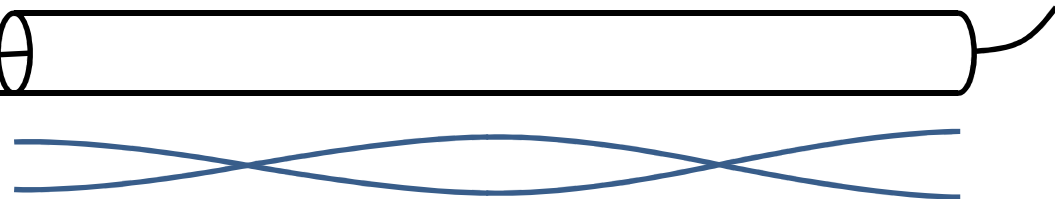
Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
 - **Open circuit: $I(L) = 0$**
 - Short circuit: $V(L) = 0$



$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi v \frac{n}{2L}$$

$$f_n = \frac{nv}{2L}$$



If $L = 1 \text{ m}$ and $v = 20 \text{ cm/ns}$...

$$n = 2$$

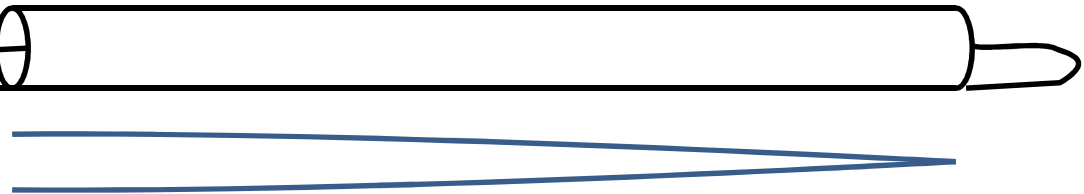
$$f = 200 \text{ MHz}$$

Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
 - Open circuit: $I(L) = 0$
 - **Short circuit: $V(L) = 0$**



$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi v \frac{n-1/2}{2L} \quad f_n = \frac{(n-\frac{1}{2})v}{2L}$$



If $L = 1 \text{ m}$ and $v = 20 \text{ cm/ns}$...

$$n = 1$$

$$f = 50 \text{ MHz}$$

Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
 - Open circuit: $I(L) = 0$
 - **Short circuit: $V(L) = 0$**



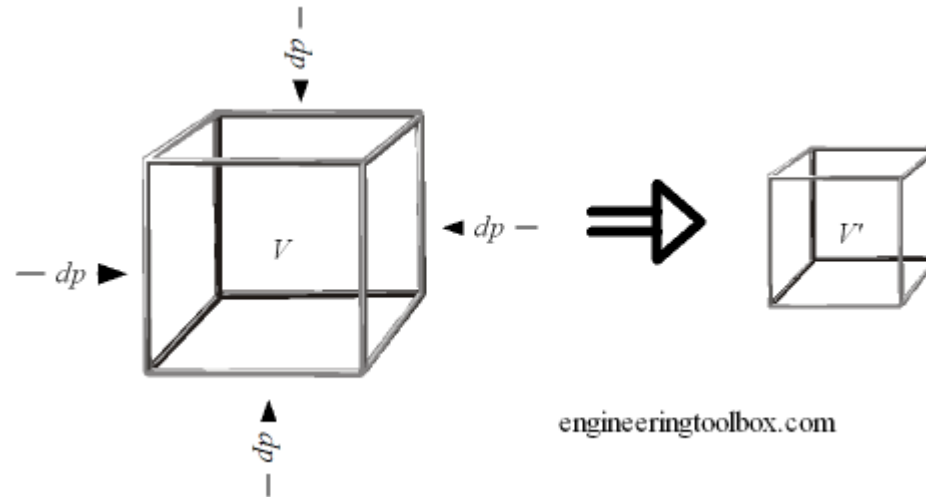
$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi v \frac{n-1/2}{2L} \quad f_n = \frac{(n-\frac{1}{2})v}{2L}$$



If $L = 1 \text{ m}$ and $v = 20 \text{ cm/ns}$...

$$n = 2 \quad f = 150 \text{ MHz}$$

Longitudinal Waves in a Gas



- Increased pressure on a volume of gas decreases its volume
- Bulk modulus of elasticity is defined

$$K = -V \frac{dp}{dV}$$

Longitudinal Waves in a Gas

- Equations of state for a gas:
 - Ideal gas law: $pV = NkT$
 - Adiabatic gas law: $pV^\gamma = \text{constant}$
- In an adiabatic process, no heat is absorbed
 - Absorbing heat would remove mechanical energy from a system
 - Propagation of sound waves through a gas is an example of an adiabatic process
- Bulk modulus calculated from equation of state:

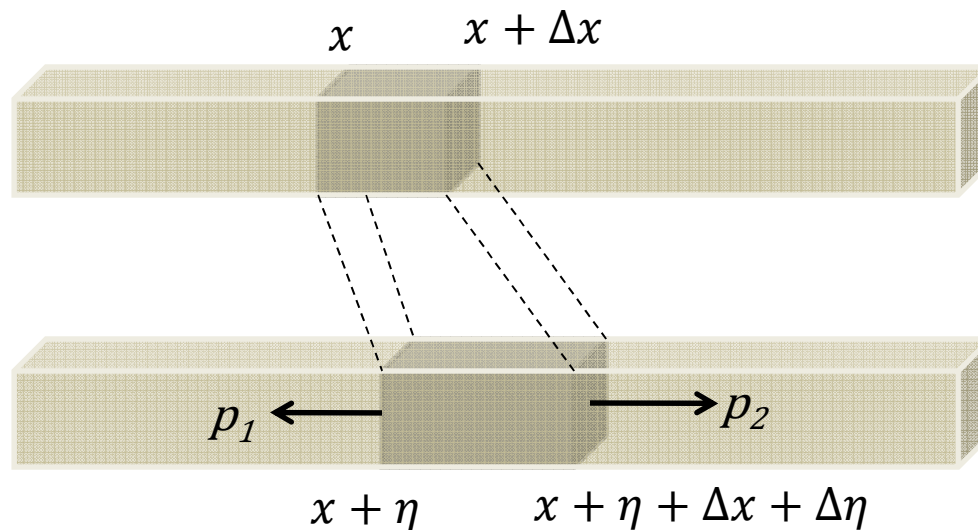
$$V^\gamma dp + \gamma p V^{\gamma-1} dV = 0$$

$$\frac{dp}{dV} = -\gamma p/V$$

$$K = -V \frac{dp}{dV} = \gamma p$$

Longitudinal Waves in a Gas

- By analogy with the solid rod, we consider an element of gas at position x of thickness Δx that is displaced by a distance $\eta(x)$:



Longitudinal Waves in a Gas

- Wave equation:

$$\frac{\partial^2 \eta}{dx^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2}$$

- For a solid rod, $v = \sqrt{Y/\rho}$
- For a gas, $v = \sqrt{K/\rho} = \sqrt{\gamma p/\rho}$
- Changes in pressure and density are very small compared with the average pressure and density.
- At standard temperature and pressure, air has

$$\gamma = 1.40$$

$$p = 101.3 \text{ kPa}$$

$$\rho = 1.2 \text{ kg/m}^3$$

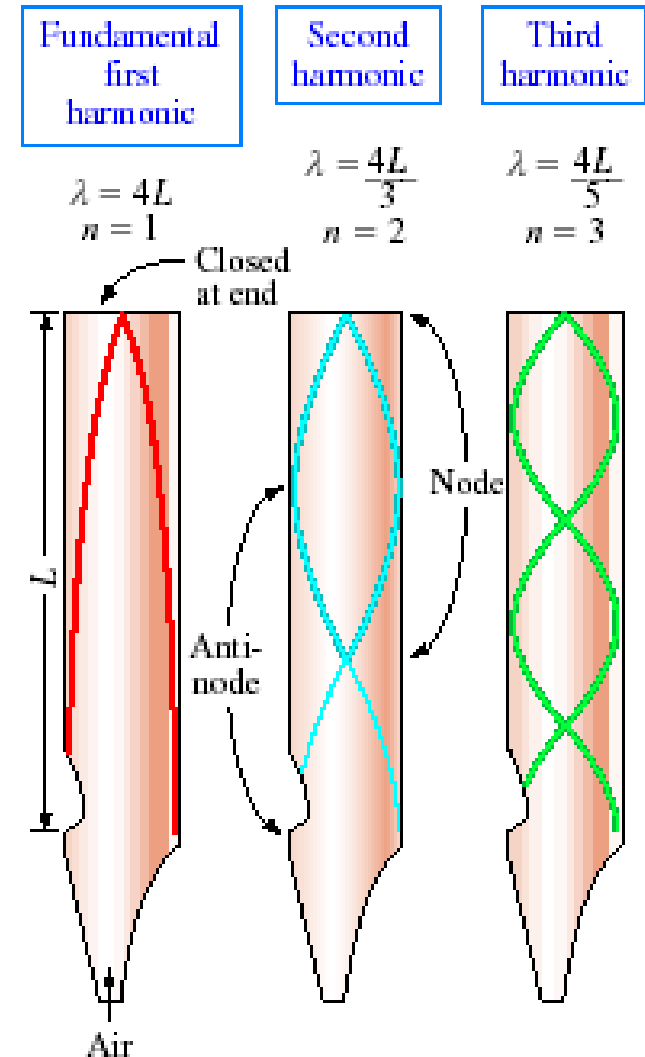
$$v = \sqrt{\frac{(1.40)(101.3 \times 10^3 \text{ N/m}^2)}{(1.2 \text{ kg/m}^3)}} = 343 \text{ m/s}$$

The Physics of Organ Pipes



Resonant Cavities

- Air under pressure enters at the bottom
 - Entering air rapidly oscillates between the pipe and the lip
 - The lower end is a displacement anti-node
- Top end can be open or closed
 - Open end is a pressure node/displacement anti-node
 - Closed end a displacement node/pressure anti-node



Wave Propagation

- The wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

- We worked out solutions that satisfied specific boundary conditions.
- A general solution is any function that is of the form
$$y(x, t) = f(x \pm vt)$$
- Are these two pictures compatible?

Wave Propagation

- Solutions for normal modes:

$$y_n(x, t) = \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$
$$\omega_n = \frac{\pi n v}{L}$$

- Trigonometric identity:

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

- This gives,

$$y_n(x, t) = \frac{1}{2} \left[\sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right) \right]$$

Wave Propagation

$$y_n(x, t) = \frac{1}{2} \left[\sin \left(\frac{n\pi x}{L} + \omega_n t \right) + \sin \left(\frac{n\pi x}{L} - \omega_n t \right) \right]$$

- Write this as

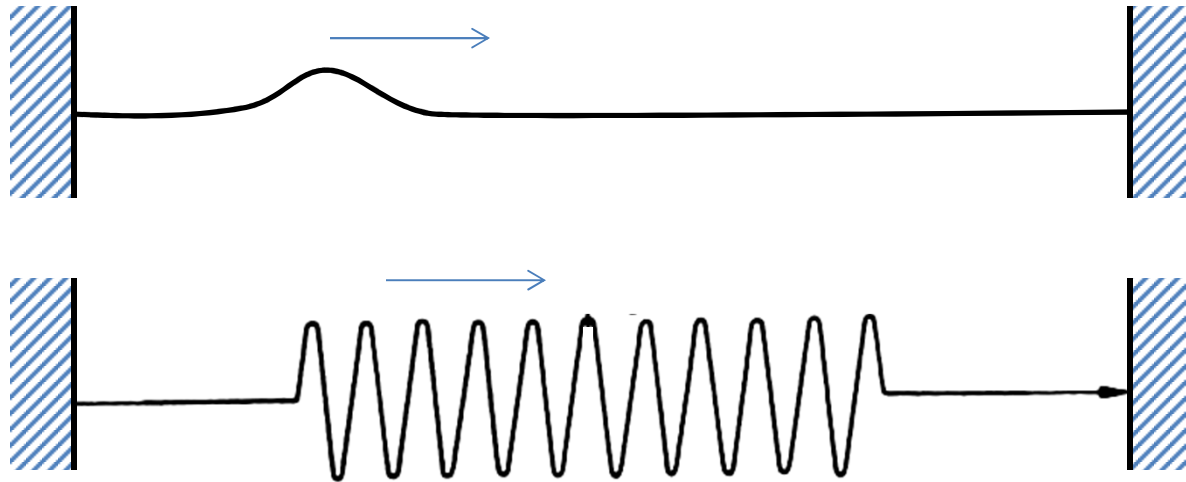
$$y_n(x, t) = \frac{1}{2} [\sin(k(x + vt)) + \sin(k(x - vt))]$$

$$k = \frac{n\pi}{L}$$

- This is the equation for two sine-waves moving in opposite directions.
- The text refers to these as “progressive waves”.
- The “standing waves” that satisfy the boundary conditions are the superposition of “progressive waves” that move in opposite directions.

Wave Propagation

- Waves can propagate in either direction.
- Easiest to visualize in terms of a pulse, or wave packet:



- If this disturbance is far from the ends, the effect is the same as letting $L \rightarrow \infty$

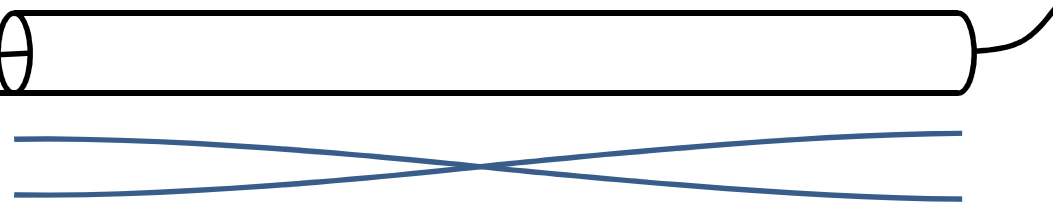
Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
 - **Open circuit: $I(L) = 0$**
 - Short circuit: $V(L) = 0$



$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi v \frac{n}{2L}$$

$$f_n = \frac{nv}{2L}$$



If $L = 1 \text{ m}$ and $v = 20 \text{ cm/ns}$...

$$n = 1$$

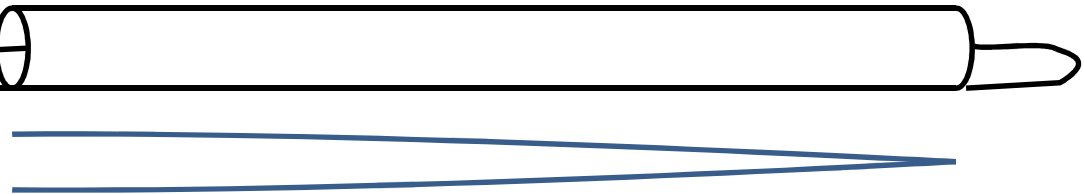
$$f = 100 \text{ MHz}$$

Transmission Lines

- A transmission line can be driven by a voltage source at one end.
- Boundary conditions at the other end:
 - Open circuit: $I(L) = 0$
 - **Short circuit: $V(L) = 0$**



$$\omega_n = \frac{2\pi v}{\lambda_n} = 2\pi v \frac{n-1/2}{2L} \quad f_n = \frac{(n-\frac{1}{2})v}{2L}$$



If $L = 1 \text{ m}$ and $v = 20 \text{ cm/ns}$...

$$n = 1$$

$$f = 50 \text{ MHz}$$

Wave Propagation

$$y(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

$y(0, t) = y(L, t) = 0$

- In general, we could write

$$y(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- In the limit where the disturbance is very far from either boundary, the Fourier sine transform is:

$$B(k) = \int_{-\infty}^{\infty} u(x) \sin(kx) dx$$

- Similarly, we can define the Fourier cosine transform:

$$A(k) = \int_{-\infty}^{\infty} u(x) \cos(kx) dx$$

Wave Propagation

$$B(k) = \int_{-\infty}^{\infty} u(x) \sin(kx) dx$$

- Similarly, we can define the Fourier cosine transform:

$$A(k) = \int_{-\infty}^{\infty} u(x) \cos(kx) dx$$

- The original function is represented by:

$$u(x) = \frac{1}{\pi} \int_0^{\infty} A(k) \cos(kx) dk + \frac{1}{\pi} \int_0^{\infty} B(k) \sin(kx) dk$$

- If $A(k) = A(-k)$ and $B(k) = -B(-k)$ then we can make this more symmetric:

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k) \cos(kx) dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} B(k) \sin(kx) dk$$

Wave Propagation

- To make this even more symmetric we can change slightly the definition of $A(k)$ and $B(k)$:

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \sin(kx) dx$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \cos(kx) dx$$

- Then,

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx) dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx) dk$$

Wave Propagation

- Previously, we interpreted the coefficients a_n as the amplitude of the normal mode with frequency ω_n
 - wavelength $\lambda_n = 2L/n$
 - wavenumber $k_n = 2\pi/\lambda_n = \pi n/L$
- Now, we interpret $A(k)$ and $B(k)$ as the amplitude for harmonic waves with wavenumbers between k and $k + dk$.
- It can be important to decompose a pulse into its frequency components because in real materials, the nature of wave propagation can depend on the frequency.

Example

- Consider a pulse that has a Gaussian shape:



$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- Special case:
 - Peak position is at $x = 0$
 - Width of the peak is $\sigma = 1$
- Other Gaussian functions can be transformed into this special case by linear change of variables.
- What is the continuous Fourier transform?

Example

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx) dx$$

- The Gaussian function $g(x)$ is an even function:

$$g(x) = g(-x)$$

- The function $\sin(kx)$ is an odd function:

$$\sin(-kx) = -\sin(kx)$$

- This integral must vanish...

$$B(k) = 0$$

Example

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2} \cos(kx) dx \end{aligned}$$

- From your table of integrals:

$$\int_{-\infty}^{\infty} e^{-ax^2} \cos bx dx = \sqrt{\frac{\pi}{a}} e^{-b^2/4a}$$

- In this case, $a = 1/2$ and $b = k$

$$A(k) = \frac{1}{2\pi} \times \sqrt{2\pi} e^{-k^2/2} = \frac{1}{\sqrt{2\pi}} e^{-k^2/2}$$

- This is a Gaussian distribution of wavenumbers $k = \omega/v$.

Notes about Fourier Transforms

- For the Gaussian pulse,

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

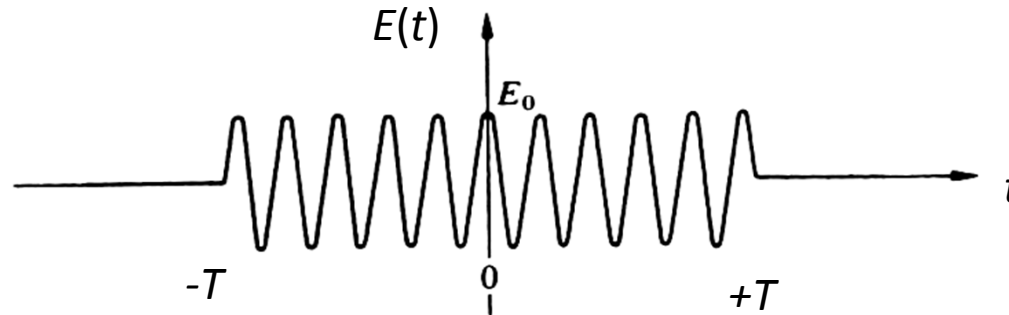
- The amplitudes of the frequency components are:

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma^2/2}, \quad B(k) = 0$$

- When the pulse is narrow, $\sigma \ll 1$, then the exponent in $A(k)$ is large for a large range of k
 - Since $\omega = v/k$, a narrow pulse has a wide range of frequency components.
- Conversely, a wide pulse has a narrow range of frequencies.

Another Example

- A photon can be described as a localized oscillation:



$$\text{At } x = 0, E(t) = \begin{cases} E_0 \cos(\omega t) & \text{when } |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$\text{At } t = 0, E(x) = \begin{cases} E_0 \cos(kx) & \text{when } |x| < cT \\ 0 & \text{otherwise} \end{cases}$$

$$A(k') = \frac{E_0}{\sqrt{2\pi}} \int_{-cT}^{cT} \cos(kx) \cos(k'x) dx$$

Another Example

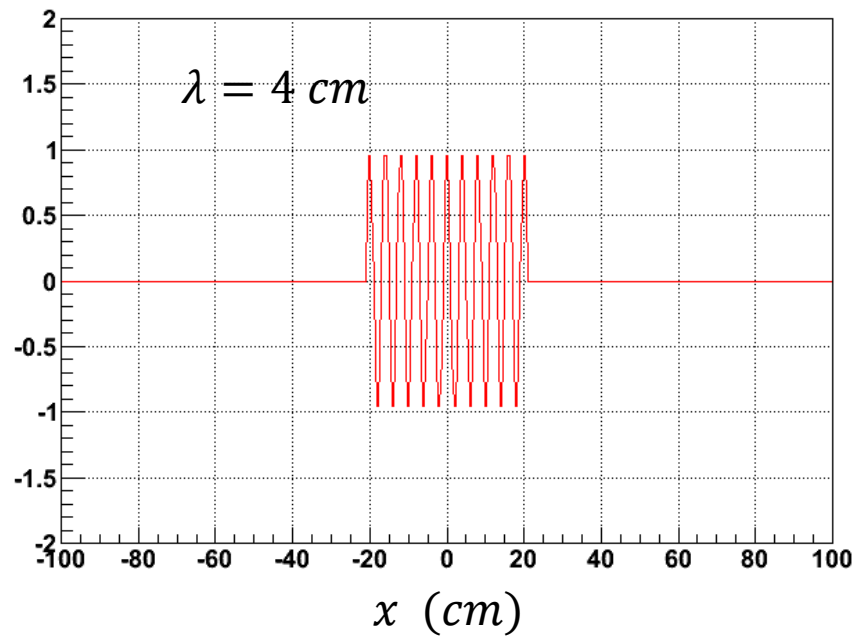
$$A(k') = \frac{E_0}{\sqrt{2\pi}} \int_{-cT}^{cT} \cos(kx) \cos(k'x) dx$$

- Trigonometric identity:

$$\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$A(k') = \frac{E_0}{\sqrt{2\pi}} \left[\frac{\sin((k - k')cT)}{k - k'} + \frac{\sin((k + k')cT)}{k + k'} \right]$$

Another Example



$$k = \frac{2\pi}{\lambda} = 1.571 \text{ cm}^{-1}$$

