PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 16 – French, Chapter 6

Spring 2015 Semester

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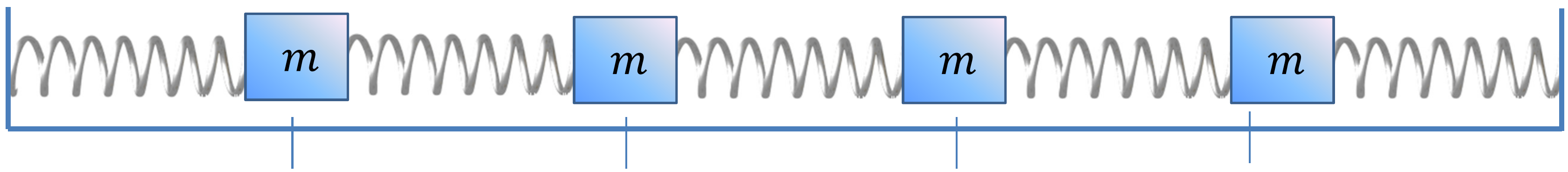
Summary



- General solution:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_k t - \theta_k)$$

- Frequencies of normal modes of oscillation:

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- Fourier coefficients:

$$a_k \cos\theta_k = \frac{2}{N}\sum_{n=1}^{N} x_n(0)\sin\left(\frac{nk\pi}{N+1}\right)$$

$$a_k \omega_k \sin\theta_k = \frac{2}{N}\sum_{n=1}^{N} \dot{x}_n(0)\sin\left(\frac{nk\pi}{N+1}\right)$$

Summary

- General solution:

$$y(x,t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) \cos(\omega_k t - \theta_k)$$

- Frequencies of normal modes of oscillation:

$$\omega_k = \frac{k\pi v}{L}$$

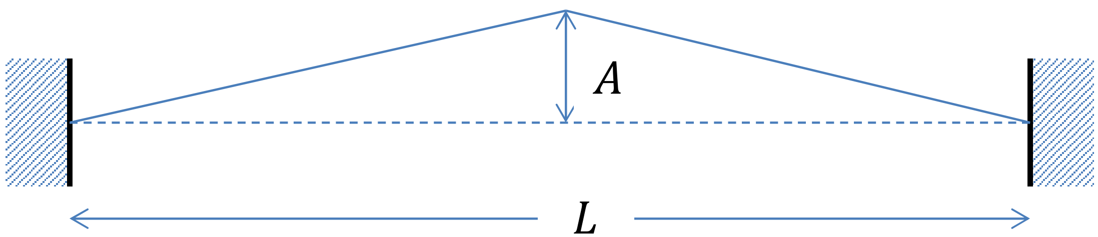
- Fourier coefficients:

$$a_k \cos\theta_k = \frac{2}{L}\int_0^L y(x,0) \sin\left(\frac{k\pi x}{L}\right) dx$$

$$a_k \omega_k \sin\theta_k = \frac{2}{L}\int_0^L \dot{y}(x,0) \sin\left(\frac{k\pi x}{L}\right) dx$$

Example

- When a string is plucked in the middle, what sound will it make?



- This is a question about the amplitudes of the different normal modes of vibration.

$$y(x,t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right)\cos(\omega_k t)$$

$$\omega_k = \frac{k\pi v}{L}$$

$$a_k = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{k\pi x}{L}\right)dx$$

Example

- The initial shape of the string is the function:

$$f(x) = \begin{cases} 2Ax/L \ when \ x < L/2 \\ 2A - 2Ax/L \ when \ x > L/2 \end{cases}$$

- Fourier coefficients:

$$a_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^{L/2} f(x)\sin\left(\frac{k\pi x}{L}\right)dx$$

$$+ \frac{2}{L}\int_{L/2}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx$$

Example

- We have only two kinds of integrals:

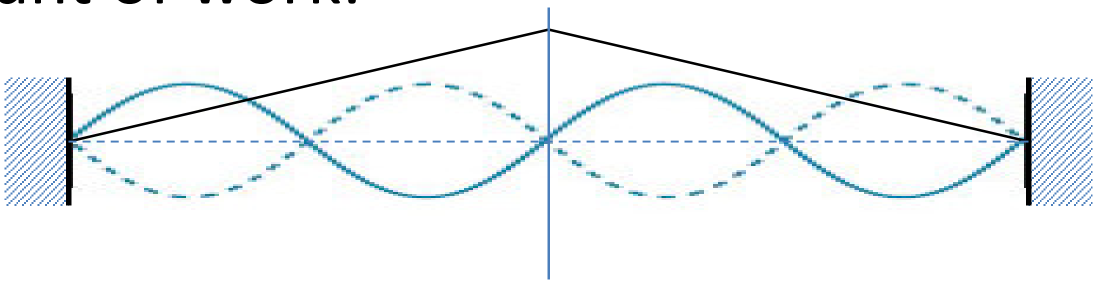
$$\int \sin\left(\frac{k\pi x}{L}\right)dx = -\frac{L}{k\pi}\cos\left(\frac{k\pi x}{L}\right)$$

$$\int x\sin\left(\frac{k\pi x}{L}\right)dx$$

$$= -\frac{L}{k\pi}\cos\left(\frac{k\pi x}{L}\right) + \frac{L^2}{k^2\pi^2}\sin\left(\frac{k\pi x}{L}\right)$$

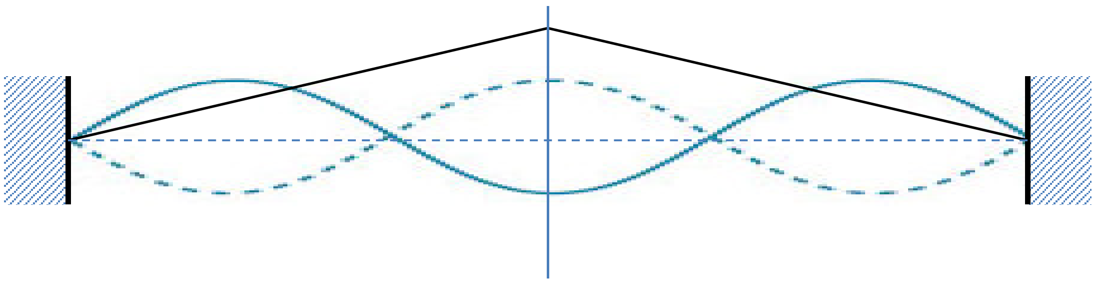
Example

- It is often useful to use symmetries to simplify the amount of work:



Left and right integrals will cancel.

$$a_2 = a_4 = a_6 = \cdots = 0$$



Left and right integrals are equal.

Example

$$a_k = \frac{8A}{L^2} \int_0^{L/2} x \sin\left(\frac{k\pi x}{L}\right) dx$$

- Use a table of integrals:

$$(91) \quad \int x \sin(ax) dx = -\frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax$$

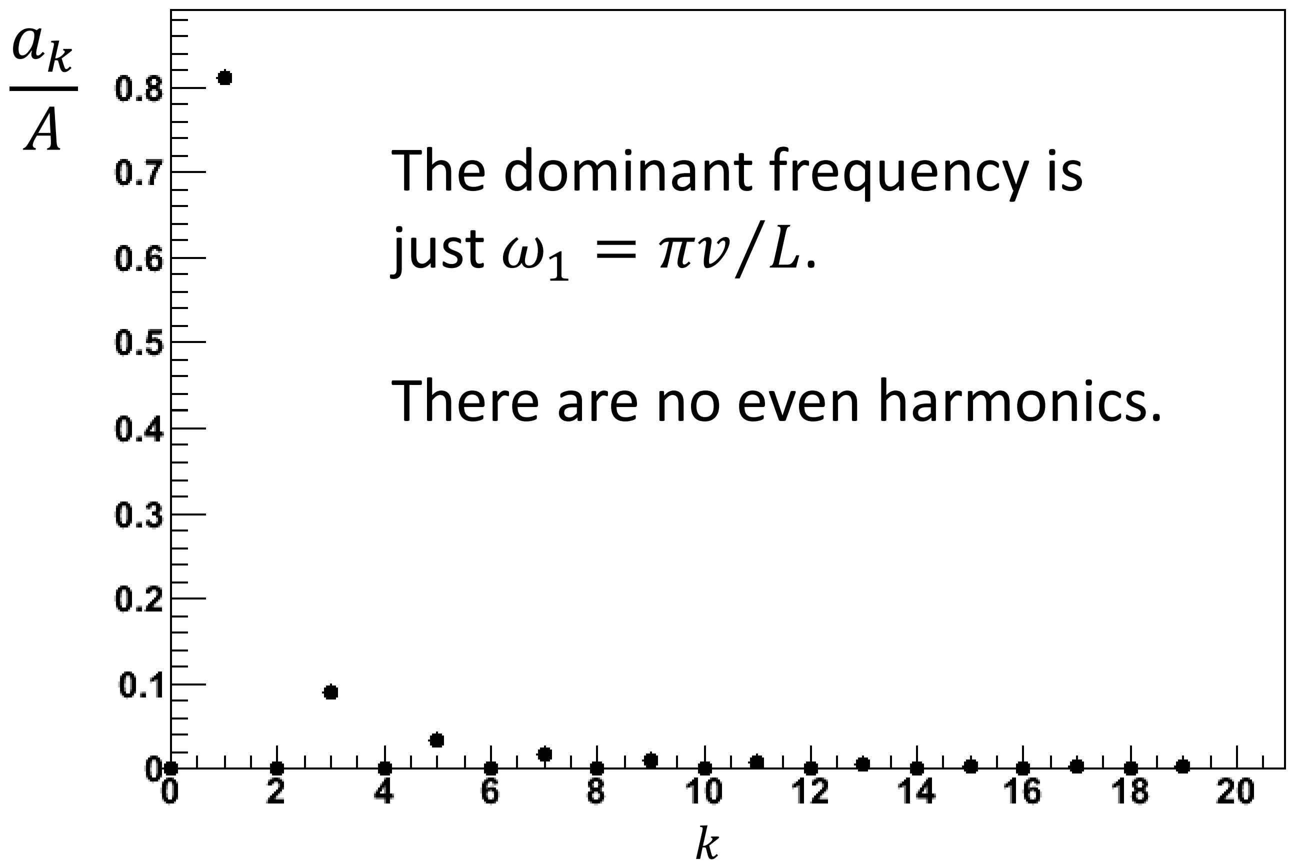
$$a_k = -\frac{4A}{k\pi} \cos\left(\frac{k\pi}{2}\right) + \frac{8A}{k^2\pi^2} \sin\left(\frac{k\pi}{2}\right)$$

- But we only care about $k = 1,3,5,7 \ldots$

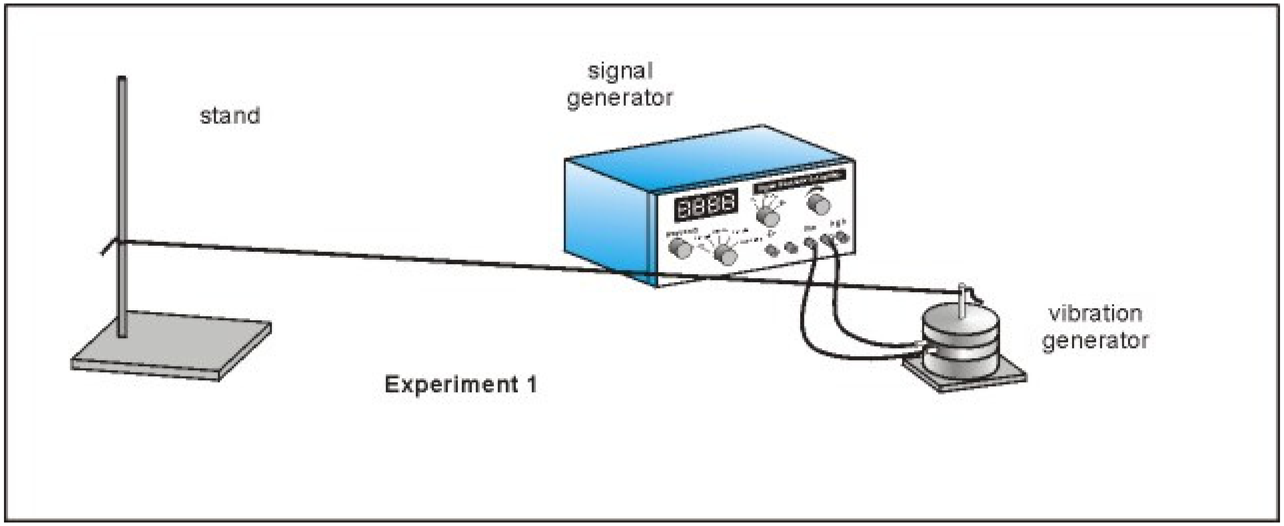
$$a_k = \frac{8A}{\pi^2}, -\frac{8A}{9\pi^2}, \frac{8A}{25\pi^2}, -\frac{8A}{49\pi^2}, \ldots$$

Example

- These are the amplitudes of each frequency component:



Forced Oscillations



- One end of the string is fixed, the other end is forced with the function $Y(t) = B\cos\omega t$.

$$y(0,t) = B\cos\omega t$$
$$y(L,t) = 0$$

- The wave equation still holds so we expect solutions to be of the form

$$y(x,t) = f(x)\cos\omega t$$

Forced Oscillations

- This time we can't constrain $f(x)$ to be zero at both ends.
- Now, let $f(x) = A\sin(kx + \alpha)$
  - The constant k is just ω/v.
  - We need to solve for A and α
- Boundary condition at $x = L$:

$$\sin\left(\frac{\omega L}{v} + \alpha\right) = 0 \quad \Rightarrow \quad \frac{\omega L}{v} + \alpha = p\pi$$

$$\alpha_p = p\pi - \frac{\omega L}{v}$$

- Condition at $x = 0$:

$$B = A_p \sin\alpha_p$$

Forced Oscillations

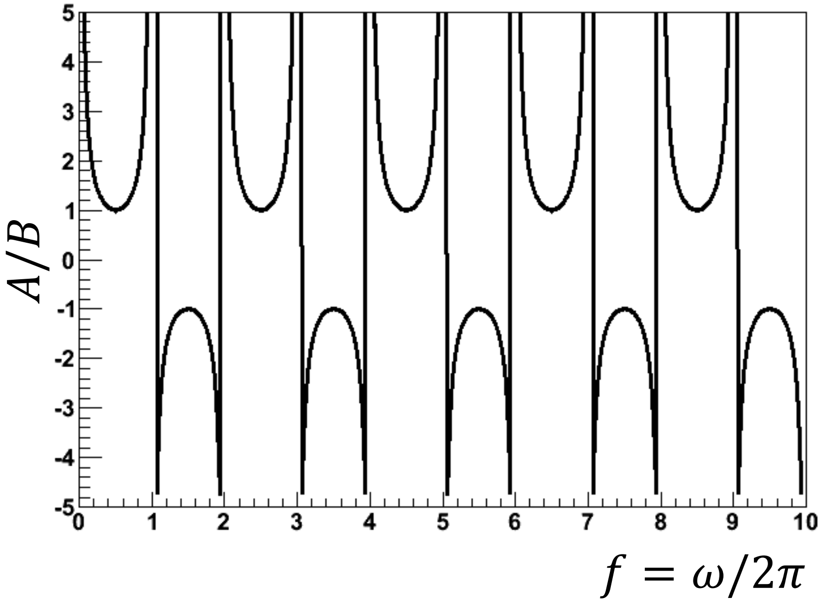
- Amplitude of oscillations:

$$A_p = \frac{B}{\sin(p\pi - \omega L/v)}$$

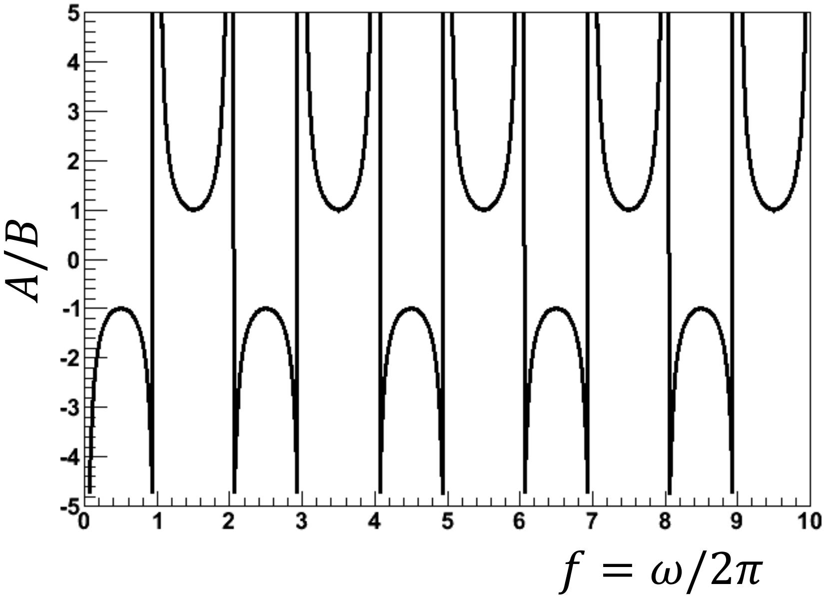
- What does this mean?
  - The driving force can excite many normal modes of oscillation
  - When $\omega = p\pi v/L$, the amplitude gets very large

Forced Oscillations

$p = 1$



$p = 2$

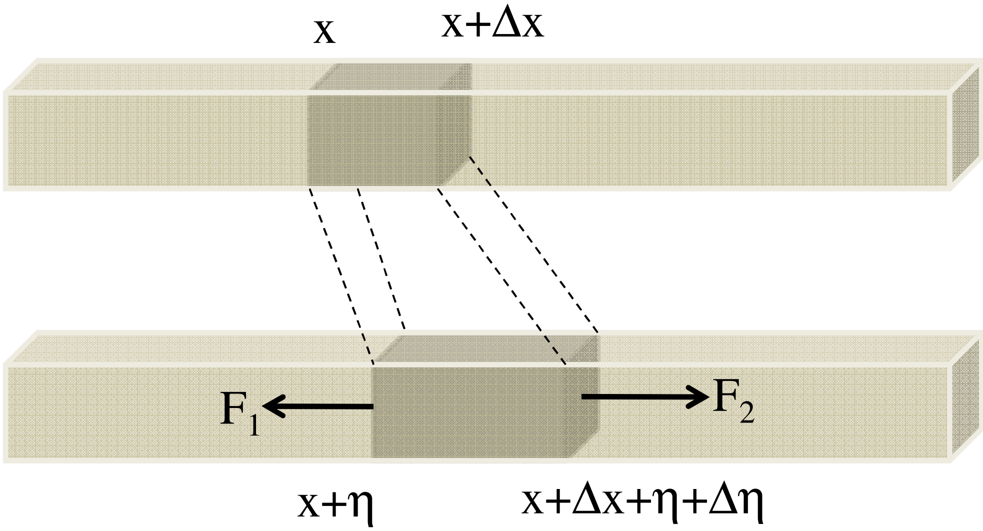


$$L = 5\,m$$

$$v = 10\,m/s$$

Other Continuous Systems

- Longitudinal waves in a solid rod:

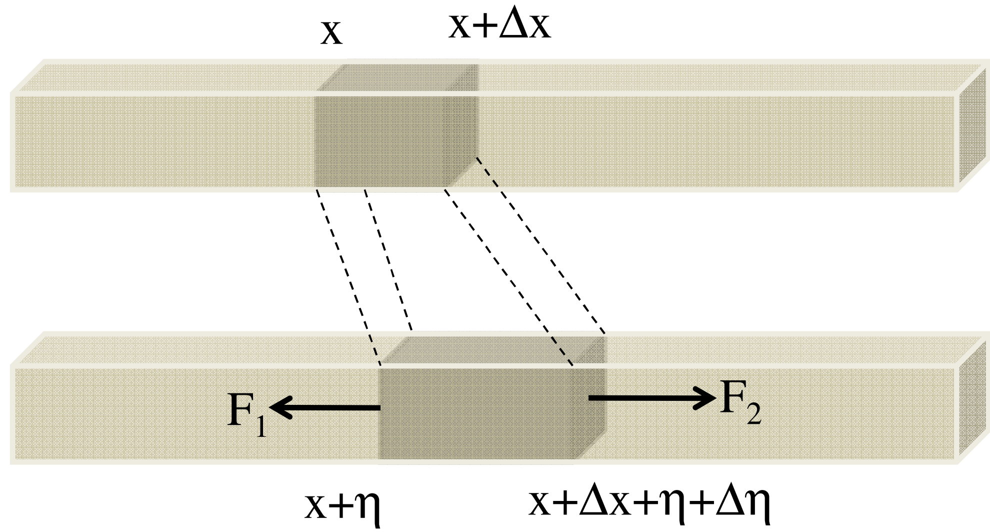


Notation:

- x labels which piece of the rod we are considering, analogous to the index n when counting discrete masses.
- η quantifies how much the element of mass has moved.

- Recall that strain was defined as the fractional increase in length of a small element: $\Delta\eta/\Delta x$
- Stress was defined as $\Delta F/A$
- These were related by $\Delta F/A = Y\ \Delta\eta/\Delta x$

Longitudinal Waves in a Solid Rod



$$\Delta F / A = Y\, \Delta\eta / \Delta x$$

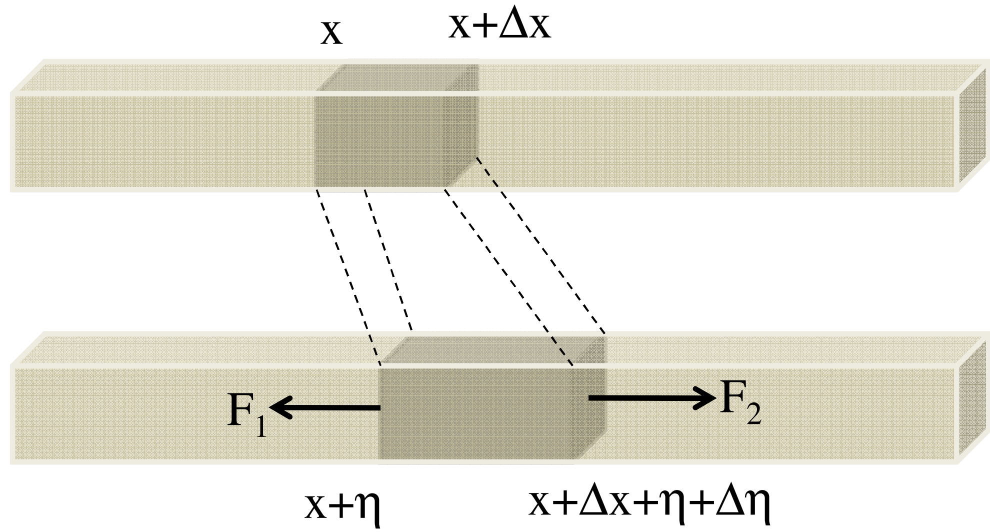
- Force on one side of the element:

$$F_1 = AY\, \Delta\eta / \Delta x = AY \partial\eta / \partial x$$

- Force on the other side of the element:

$$F_2 = F_1 + AY \frac{\partial^2 \eta}{dx^2} \Delta x$$

Longitudinal Waves in a Solid Rod



- Newton's law:

$$m\ddot{\eta} = F_2 - F_1$$

$$F_2 - F_1 = AY\frac{\partial^2\eta}{dx^2}\Delta x = \rho A\Delta x\ \frac{\partial^2\eta}{\partial t^2}$$

- Wave equation:

$$\frac{\partial^2\eta}{dx^2} = \frac{\rho}{Y}\frac{\partial^2\eta}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2}$$

Longitudinal Normal Modes

- What is the solution for a rod of length L?

$$\frac{\partial^2\eta}{dx^2} = \frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2} \qquad v = \sqrt{Y/\rho}$$

- Boundary conditions:
  - Suppose one end is fixed

$$\eta(0) = 0$$

  - No force at the free end of the rod so the stress is zero there. Strain ∝ stress, so the strain is also zero.

$$F = AY\partial\eta/\partial x$$

$$\frac{\partial\eta}{\partial x}_{x=L} = 0$$

- Look for solutions that are of the form

$$\eta(x) = f(x)\cos\omega t$$

Longitudinal Normal Modes

$$\eta(x) = f(x)\cos\omega t$$

- Inspired by the continuous string problem, we let

$$f(x) = A\sin(kx)$$

- Derivatives:

$$\frac{\partial^2\eta}{\partial x^2} = -k^2\eta$$

$$\frac{\partial^2\eta}{\partial t^2} = -\omega^2\eta$$

$$\frac{\partial^2\eta}{dx^2} = \frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2} \Rightarrow k = \frac{\omega}{v}$$

Longitudinal Normal Modes

$$f(x) = A\sin\left(\frac{\omega x}{v}\right)$$

- This automatically satisfies the boundary condition at $x = 0$.
- At $x = L$, $\partial\eta/\partial x = 0$:

$$\frac{\partial\eta}{\partial x}_{x=L} \propto \cos\left(\frac{\omega L}{v}\right) = 0$$

- This means that $\frac{\omega L}{v} = (n - \frac{1}{2})\pi$
- Angular frequencies of normal modes are

$$\omega_n = \frac{\pi}{L}\,(n - \frac{1}{2})\sqrt{Y/\rho}$$

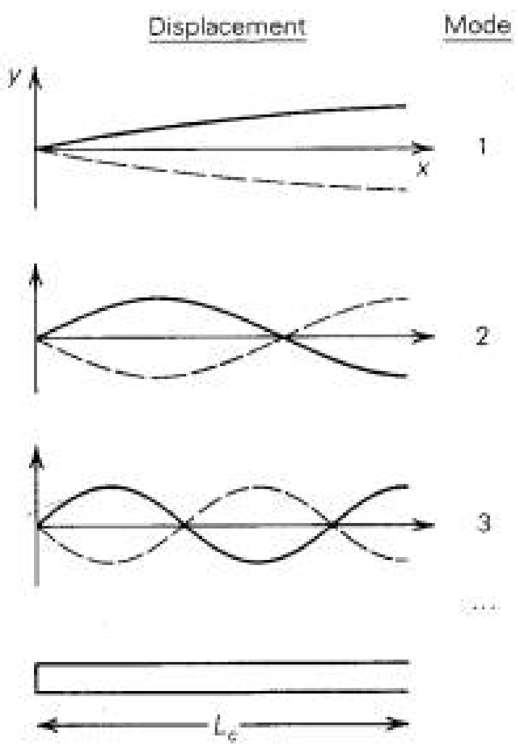
- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L}\sqrt{Y/\rho}$$

Longitudinal Normal Modes

- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L}\sqrt{Y/\rho}$$



Lowest possible frequency:

$$\nu_1 = \frac{1}{4L}\sqrt{\frac{Y}{\rho}}$$

Frequencies of Metal Chimes

- Suppose a set of chimes were made of copper rods, with lengths between 30 and 40 cm, rigidly fixed at one end.
- What frequencies should we expect if

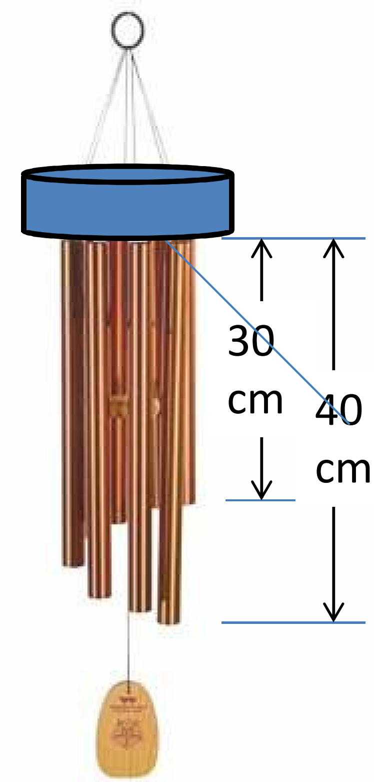
$$Y = 117 \times 10^9 \ \mathrm{N} \cdot m^{-2}$$

$$\rho = 8.96 \times 10^3 \ \mathrm{kg} \cdot m^{-3}$$

$$\nu_1 = \frac{1}{4L}\sqrt{\frac{117 \times 10^9 \ \mathrm{N} \cdot m^{-2}}{8.96 \times 10^3 \ \mathrm{kg} \cdot m^{-3}}}$$

$$= 2260 \ - 3010 \ Hz$$

(highest octave on a piano)



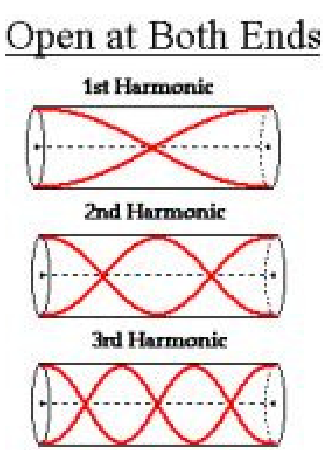
Frequencies of Metal Chimes

- If the metal rods were not fixed at one end then the boundary conditions at both ends would be:

$$\frac{\partial \eta}{\partial x} = 0$$

- Allowed frequencies of normal modes:

$$\nu_n = \frac{n}{2L}\sqrt{Y/\rho}$$



Harmonic	Wavelength λ	Frequency f
1st	$2L$	f_1
2nd	L	$2f_1$
3rd	$2L/3$	$3f_1$