Purdue University Department of Physics

# Physics 42200

# **Waves & Oscillations**

Lecture 14 – French, Chapter 6

Spring 2015 Semester

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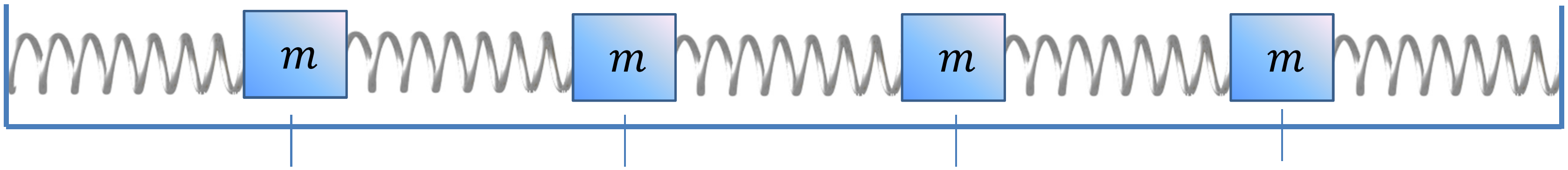
# Vibrations of Continuous Systems




- Equations of motion for masses in the middle:

$$m\,\ddot{x}_i + 2kx_i \; - k(x_{i-1} + x_{i+1}) = 0$$

$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0$$

- Proposed solution:

$$x_i(t) = A_i \cos \omega t$$

$$\frac{A_{i-1} + A_{i+1}}{A_i} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- We solved this to determine $A_i$ and $\omega_j$…

# Vibrations of Continuous Systems

- Amplitude of mass $n$ for normal mode $k$:

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

- Frequency of normal mode $k$:

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- Solution for normal modes:

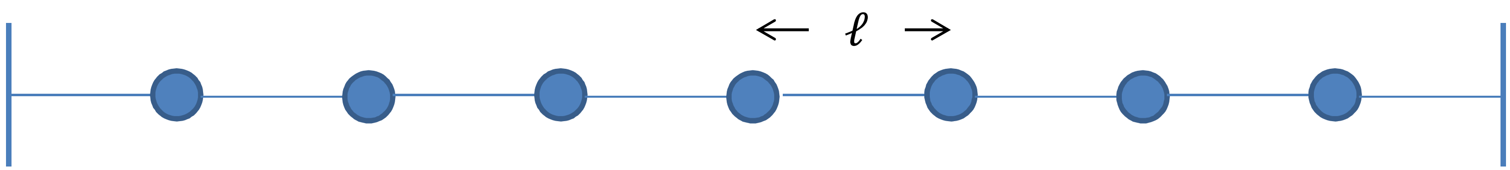
$$q_{n,k}(t) = A_{n,k} \cos \omega_k t$$

- General solution:

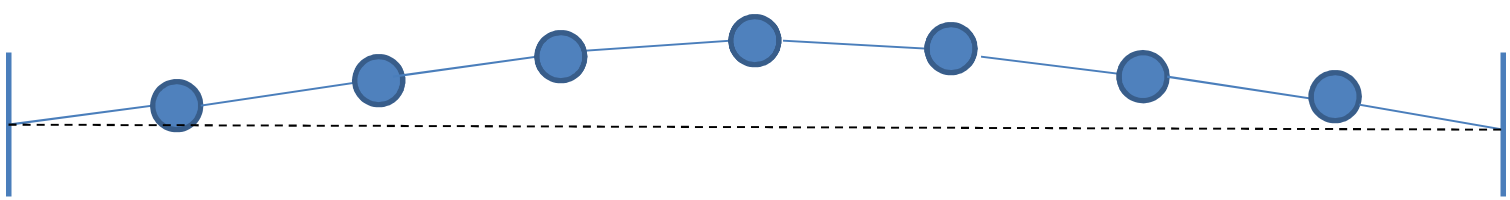
$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \theta_k)$$

# Another Example

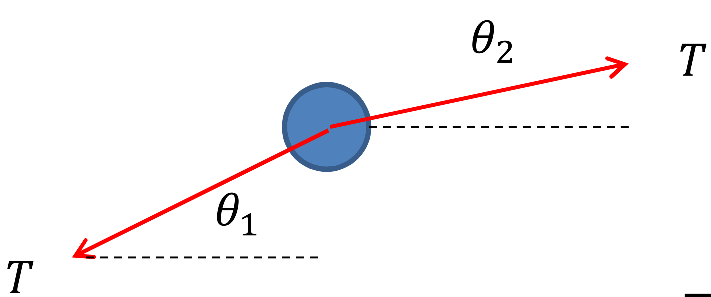
- Discrete masses on an elastic string with tension $T$:




- Consider transverse displacements:

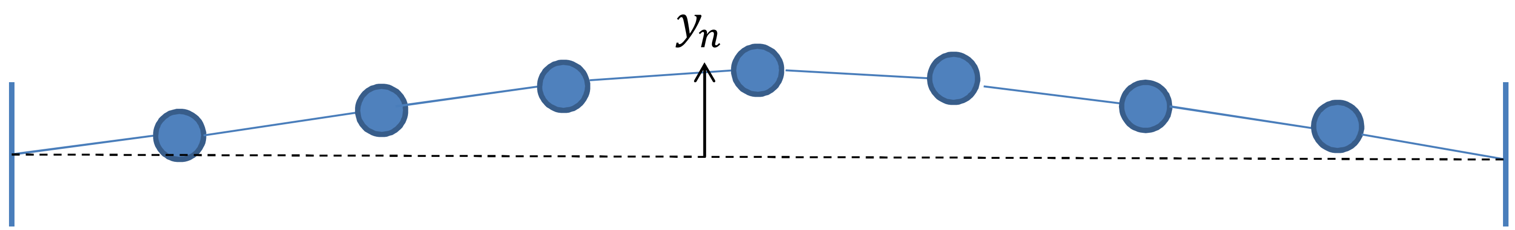


- Vertical force on one mass:




$$F_n = T\sin\theta_2 - T\sin\theta_1$$
$$= T(\theta_2 - \theta_1)$$
$$= \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

# Another Example




- Equation of motion for mass $n$:

$$m\,\ddot{y}_n = F_n = \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

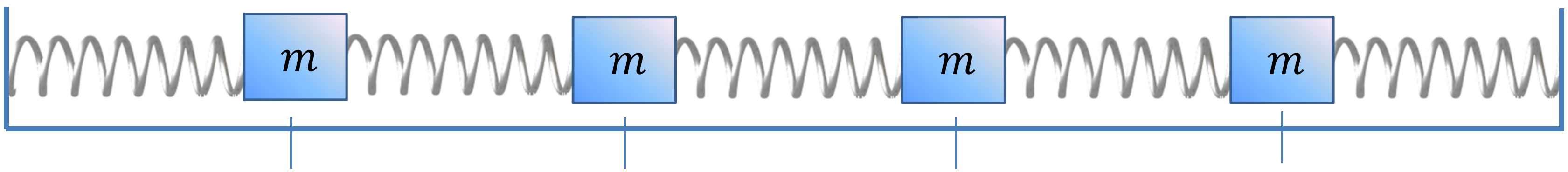
$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = \frac{T}{m\ell}$$

- Normal modes:

$$y_{n,k}(t) = A_{n,k}\cos(\omega_k t - \theta_k)$$

# Example

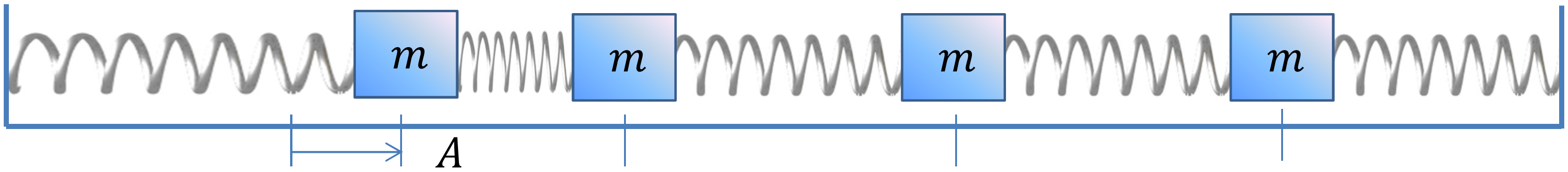



- Solutions are of the form

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_k t - \theta_k)$$

- The constants $a_k$ and $\theta_k$ must be chosen to satisfy the initial conditions.
- Consider, for example, an initial state where all masses are in their equilibrium position except for the mass at $x_1$ which is initially displaced by a distance $A$ ...

# Example




- Consider, for example, an initial state where all masses are in their equilibrium position except for the mass at $x_1$ which is initially displaced by a distance $A$ ...

$$x_1(0) = \sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{N+1}\right) \cos\theta_k = A$$

$$x_2(0) = \sum_{k=1}^{N} a_k \sin\left(\frac{2k\pi}{N+1}\right) \cos\theta_k = 0$$

$$\vdots$$

$$x_N(0) = \sum_{k=1}^{N} a_k \sin\left(\frac{Nk\pi}{N+1}\right) \cos\theta_k = 0$$

$$\dot{x}_n(0) = 0$$

# Example

- We have $2N$ equations
  - initial positions of $N$ masses
  - initial velocities of $N$ masses
- We have $2N$ unknowns: $a_k$ and $\theta_k$
- How do we solve this linear system of equations?
- Properties of the normal modes:
  - Eigenvalues: $\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$
  - Eigenvectors: $A_{n,k} = \sin\left(\frac{nk\pi}{N+1}\right)$
- Eigenvectors are orthogonal:

$$\sum_{k=1}^{N} A_{n,k} A_{m,k} = 0 \text{ when } n \neq m$$

# Discrete Sine Transform

- The eigenvectors are orthogonal so it must be true that

$$\sum_{k=1}^{N} \sin\left(\frac{nk\pi}{N+1}\right) \sin\left(\frac{mk\pi}{N+1}\right) = 0$$

when $n \neq m$.

This term sums to zero…

- When $n = m$ we just have

$$\sum_{k=1}^{N} \sin^2\left(\frac{nk\pi}{N+1}\right) = \sum_{k=1}^{N} \frac{1}{2}\left(1 + \cos\left(\frac{2nk\pi}{N+1}\right)\right) = \frac{N}{2}$$

# Discrete Sine Transform

- We can summarize this in a useful form:

$$\sum_{k=1}^{N} \sin\left(\frac{nk\pi}{N+1}\right) \sin\left(\frac{mk\pi}{N+1}\right) = \frac{N}{2}\ \delta_{nm}$$

- The symbol $\delta_{nm}$ is called the Kronecker Delta:

$$\delta_{nm} = \begin{cases} 0 & \text{when } n \neq m \\ 1 & \text{when } n = m \end{cases}$$

- How will this help us solve for the constants of integration, given the initial conditions?

# Example

- General solution:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_k t - \theta_k)$$

- At time $t = 0$,

$$x_n(0) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\theta_k)$$

- Consider the expression:

$$\sum_{n=1}^{N} x_n(0) \sin\left(\frac{nk'\pi}{N+1}\right) = \sum_{n,k=1}^{N} a_k \cos\theta_k \sin\left(\frac{nk'\pi}{N+1}\right)\sin\left(\frac{nk\pi}{N+1}\right)$$

$$= \frac{N}{2}\sum_{k=1}^{N} a_k \cos\theta_k \, \delta_{k'k} = \frac{N}{2} a_{k'} \cos\theta_{k'}$$

# Example

- Likewise, consider the time derivatives:

$$\dot{x}_n(t) = -\sum_{k=1}^{N} a_k \omega_k \sin\left(\frac{nk\pi}{N+1}\right) \sin(\omega_k t - \theta_k)$$

$$\dot{x}_n(0) = \sum_{k=1}^{N} \underbrace{a_k \omega_k \sin\theta_k}_{\text{constants}} \sin\left(\frac{nk\pi}{N+1}\right)$$

$$\sum_{n=1}^{N} \dot{x}_n(0) \sin\left(\frac{nk'\pi}{N+1}\right) = \frac{N}{2} a_{k'} \omega_{k'} \sin\theta_{k'}$$

- If the initial velocities were all zero, then

$$\theta_k = 0 \text{ for } k = 1, \dots, N$$

# Example

- Now we know that $\theta_k$ are all zero…

$$\sum_{n=1}^{N} x_n(0)\sin\left(\frac{nk'\pi}{N+1}\right) = \frac{N}{2}a_{k'}$$

$$a_k = \frac{2}{N}\sum_{n=1}^{N} x_n(0)\sin\left(\frac{nk\pi}{N+1}\right)$$

- In this example, $x_1(0) = A, x_{n\neq 1}(0) = 0$
- Therefore,

$$a_k = \frac{2A}{N}\sin\left(\frac{k\pi}{N+1}\right)$$

And we're done!

# A slightly different example…

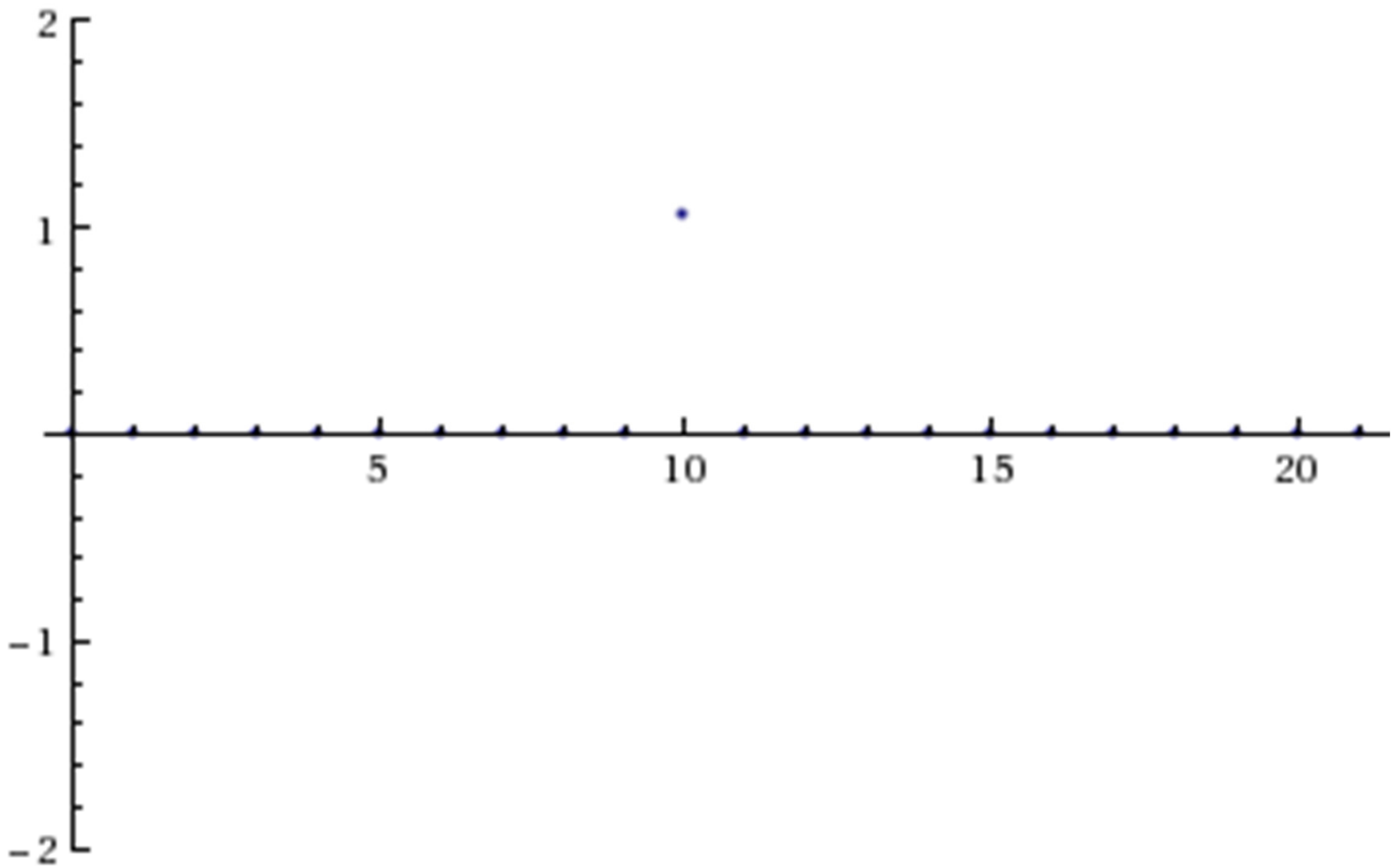
- Instead of the mass at one end being initially displaced, suppose it was the mass in the middle. In this case,

$$a_k = \frac{2A}{N} \sin\left(\frac{(N/2)k\pi}{N+1}\right)$$

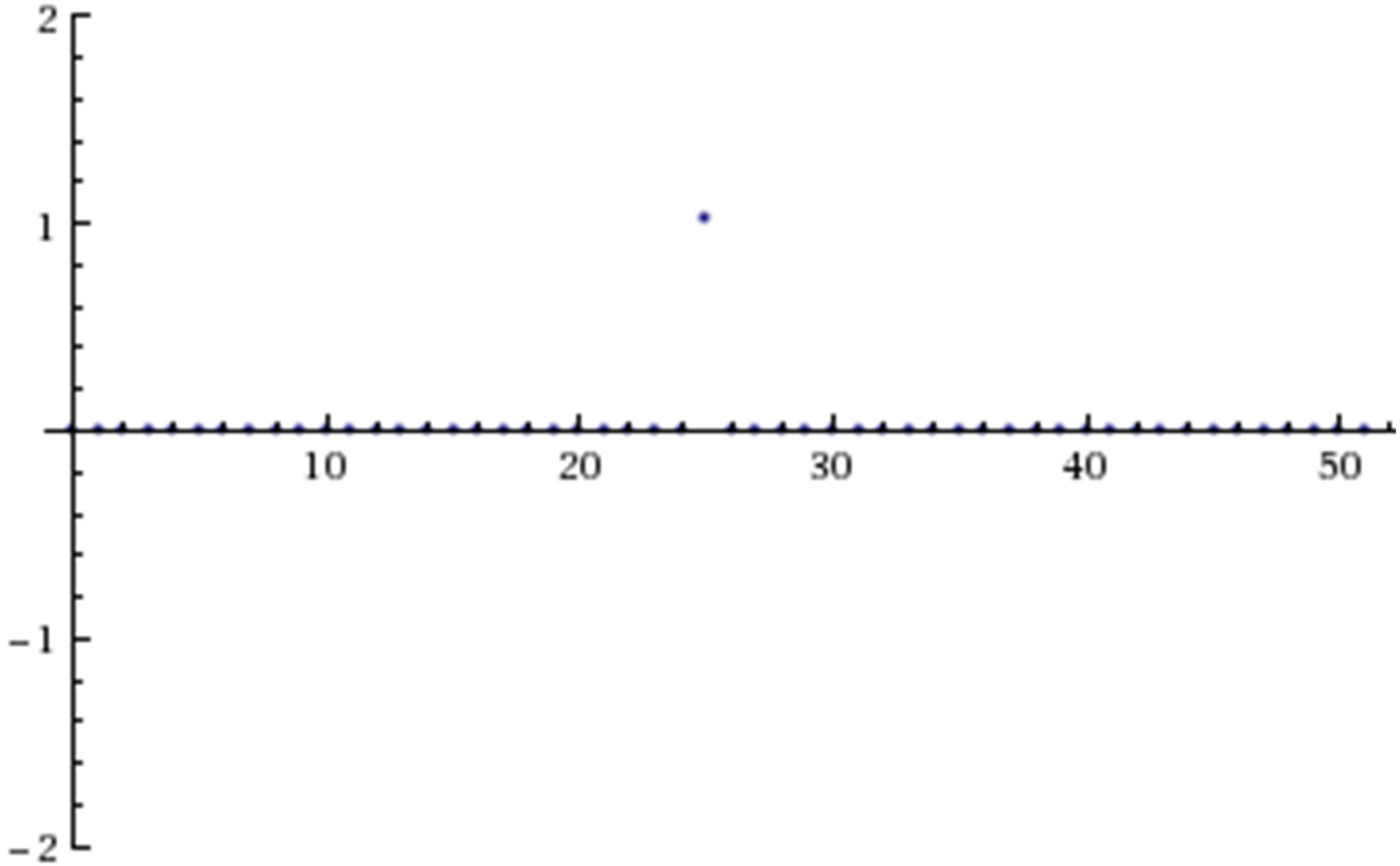
$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t)$$

# Example with N=20



# Example with N=50



# Review

- We calculated the eigenvalues for a system with $N$ identical masses

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- We found the normal modes of vibration (eigenvectors):

$$A_{n,k} = \sin\left(\frac{nk\pi}{N+1}\right)$$

- The general form of the solution is

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_k t - \theta_k)$$

# Review

- We determined the constants of integration from the initial conditions:

$$a_k \cos\theta_k = \frac{2}{N}\sum_{n=1}^{N} x_n(0)\sin\left(\frac{nk\pi}{N+1}\right)$$

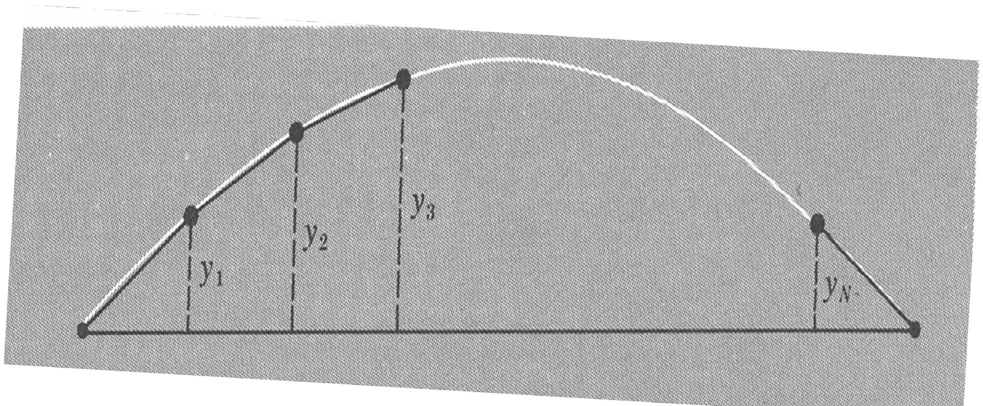
$$a_k \sin\theta_k = \frac{2}{N\omega_k}\sum_{n=1}^{N} \dot{x}_n(0)\sin\left(\frac{nk\pi}{N+1}\right)$$

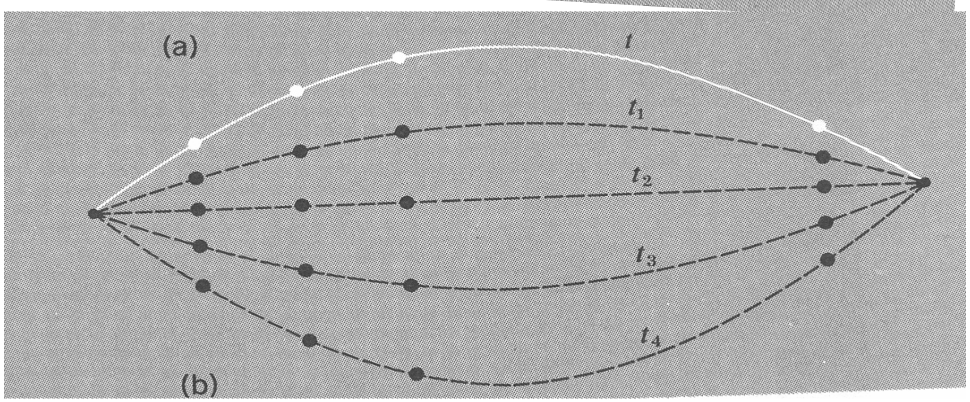
- Put these back into the general form of the solution:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right)\cos(\omega_k t - \theta_k)$$

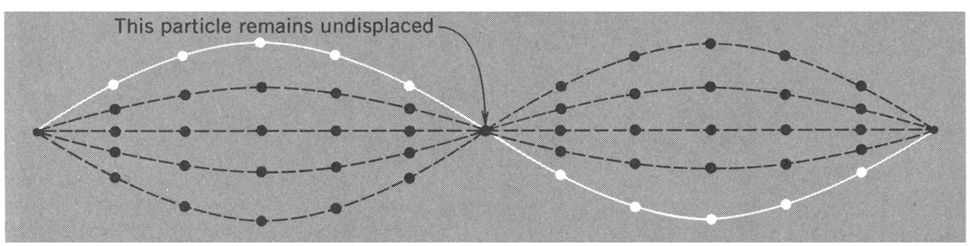
And we're done…

# Masses on a String




First normal mode




Second normal mode

# Continuous Systems

- What happens when the number of masses goes to infinity, while the linear mass density remains constant?

$$m\,\ddot{y}_n = \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

$$\frac{m}{\ell} \to \mu$$

$$\frac{y_{n+1} - y_n}{\ell} \to \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \qquad \frac{(y_n - y_{n-1})}{\ell} \to \left(\frac{\partial y}{\partial x}\right)_x$$

$$\mu\ell\,\frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

# Continuous Systems

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\ell}$$

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}$$

The Wave Equation: $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$ $$v = \sqrt{T/\mu}$$

# Solutions

- When we had $N$ masses, the solutions were
$$y_{n,k}(t) = A_{n,k} \cos(\omega_k t - \delta_k)$$
  - $n$ labels the mass along the string
  - With a continuous system, $n$ is replaced by $x$.
- Proposed solution to the wave equation for the continuous string:
$$y(x,t) = f(x) \cos \omega t$$
- Derivatives:
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x) \cos \omega t$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} \cos \omega t$$

# Solutions

- Substitute into the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\omega^2}{v^2} f(x)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{v^2} f(x) = 0$$

- This is the same differential equation as for the harmonic oscillator.
- Solutions are $f(x) = A\sin(\omega x/v) + B\cos(\omega x/v)$

# Solutions

$$f(x) = A\sin(\omega x/v) + B\cos(\omega x/v)$$

- Boundary conditions at the ends of the string:

$$f(0) = f(L) = 0$$

$$f(x) = A\sin(\omega x/v) \text{ where } \omega L/v = n\pi$$

- Solutions can be written:

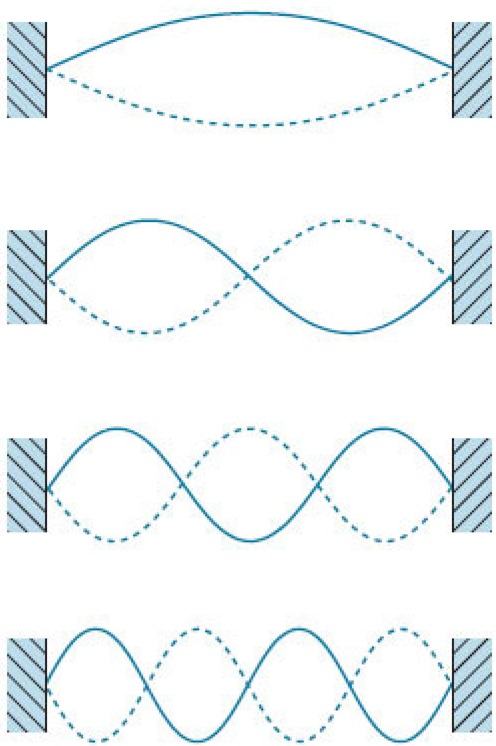
$$f_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right)$$

- Complete solution describing the motion of the whole string:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t$$

# Properties of the Solutions

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t$$



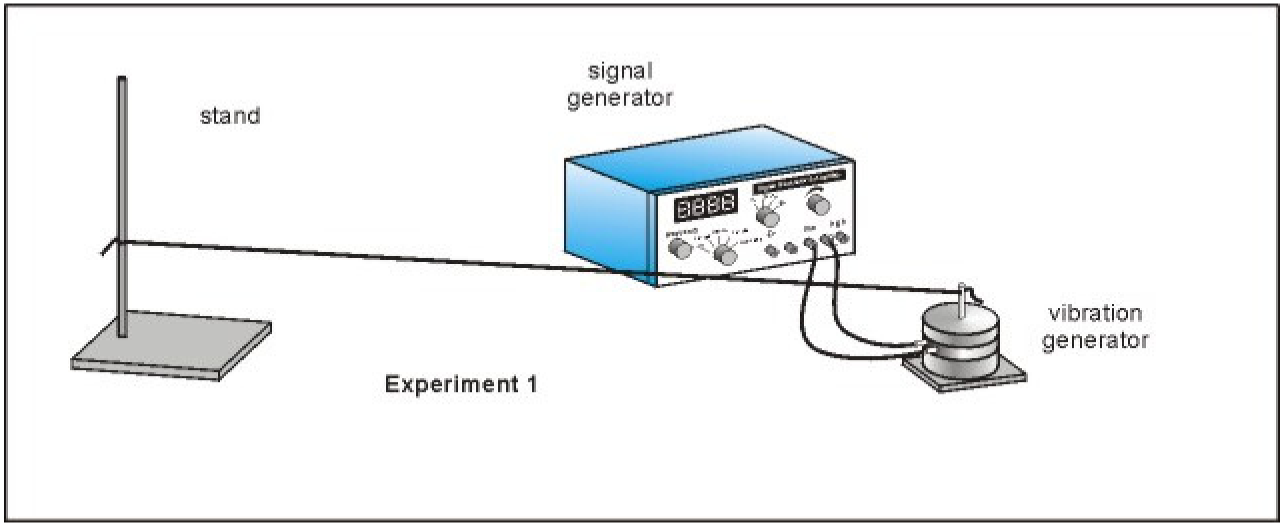
| mode | wavelength | frequency |
|---|---|---|
| first | $2L$ | $\frac{v}{2L}$ |
| second | $L$ | $\frac{v}{L}$ |
| third | $\frac{2L}{3}$ | $\frac{3v}{2L}$ |
| fourth | $\frac{L}{2}$ | $\frac{2v}{L}$ |

$$\lambda_n = \frac{2L}{n}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

# Forced Oscillations




- One end of the string is fixed, the other end is forced with the function $Y(t) = B\cos\omega t$.

$$y(0,t) = B\cos\omega t$$
$$y(L,t) = 0$$

- The wave equation still holds so we expect solutions to be of the form

$$y(x,t) = f(x)\cos\omega t$$

# Forced Oscillations

- This time we can't constrain $f(x)$ to be zero at both ends.
- Now, let $f(x) = A\sin(kx + \alpha)$
  - The constant $k$ is just $\omega/v$.
  - We need to solve for $A$ and $\alpha$
- Boundary condition at $x = L$:

$$\sin\left(\frac{\omega L}{v} + \alpha\right) = 0 \quad \Rightarrow \quad \frac{\omega L}{v} + \alpha = p\pi$$

$$\alpha_p = p\pi - \frac{\omega L}{v}$$

- Condition at $x = 0$:

$$B = A_p \sin\alpha_p$$

# Forced Oscillations

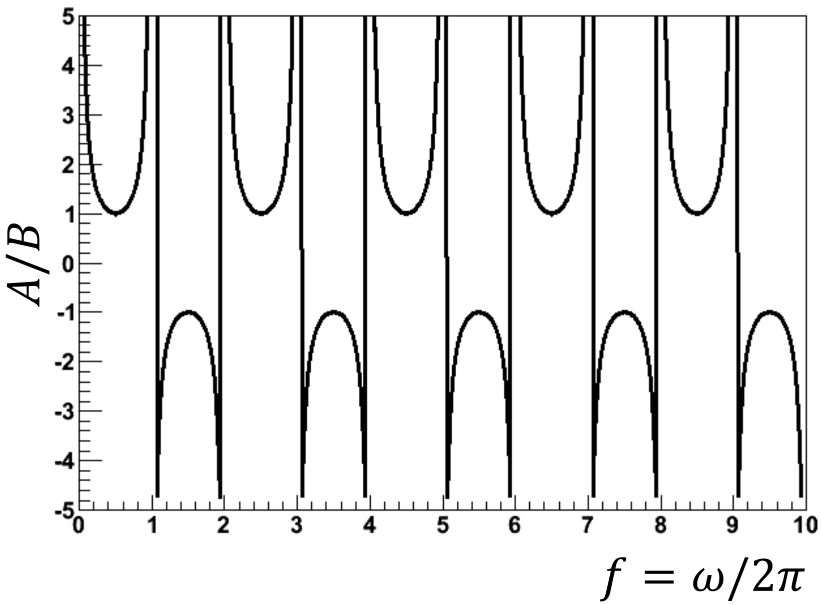
- Amplitude of oscillations:

$$A_p = \frac{B}{\sin(p\pi - \omega L/v)}$$

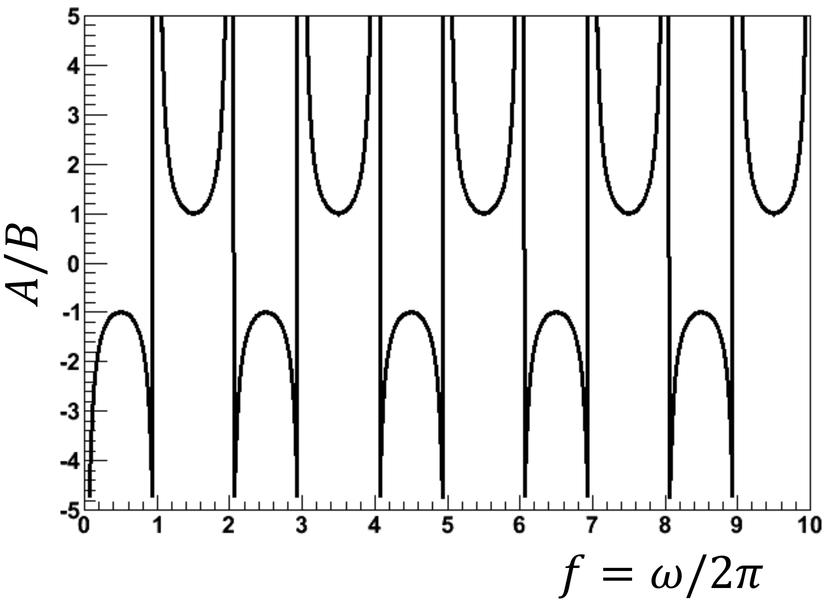
- What does this mean?
  - The driving force can excite many normal modes of oscillation
  - When $\omega = p\pi v/L$, the amplitude gets very large

# Forced Oscillations

$p = 1$




$p = 2$




$$L = 5\,m$$

$$v = 10\,m/s$$