

Physics 42200
Waves & Oscillations

Lecture 12 – French, Chapter 5

Spring 2015 Semester

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The Eigenvalue Problem

- If A is an $n \times n$ matrix and $\vec{u}$ is a vector, find the numbers λ that satisfy

$$A \vec{u} = \lambda \vec{u}$$

- Re-write the equation this way:

$$(A - \lambda I) \vec{u} = 0$$

- This is true only if

$$\det(A - \lambda I) = 0$$

- For a 2×2 matrix, this is:

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = 0$$

- This is a second order polynomial in λ . Use the quadratic formula to find the roots.

The Eigenvalue Problem

- The eigenvectors are vectors $\vec{u}_i$ such that
$$(A - \lambda_i I)\vec{u}_i = 0$$
- There are n eigenvalues and n eigenvectors
- If $\vec{u}_i$ is an eigenvector, then $\alpha\vec{u}_i$ is also an eigenvector.
- Sometimes it is convenient to choose the eigenvectors so that they have unit length:

$$\hat{u}_i \cdot \hat{u}_i = 1$$

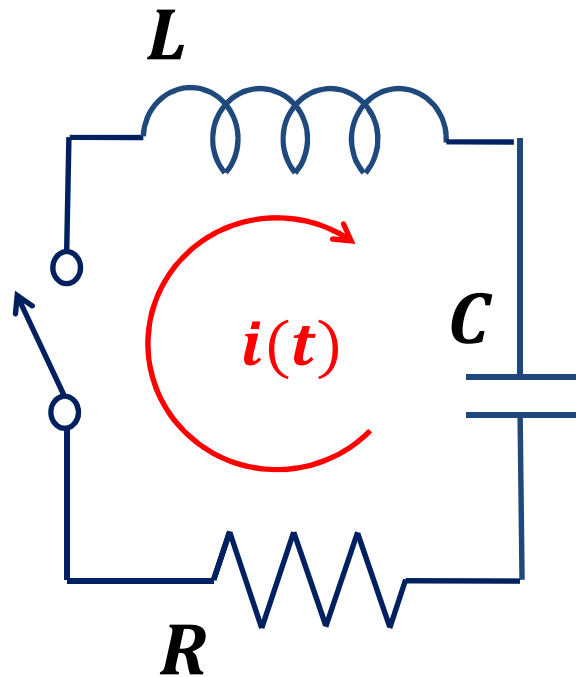
- Eigenvectors are orthogonal:

$$\vec{u}_i \cdot \vec{u}_j = 0 \text{ when } i \neq j$$

- An arbitrary vector $\vec{v}$ can be written as a linear combination of the eigenvectors:

$$\vec{v} = a_1 \hat{u}_1 + a_2 \hat{u}_2 + \cdots$$

A Circuit with One Loop



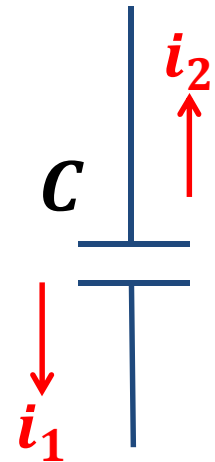
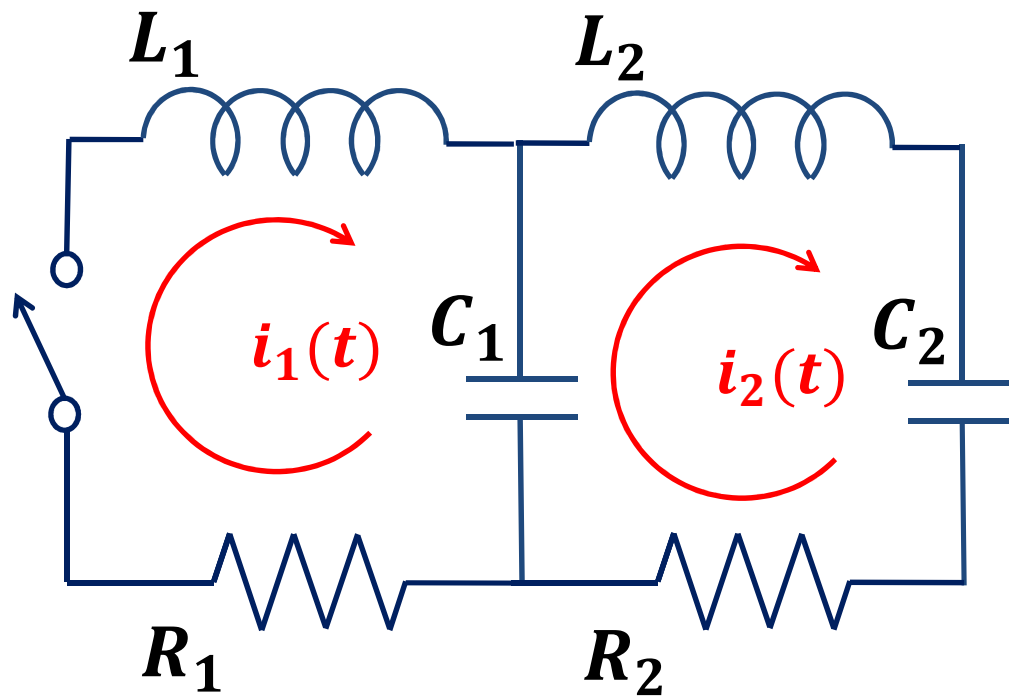
$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0$$

$$\frac{d^2 i}{dt^2} + \gamma \frac{di}{dt} + (\omega_0)^2 i(t) = 0$$

A diagram of a capacitor labeled C with a red arrow pointing downwards through it, labeled $i(t)$.

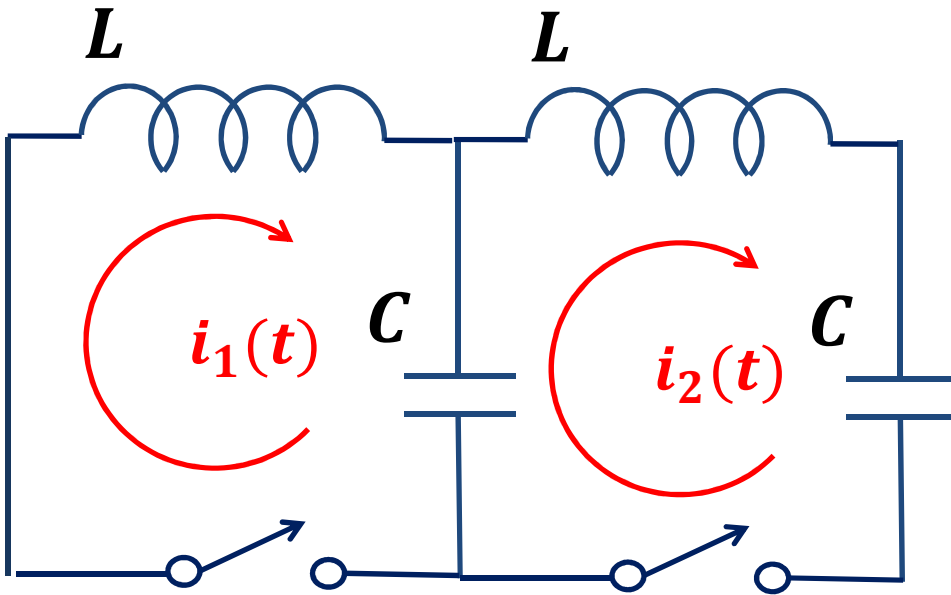
$$-\frac{1}{C} \int i(t) dt$$

A Circuit with Two Loops



$$-\frac{1}{C} \int (i_1 - i_2) dt$$

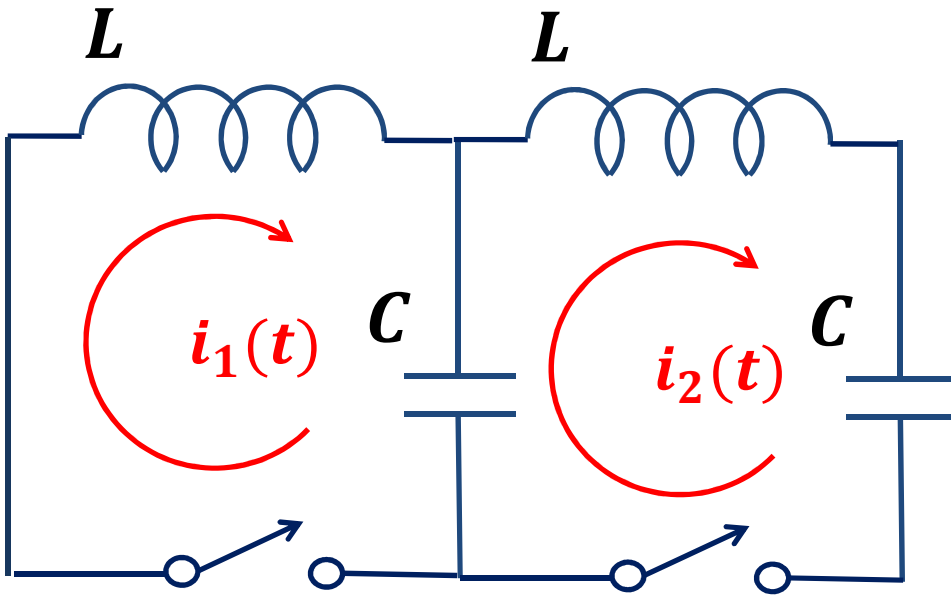
Example



$$-L \frac{di_1}{dt} - \frac{1}{C} \int (i_1 - i_2) dt = 0$$

$$-L \frac{di_2}{dt} - \frac{1}{C} \int i_2 dt - \frac{1}{C} \int (i_2 - i_1) dt = 0$$

Example



$$\frac{d^2 i_1}{dt^2} + (\omega_0)^2 (i_1 - i_2) = 0$$
$$\frac{d^2 i_2}{dt^2} + (\omega_0)^2 (2i_2 - i_1) = 0$$

Normal Modes of Oscillation

- What are the frequencies of the normal modes of oscillation?

- Let $\vec{l}(t) = \vec{l} \cos \omega t$

- Then $\frac{d^2 \vec{l}}{dt^2} = -\omega^2 \vec{l}(t)$

- Substitute into the pair of differential equations:

$$(-\omega^2 + (\omega_0)^2)i_1 - (\omega_0)^2 i_2 = 0$$

$$(-\omega^2 + 2(\omega_0)^2)i_2 - (\omega_0)^2 i_1 = 0$$

- Write it as a matrix:

$$\begin{pmatrix} (\omega_0)^2 - \omega^2 & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

Eigenvalue Problem

$$\begin{pmatrix} (\omega_0)^2 - \omega^2 & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

- For simplicity, let $\lambda = \omega^2$ and calculate the determinant:

$$\begin{vmatrix} (\omega_0)^2 - \lambda & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \lambda \end{vmatrix} = (\lambda - (\omega_0)^2)(\lambda - 2(\omega_0)^2) - (\omega_0)^4 \\ = \lambda^2 - 3\lambda(\omega_0)^2 + (\omega_0)^4 = 0$$

- Roots of the polynomial:

$$\lambda = \frac{3}{2}(\omega_0)^2 \pm \frac{1}{2}\sqrt{9(\omega_0)^4 - 4(\omega_0)^4} \\ \omega^2 = (\omega_0)^2 \left(\frac{3 \pm \sqrt{5}}{2} \right)$$

Eigenvalue Problem

- The eigenvectors are obtained by substituting in each eigenvalue.

- When $\omega^2 = (\omega_0)^2 \left(\frac{3+\sqrt{5}}{2} \right)$

$$\frac{(\omega_0)^2}{2} \begin{pmatrix} -1 - \sqrt{5} & -2 \\ -2 & 1 - \sqrt{5} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

$$i_1 = \left(\frac{1 - \sqrt{5}}{2} \right) i_2$$

- First normal mode of oscillation:

$$\vec{q}_1 = \mathbf{A} \begin{pmatrix} 1 - \sqrt{5} \\ 2 \end{pmatrix} \cos(\omega_1 t + \alpha)$$

Eigenvalue Problem

- The eigenvectors are obtained by substituting in each eigenvalue.

- When $\omega^2 = (\omega_0)^2 \left(\frac{3-\sqrt{5}}{2} \right)$

$$\frac{(\omega_0)^2}{2} \begin{pmatrix} -1 + \sqrt{5} & -2 \\ -2 & 1 + \sqrt{5} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

$$i_1 = \left(\frac{1 + \sqrt{5}}{2} \right) i_2$$

- Second normal mode of oscillation:

$$\vec{q}_2 = \mathbf{B} \begin{pmatrix} 1 + \sqrt{5} \\ 2 \end{pmatrix} \cos(\omega_2 t + \boldsymbol{\beta})$$

Eigenvalue Problem

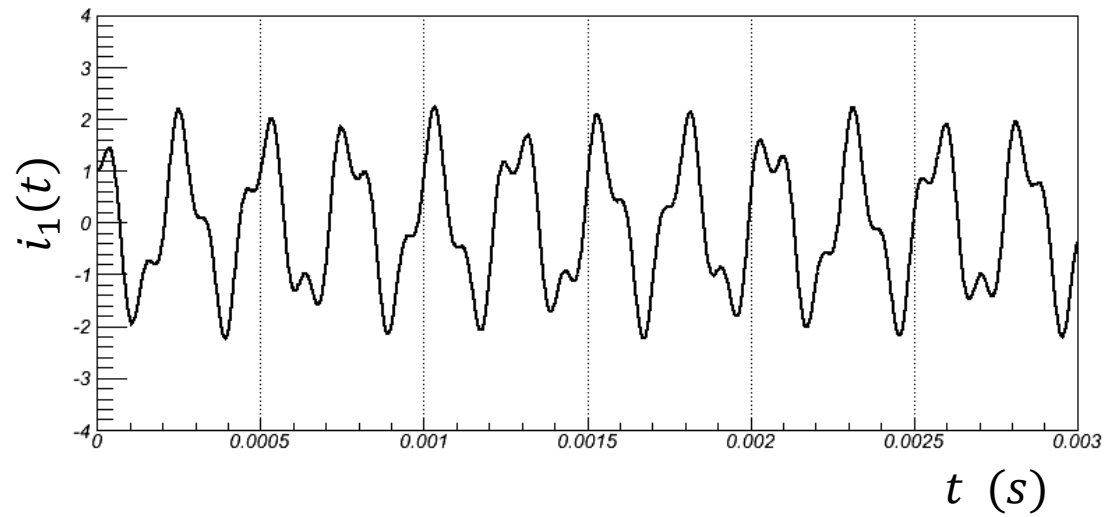
- The original “coordinates” are the sum of the normal modes of oscillation:

$$i_1(t) = A(1 - \sqrt{5}) \cos(\omega_1 t + \alpha) + B(1 + \sqrt{5}) \cos(\omega_2 t + \beta)$$

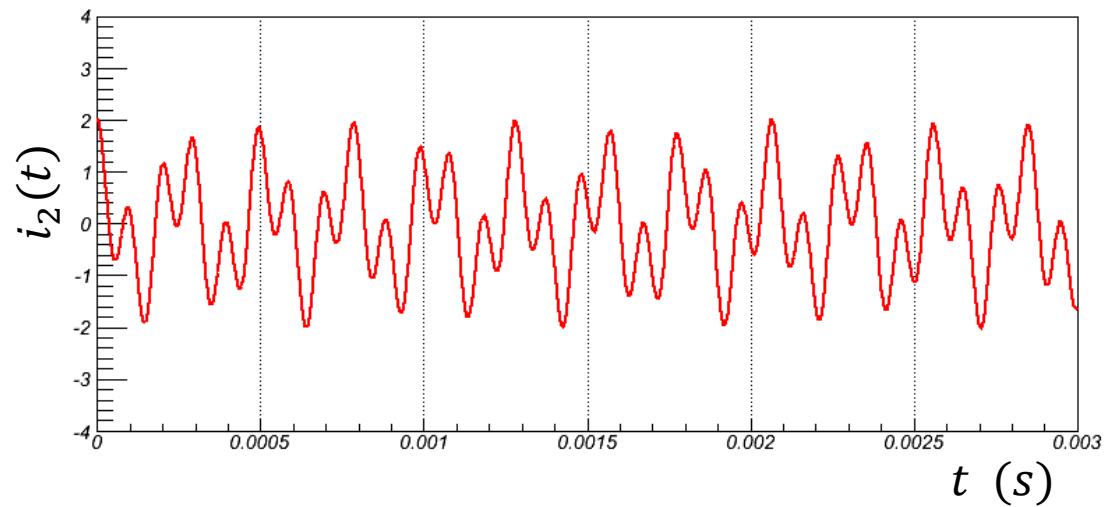
$$i_2(t) = 2A \cos(\omega_1 t + \alpha) + 2B \cos(\omega_2 t + \beta)$$

- The constants of integration can be chosen to satisfy the initial conditions
 - For example, suppose that $i_1(0) = i_0$ and $i_2(0) = 0$
 - Then $A = -B$, $2A = i_0 \rightarrow A = \frac{i_0}{2}$, $B = -\frac{i_0}{2}$

Two Loop Circuit

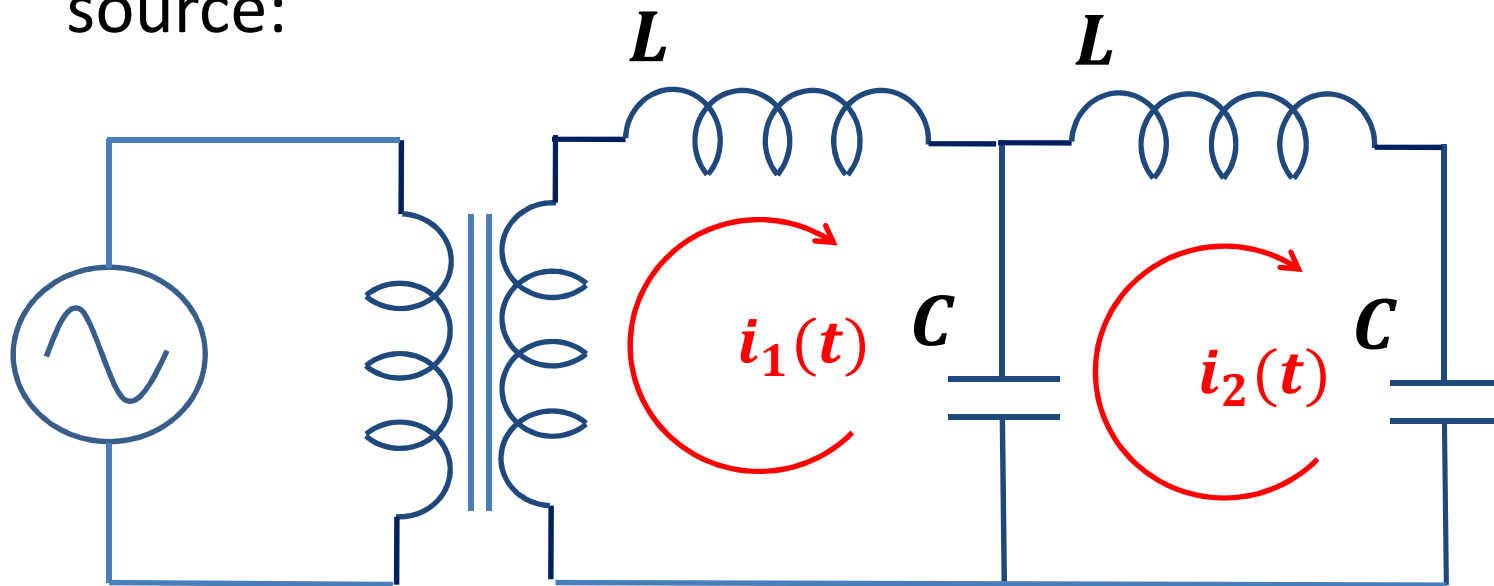


$$f_0 = \frac{\omega_0}{2\pi} = 1 \text{ kHz}$$
$$i_0 = 1 \text{ A}$$



Forced Coupled Circuit

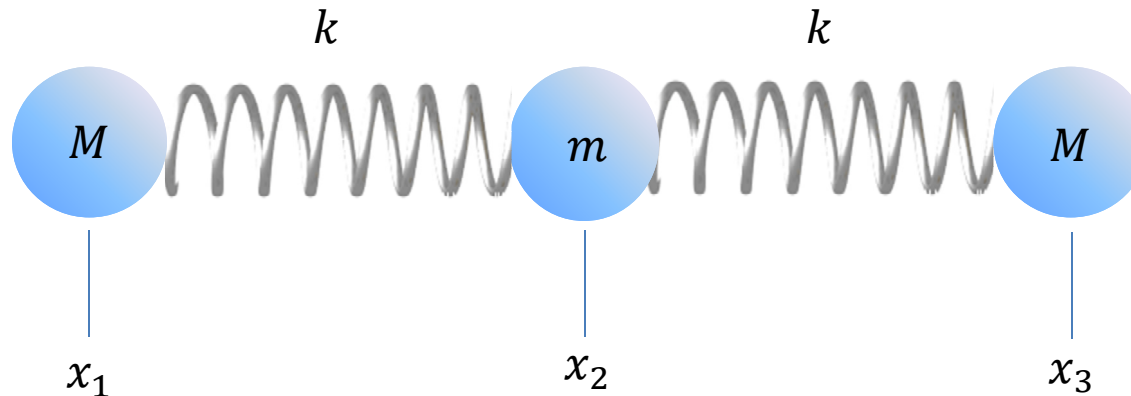
- If the two loops were driven with a sinusoidal voltage source:



- Resonance would occur at the frequency of each normal mode:

$$\omega^2 = (\omega_0)^2 \left(\frac{3 \pm \sqrt{5}}{2} \right)$$

Three Masses



- Forces on each mass:

$$F_1 = -k(x_1 - x_2)$$

$$F_2 = k(x_1 - x_2) - k(x_2 - x_3) = kx_1 - 2kx_2 + kx_3$$

$$F_3 = -k(x_3 - x_2)$$

Three Masses

- Equations of motion:

$$F_1 = M\ddot{x}_1 = -k(x_1 - x_2)$$

$$F_2 = m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3$$

$$F_3 = M\ddot{x}_3 = -k(x_3 - x_2)$$

- Let $\omega_0^2 = \frac{k}{M}$ and $\omega_0'^2 = \frac{k}{m}$

- Then,

$$\ddot{x}_1 + \omega_0^2 x_1 - \omega_0^2 x_2 = 0$$

$$\ddot{x}_2 - \omega_0'^2 x_1 + 2\omega_0'^2 x_2 - \omega_0'^2 x_3 = 0$$

$$\ddot{x}_3 - \omega_0^2 x_2 + \omega_0^2 x_3 = 0$$

- Write this as a matrix...

Three Masses

- Assume that solutions are of the form

$$x_i(t) = A_i \cos(\omega t + \varphi)$$

- Then $\ddot{x}_i = -\omega^2 x_i$ and

$$\begin{pmatrix} -\omega^2 + \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0'^2 & -\omega^2 + 2\omega_0'^2 & -\omega_0'^2 \\ 0 & -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

- Let $\lambda = \omega^2$. Then this will be true if

$$\begin{vmatrix} \lambda - \omega_0^2 & \omega_0^2 & 0 \\ \omega_0'^2 & \lambda - 2\omega_0'^2 & \omega_0'^2 \\ 0 & \omega_0^2 & \lambda - \omega_0^2 \end{vmatrix} = 0$$

Three Masses

- Expand the determinant:

$$\begin{vmatrix} \lambda - \omega_0^2 & \omega_0^2 & 0 \\ \omega_0'^2 & \lambda - 2\omega_0'^2 & \omega_0'^2 \\ 0 & \omega_0^2 & \lambda - \omega_0^2 \end{vmatrix}$$
$$\begin{aligned} &= (\lambda - \omega_0^2) [(\lambda - 2\omega_0'^2)(\lambda - \omega_0^2) - \omega_0'^2 \omega_0^2] \\ &\quad - \omega_0^2 \omega_0'^2 (\lambda - \omega_0^2) \\ &= (\lambda - \omega_0^2) [(\lambda - 2\omega_0'^2)(\lambda - \omega_0^2) - 2\omega_0'^2 \omega_0^2] \\ &\quad = \lambda(\lambda - \omega_0^2)(\lambda - \omega_0^2 - 2\omega_0'^2) \end{aligned}$$

Three Masses

$$\lambda(\lambda - \omega_0^2)(\lambda - \omega_0^2 - 2\omega_0'^2) = 0$$

- The roots are

$$\lambda = \omega^2 = 0$$

$$\lambda = \omega^2 = \omega_0^2$$

$$\lambda = \omega^2 = \omega_0^2 + 2\omega_0'^2$$

- What motion does this correspond to?
- Calculate the eigenvectors...

Three Masses

- When $\lambda = 0$:

$$\begin{pmatrix} \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0'^2 & 2\omega_0'^2 & -\omega_0'^2 \\ 0 & -\omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

- This tells us that

$$x_2 = x_1$$

$$x_3 = x_2$$

- All masses move in the same direction at once.
- This is just a translation of the entire system.

Three Masses

- When $\lambda = \omega_0^2$

$$\begin{pmatrix} 0 & -\omega_0^2 & 0 \\ -\omega_0'^2 & 2\omega_0'^2 - \omega_0^2 & -\omega_0'^2 \\ 0 & -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

- This tells us that

$$x_2 = 0$$

$$x_1 = -x_3$$

- Motion is described by

$$x_1(t) = A \cos(\omega_0 t + \varphi)$$

$$x_2(t) = 0$$

$$x_3(t) = -A \cos(\omega_0 t + \varphi)$$

Three Masses

- When $\lambda = \omega_0^2 + 2\omega_0'^2$

$$\begin{pmatrix} -2\omega_0'^2 & -\omega_0^2 & 0 \\ -\omega_0'^2 & \omega_0^2 & -\omega_0'^2 \\ 0 & -\omega_0^2 & -2\omega_0'^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

- This tells us that

$$2\omega_0'^2 x_1 = -\omega_0^2 x_2$$

$$\omega_0^2 x_2 = -2\omega_0'^2 x_2$$

- Which means that

$$x_1 = x_3$$

$$x_2 = -\frac{2\omega_0'^2}{\omega_0^2} x_1$$