In[1]:= (* Transfer matrix for a distance d in a medium with index of refraction n *)

$$\mathrm{tm}[d\_, n\_] := \begin{pmatrix} 1 & 0 \\ d/n & 1 \end{pmatrix}$$

In[2]:= (* Refraction matrix for a spherical surface of radius r with a change in index of refraction dn = nt - ni *)

$$\mathrm{rm}[r\_, dn\_] := \begin{pmatrix} 1 & -dn/r \\ 0 & 1 \end{pmatrix}$$

In[3]:= (* Initial ray starting at a distance so from the first vertex *)

$$\mathrm{yo} := \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

In[4]:= (* Propagate the ray to the first virtex *)

```
y1 = tm[so, 1].yo
```

Out[4]= $\{\{\alpha\}, \{\mathrm{so}\,\alpha\}\}$

In[7]:= (* Refract the ray into the sphere *)

```
y2 = rm[r, n - 1].y1
```

Out[7]= $$\left\{\left\{\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right\}, \{\mathrm{so}\,\alpha\}\right\}$$

In[8]:= (* Propagate the ray through the sphere *)

```
y3 = tm[2 r, n].y2
```

Out[8]= $$\left\{\left\{\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right\}, \left\{\mathrm{so}\,\alpha + \frac{2\,r\left(\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right)}{n}\right\}\right\}$$

In[9]:= (* Refract out of the sphere *)

```
y4 = rm[-r, 1 - n].y3
```

Out[9]= $$\left\{\left\{\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r} - \frac{(-1+n)\left(\mathrm{so}\,\alpha + \frac{2\,r\left(\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right)}{n}\right)}{r}\right\}, \left\{\mathrm{so}\,\alpha + \frac{2\,r\left(\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right)}{n}\right\}\right\}$$

In[10]:= (* Propagate ray to the image point *)

```
yi = tm[si, 1].y4
```

Out[10]= $$\left\{\left\{\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r} - \frac{(-1+n)\left(\mathrm{so}\,\alpha + \frac{2\,r\left(\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right)}{n}\right)}{r}\right\},\right.$$
$$\left.\left\{\mathrm{so}\,\alpha + \frac{2\,r\left(\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right)}{n} + \mathrm{si}\left(\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r} - \frac{(-1+n)\left(\mathrm{so}\,\alpha + \frac{2\,r\left(\alpha + \frac{(1-n)\,\mathrm{so}\,\alpha}{r}\right)}{n}\right)}{r}\right)\right\}\right\}$$

In[13]:= (* The condition for an image is that the ray crosses the optical axis at the image point. So we need to solve for si as a function of so. *)

```
solution = Solve[yi[[2]] == 0, {si}]
```

Out[13]= $$\left\{\left\{\mathrm{si} \to \frac{r\,(2\,r + 2\,\mathrm{so} - n\,\mathrm{so})}{-2\,r + n\,r - 2\,\mathrm{so} + 2\,n\,\mathrm{so}}\right\}\right\}$$

```
(* When the object is placed at the focal point of the first refracting surface,
we will have... *)
```

In[15]:=
```
FullSimplify[si /. solution /. {so → r / (n - 1)}]
```

Out[15]=

$$\left\{\frac{r}{-1+n}\right\}$$