PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 6 – French, Chapter 3

Spring 2014 Semester

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Damped Harmonic Motion

- Viscous damping: $F = -b\dot{x}$ (force proportional to velocity)
- Differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(t) = Ae^{\alpha t}$$

$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$

$$\alpha^2 m + \alpha b + k = 0$$

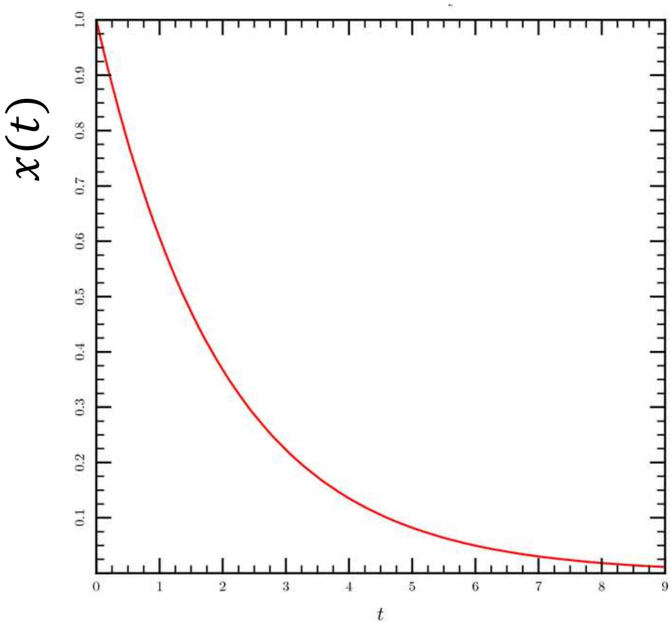
$$\alpha = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

- Three possible forms of the solution depend on the sign of the discriminant.

Overdamped Case

$$\frac{b^2}{4m^2} - \frac{k}{m} = \frac{\gamma^2}{4} - (\omega_0)^2 > 0$$

$$x(t) = \boldsymbol{A} e^{-\frac{\gamma}{2}t} e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}} + \boldsymbol{B} e^{-\frac{\gamma}{2}t} e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

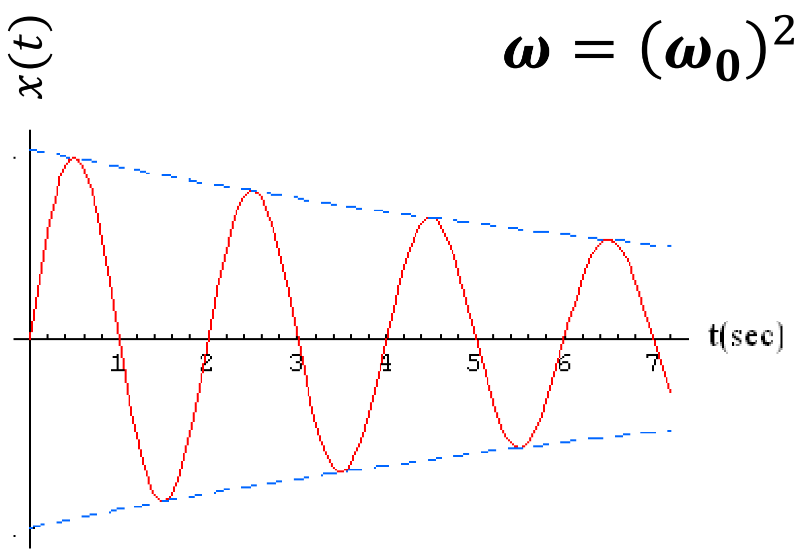


Underdamped Case

$$\frac{b^2}{4m^2} - \frac{k}{m} = \frac{\gamma^2}{4} - (\omega_0)^2 < 0$$

$$x(t) = \boldsymbol{A} e^{-\frac{\gamma}{2}t} \sin \omega t + \boldsymbol{B} e^{-\frac{\gamma}{2}t} \cos \omega t$$

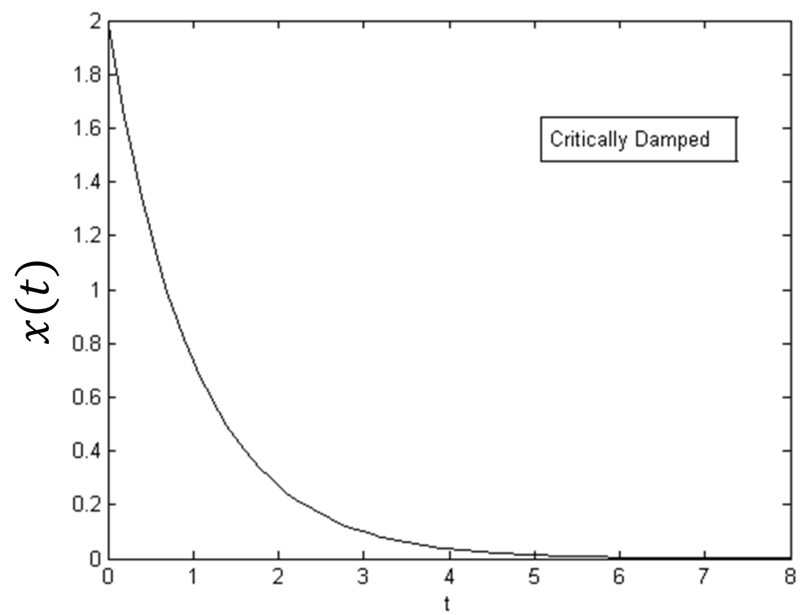
$$\boldsymbol{\omega} = (\boldsymbol{\omega_0})^2 - \frac{\boldsymbol{\gamma}^2}{4}$$



Critically Damped Case

$$\frac{b^2}{4m^2} - \frac{k}{m} = 0$$

$$x(t) = (\boldsymbol{A} + \boldsymbol{B}t)e^{-\frac{\gamma}{2}t}$$

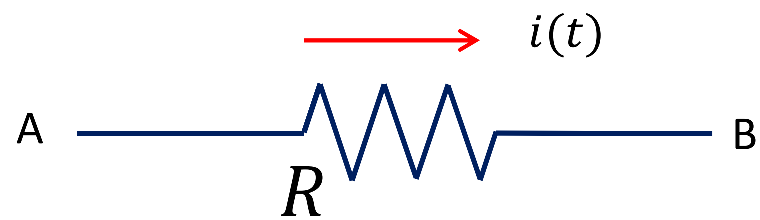


RLC Circuits

Review of electricity and magnetism:

- Capacitors store energy in an electric field
- Resistors dissipate energy by heating
- Inductors store energy in a magnetic field
- Voltage
  - Energy per unit charge
  - SI units: *Volt = J/C*
- Current
  - Charge passing a point in a circuit per unit time
  - SI units: *Ampere = C/s*

Resistors

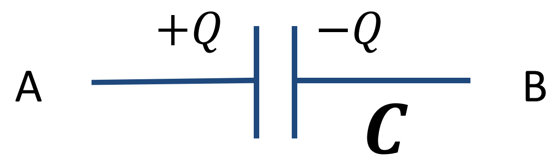


- Current flows from A to B in the direction indicated by the arrow.
- Charges at point B have less energy than at point A because some of their energy was dissipated as heat.
- Potential difference:

$$\Delta V = V_A - V_B = i(t)\, R$$

- Resistance, R, is measured in *ohms* in SI units

Capacitors



- Potential difference:

$$\Delta V = V_A - V_B = \frac{Q}{C}$$

- No current can flow across the capacitor, so any charge that flows onto the plates accumulates there:

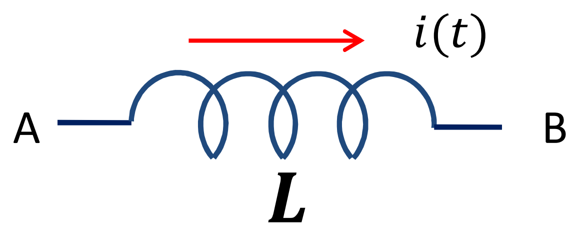
$$Q(t) = Q_0 + \int_0^t i(t)dt$$

- Potential difference:

$$\Delta V = \frac{1}{C}\int_0^t i(t)dt$$

- SI units for capacitance: *farad*

Inductors



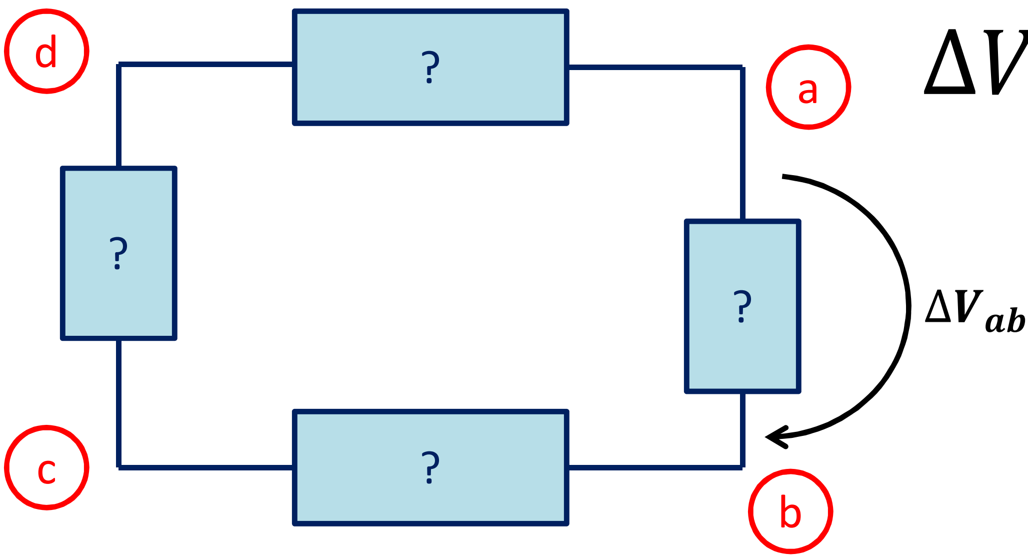
- The inductor will establish a potential difference that opposes any change in current.
  - If $di/dt < 0$ then $V_B > V_A$
    - magnetic field is being converted to energy
  - If $di/dt > 0$ then $V_B < V_A$
    - energy is being stored in the magnetic field

$$\Delta V = V_A - V_B = L\frac{di}{dt}$$

- SI units for inductance: *henry*

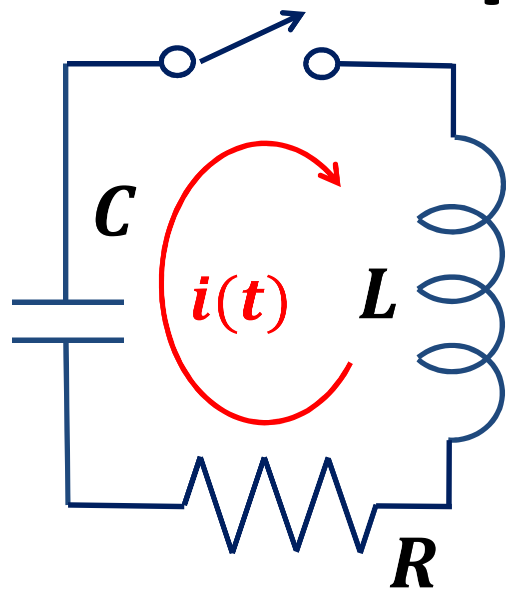
Kirchhoff's Loop Rule

- The sum of the potential differences around a loop in a circuit must equal zero:



$$\Delta V_{ab} + \Delta V_{bc} + \Delta V_{cd} + \Delta V_{da} = 0$$

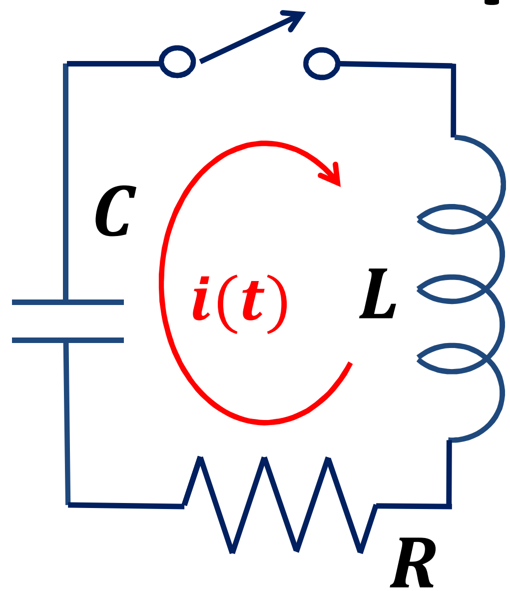
Kirchhoff's Loop Rule



Sum of potential differences:

$$-L\frac{di}{dt}$$

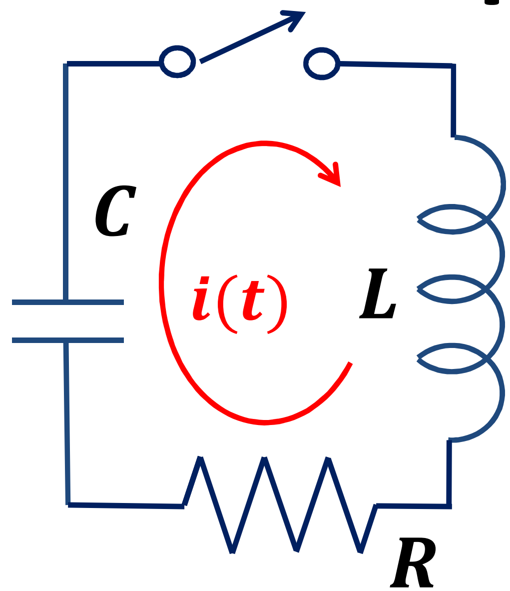
Kirchhoff's Loop Rule



Sum of potential differences:

$$-L\frac{di}{dt} - i(t)R$$

Kirchhoff's Loop Rule



Sum of potential differences:

$$-L\frac{di}{dt} - i(t)R - \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Initial charge, Q_0, defines the initial conditions.

Kirchhoff's Loop Rule

$$L\frac{di}{dt} + i(t)R + \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Differentiate once with respect to time:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

This is of the same form as the equation for a damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = 0$$

Solutions

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

Suppose $i(t) = Ae^{\alpha t}$

Then $\frac{di}{dt} = \ \alpha Ae^{\alpha t}$ and $\frac{d^2i}{dt^2} = \alpha^2 Ae^{\alpha t}$

Substitute into the differential equation:

$$\left(\alpha^2 L + \alpha R + \frac{1}{C}\right)Ae^{\alpha t} = 0$$

True for any t only if $\alpha^2 L + \alpha R + \frac{1}{C} = 0.$

Solutions

$$\alpha^2 L + \alpha R + \frac{1}{C} = 0$$

Roots of the polynomial:

$$\alpha = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Define some new symbols:

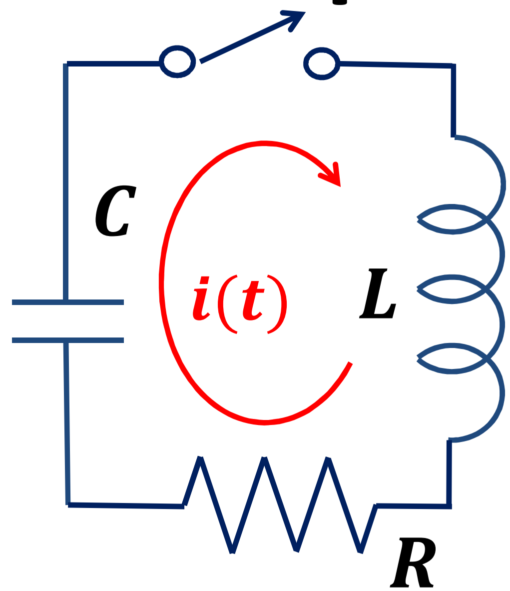
$$\omega_0 = \sqrt{1/_{LC}}$$

$$\gamma = R/L$$

Then the roots can be written

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

Example



- If $L = 2.2\ \mu H$ and $C = 10\ nF$ what is the frequency of oscillations when $R = 0$?
- What is the largest value of R that will still allow the circuit to oscillate?

Example

$$\omega_0 = \sqrt{1/_{LC}} = \sqrt{\frac{1}{(2.2\ \mu H)(0.01\ \mu F)}} = 6.74 \times 10^6\ s^{-1} = 2.15\ MHz$$

$$\gamma = R/L$$

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

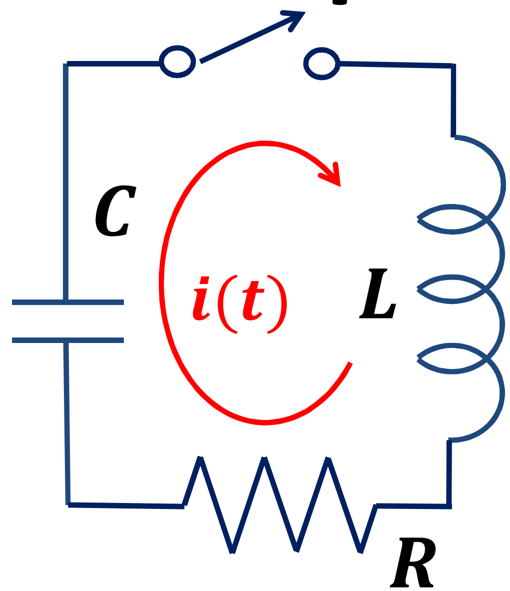
- Critical damping:

$$\frac{\gamma^2}{4} - (\omega_0)^2 = 0$$

$$\gamma = \frac{R}{L} = 2\sqrt{\frac{1}{LC}}$$

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{2.2\ \mu H}{0.01\ \mu F}} = 29.7\ \Omega$$

Example



$$L = 2.2\ \mu H$$
$$C = 10\ nF$$
$$R = 2\ \Omega$$

- Suppose the initial charge on the capacitor was 10 nC… What voltage is measured across R as a function of time?

Example

- Calculate the discriminant:

$$\frac{R^2}{4L^2} - \frac{1}{LC} = \frac{(2\ \Omega)^2}{4(2.2\ \mu H)^2} - \frac{1}{(2.2\ \mu H)(0.01\ \mu F)} < 0$$

- The circuit will oscillate with frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 1.07\ MHz$$

- Time constant:

$$\frac{\gamma}{2} = \frac{R}{2L} = \frac{(2\ \Omega)}{2(2.2\ \mu H)} = 4.55 \times 10^5\ s^{-1}$$

- Current will be:

$$i(t) = \ i_0 e^{-\gamma t/2} \cos(\omega t + \varphi)$$

Example

$$i(t) = i_0 e^{-\gamma t/2} \cos(\omega t + \varphi)$$

- Initial conditions:
  - $i(0) = 0$ because the inductor produces a potential difference that opposes the change in current.
  - Therefore, $\varphi = \pi/2$
  - Initial potential across capacitor:

$$\Delta V = \frac{Q}{C} = \frac{10\ nC}{10\ nF} = 1\ Volt$$

  - Initial voltage across inductor:

$$\Delta V = L\frac{di}{dt} = 1\ Volt$$

$$\frac{di}{dt} = \frac{1\ V}{2.2\ \mu H} = 4.55 \times 10^5\ A/s = i_0\omega$$

  - $i_0 = \frac{4.55 \times 10^5\ A/s}{6.73 \times 10^6\ /s} = 68\ mA$

Example

- Current in circuit:

$$i(t) = i_0 e^{-\gamma t/2} \sin \omega t$$

where $\gamma/2 = 4.55 \times 10^5\, s^{-1}$ and $i_0 = 68\, mA$

- Potential difference across the resistor:

$$\Delta V = i(t)R$$

$$v(t) = v_0 e^{-\gamma t/2} \sin \omega t$$

where $v_0 = 136\, mV$.