PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 5 – French, Chapter 3

Spring 2014 Semester

Matthew Jones

Announcement

- Office hours:
  - Nominally 2:00 to 3:00 Tuesdays and Thursdays
  - If I am not in my office (Phys 378) then I might be in my lab (Phys 337). If the door is closed, use the "secret knock" (4-2-2)…
  - The teaching assistants are:
    - Jingchen Liang (liang101@purdue.edu)
    - Yongjin Park (park626@purdue.edu)
      - Tuesday 12:00 – 2:00 pm in room 031

Still More Oscillating Systems

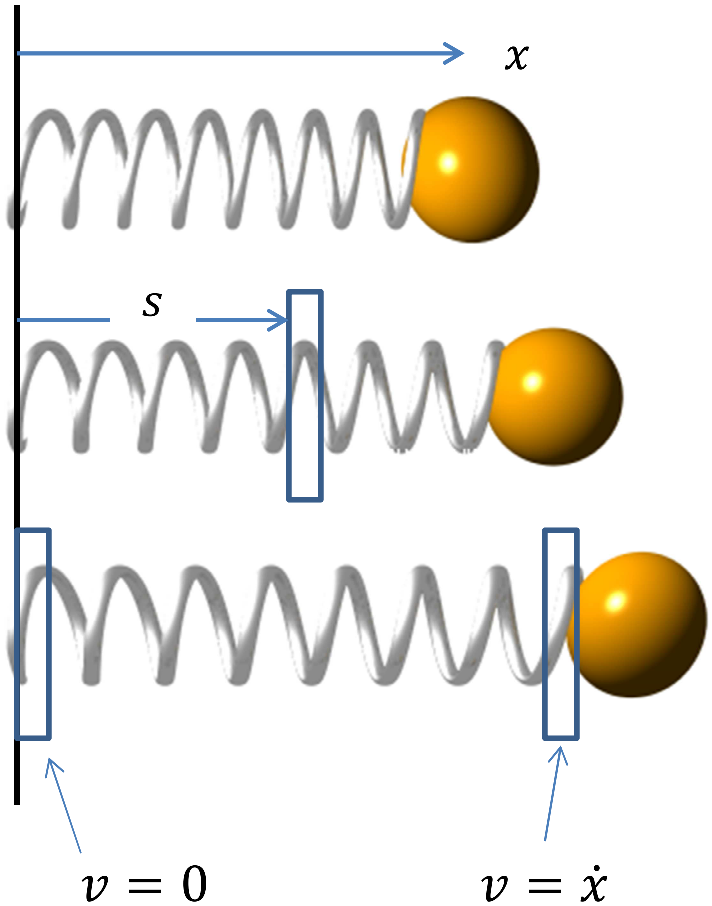
- Consider a mass-spring system:

$$E = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2$$

- How do we account for the mass of the spring?
  - Hooke's constant, k, already accounts for all the poter energy stored in the spring
  - All pieces of the spring are in motion and give an additional contribution to the kinetic energy
  - We need to calculate this and add it to the kinetic energy of the mass.

Physical Spring

- When the mass is in motion, how much kinetic energy does the spring have?



1. The spring has a total length x and total mass M
2. The velocity of the fixed end of the spring is always zero
3. The velocity of the moving end of the spring is given by $\dot{x}$
4. At a distance s from the fixed end, the velocity will be
$$v = \frac{s}{x}\dot{x}$$
5. The mass of an element of length ds will be
$$dM = \frac{M}{x}ds$$

Physical Spring

- Kinetic energy of one element of the spring:

$$dT = \frac{1}{2}v^2 dM = \frac{1}{2}\left(\frac{s}{x}\dot{x}\right)^2 \frac{M}{x} ds$$

- We get the total kinetic energy by integrating over the length of the spring:

$$T_{spring} = \frac{M}{2x^3}(\dot{x})^2 \int_0^x s^2 ds = \frac{M}{6x^3}(\dot{x})^2 s^3 \Big|_0^x$$

$$= \frac{M}{6}(\dot{x})^2$$

- Total kinetic energy is $T = T_{mass} + T_{spring}$

Physical Spring

- Total kinetic energy:

$$T = \frac{1}{2}m(\dot{x})^2 + \frac{1}{6}M(\dot{x})^2 = \frac{1}{2}\left(m + \frac{M}{3}\right)(\dot{x})^2$$

- Potential energy:

$$V = \frac{1}{2}kx^2$$

- Total energy:

$$E = T + V = \frac{1}{2}\left(m + \frac{M}{3}\right)(\dot{x})^2 + \frac{1}{2}kx^2$$

- We know that when $E = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}kx^2$ the frequency is $\omega = \sqrt{k/m}$
- Therefore, the oscillation frequency of the physical spring must be

$$\omega = \sqrt{\frac{k}{m + M/3}}$$

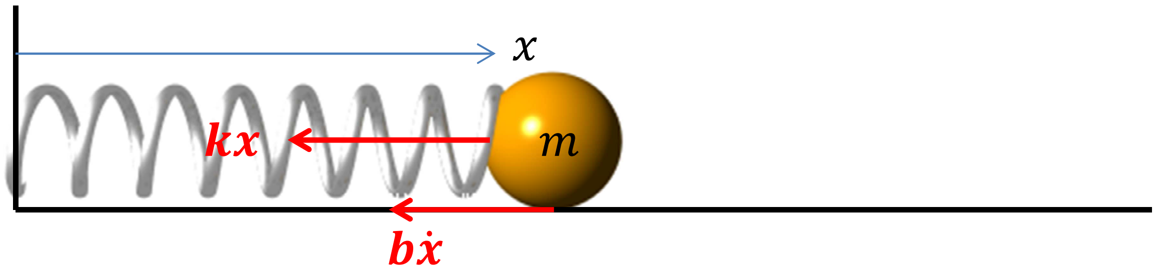
Damping

- Most oscillating physical systems dissipate their energy over time
- We will consider the special cases where the force is a function of velocity

$$F = -b_1 v - b_2 v^2$$

- The drag force is in the opposite direction of the velocity
- Typical of an object moving through a fluid
  - Moving quickly through air: turbulent drag ($b_2 v^2$ is important)
  - Moving slowly through water: viscous drag ($b_1 v$ is important)
- When v is small enough, or b_2 is small enough, only the first term is important.

Oscillating System with Drag



- Newton's second law:

$$m\ddot{x} = -kx - b\dot{x}$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

- What is the solution to this differential equation?
- Let $x(t) = Ae^{\alpha t}$
  - then $\dot{x}(t) = \alpha A e^{\alpha t}$ and $\ddot{x}(t) = \alpha^2 A e^{\alpha t}$
- Substitute it into the differential equation:

$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$

Oscillating Systems with Drag

$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$

This is true for any value of t only when

$$\alpha^2 m + \alpha b + k = 0$$

Use the quadratic formula:

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

which we write as

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

Oscillating Systems with Damping

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}} + Be^{-\frac{\gamma}{2}t}e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

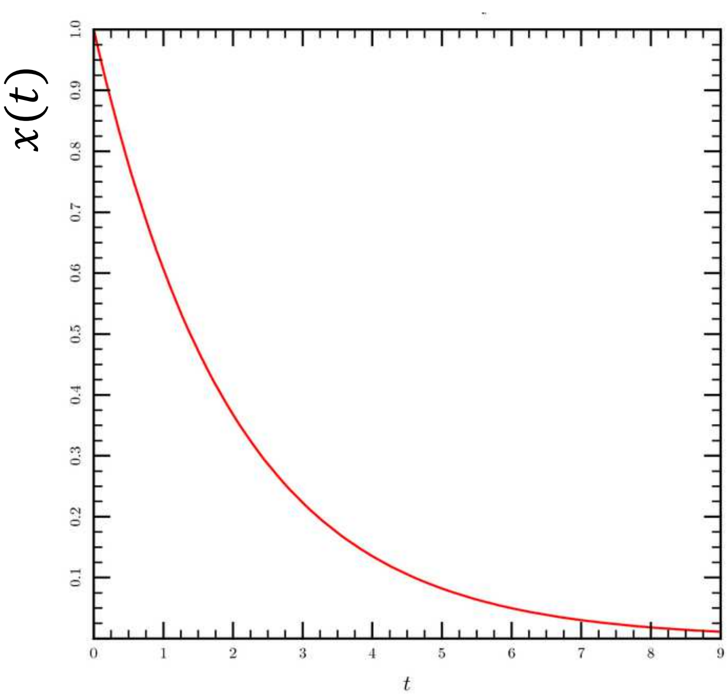
Three cases to consider:

1. $\frac{\gamma^2}{4} - (\omega_0)^2 > 0$
2. $\frac{\gamma^2}{4} - (\omega_0)^2 < 0$
3. $\frac{\gamma^2}{4} - (\omega_0)^2 = 0$

Oscillating Systems with Damping

$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}} + Be^{-\frac{\gamma}{2}t}e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

When $\boldsymbol{\frac{\gamma^2}{4} - (\omega_0)^2 > 0}$ both exponents are real and negative



The mass does not oscillate

It gradually approaches the equilibrium position at $x = 0$.

Oscillating Systems with Damping

$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}} + Be^{-\frac{\gamma}{2}t}e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

When $\frac{\boldsymbol{\gamma}^2}{\mathbf{4}} - (\boldsymbol{\omega}_0)^2 < \mathbf{0}$ we can write:

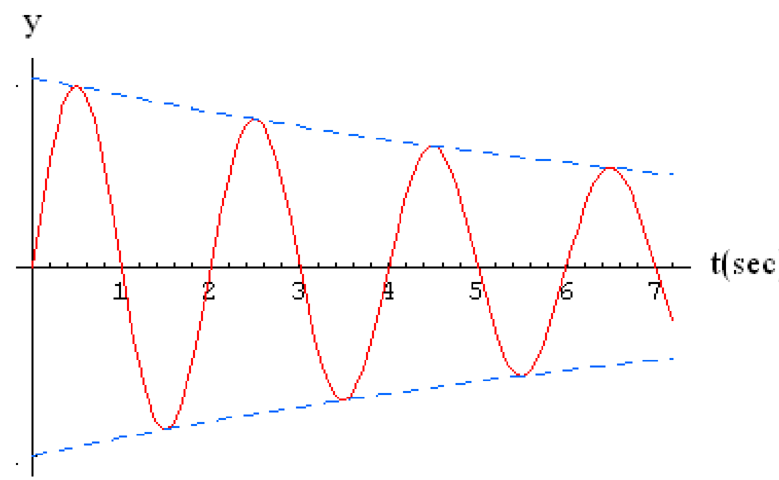
$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}} + Be^{-\frac{\gamma}{2}t}e^{-it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}}$$

$$= Ce^{-\frac{\gamma}{2}t}\cos(\omega t + \varphi)$$

where $\omega = \sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}$

Oscillation frequency gradually decreases.

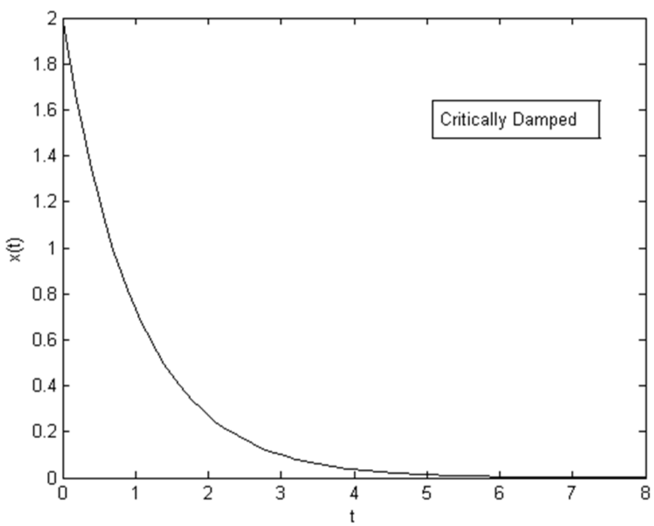
Oscillation frequency is slightly lower than the un-damped oscillator.



Oscillating Systems with Damping

- Third possibility: $\frac{\gamma}{2} - \omega_0 = 0$
- In this case the solution is slightly different:

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$



fastest return to equilibrium position without oscillating.

Notation

- We just introduced a lot of notation:

$$\omega_0 = \sqrt{k/m}$$

$$\gamma = \frac{b}{m}$$

$$\omega = \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

- None of these express a "fundamental physical law"
- They are just definitions
- If we had defined $\gamma' = \frac{b}{2m}$ then we could write

$$x(t) = Ae^{-\gamma' t}e^{t\sqrt{(\gamma')^2-(\omega_0)^2}} + Be^{-\gamma' t}e^{-t\sqrt{(\gamma')^2-(\omega_0)^2}}$$

- Maybe this is simpler or easier to remember, but it is still rather arbitrary and chosen only for convenience.

Suggestion

- Do not memorize these formulas…
- Instead, memorize this procedure:

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(t) = Ae^{\alpha t}$$

$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$

$$\alpha^2 m + \alpha b + k = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

- Then define any new variables you introduce:

$$x(t) = Ce^{-\frac{\gamma}{2}t}\cos(\omega t + \varphi) \text{ (when } \frac{b^2}{4m^2} - \frac{k}{m} < 0)$$

- Recognize that there are three possible forms of the solution.

Example

- Suppose a 1 kg mass oscillates with frequency f and the amplitude of oscillations decreases by a factor of $\frac{1}{2}$ in time T. What differential equation describes the motion?

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$m = 1\,kg$$

$$b = \frac{2m\log 2}{T}$$

$$k = m\left(4\pi^2 f^2 + \left(\frac{\log 2}{T}\right)^2\right)$$