PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 36 – Diffraction

Spring 2014 Semester

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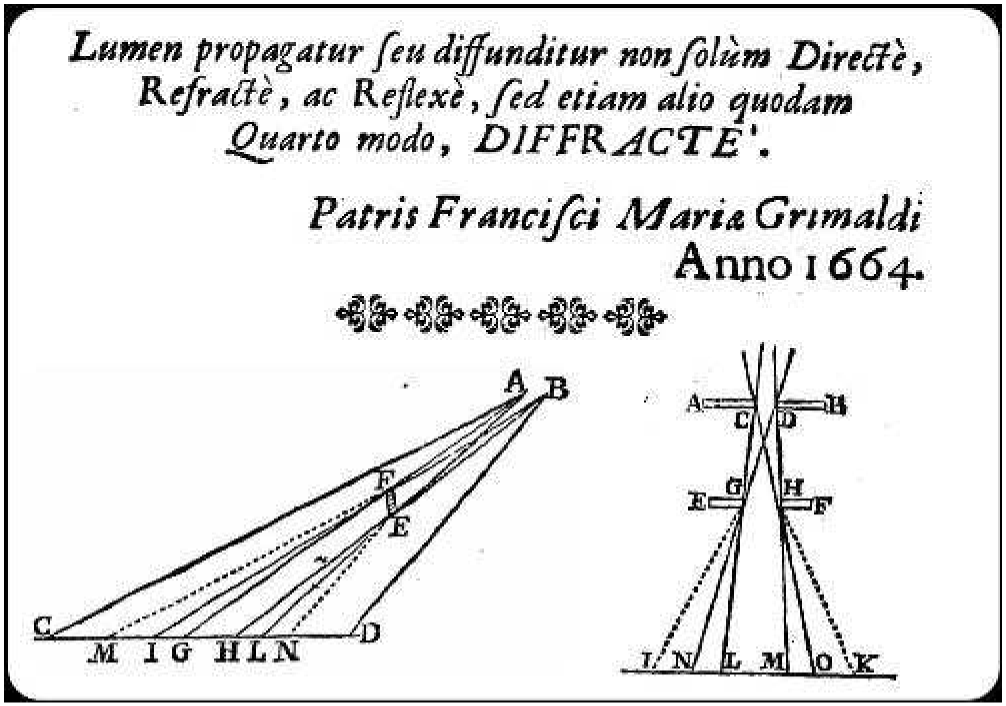
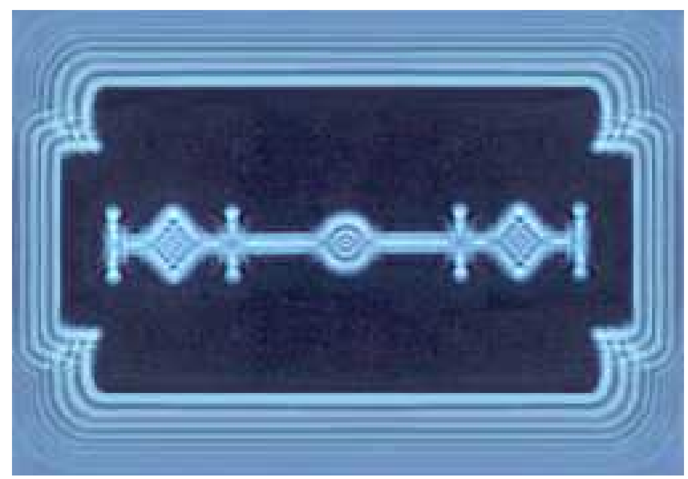
Diffraction


Lumen propagatur ſeu diffunditur non ſolùm Directè, Refractè, ac Reflexè, ſed etiam alio quodam Quarto modo, DIFFRACTE'.

Patris Franciſci Mariæ Grimaldi
Anno 1664.



“Light transmitted or diffused, not only directly, refracted, and reflected, but also in some other way in the fourth, *breaking*.”

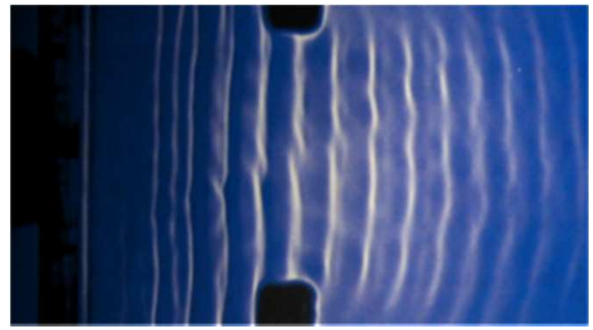
Huygens-Fresnel Principle

- Huygens:
  - Every point on a wave front acts as a point source of secondary spherical waves that have the same phase as the original wave at that point.
- Fresnel:
  - The amplitude of the optical field at any point in the direction of propagation is the superposition of all wavelets, considering their amplitudes and relative phases.

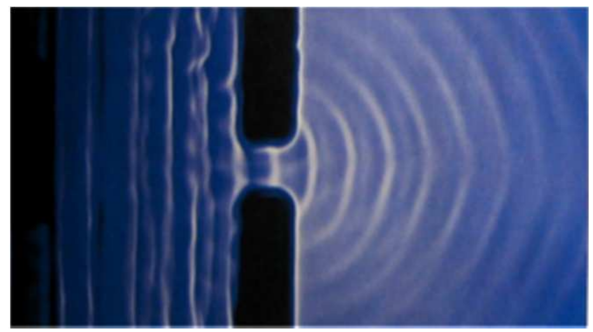
Single Slit Diffraction

Examples with water waves

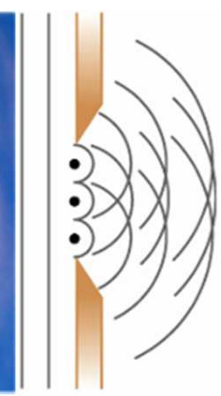
- Wide slit: waves are unaffected



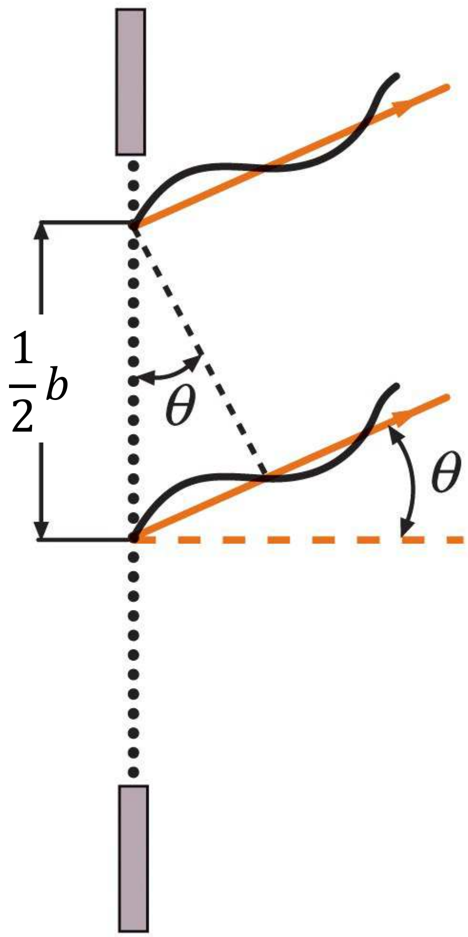
- Narrow slit: source of spherical waves



- In between: multiple interfering point sources



Single Slit Diffraction



Think of the slit as a number of point sources with equal amplitude. Divide the slit into two pieces and think of the interference between light in the upper half and light in the lower half.

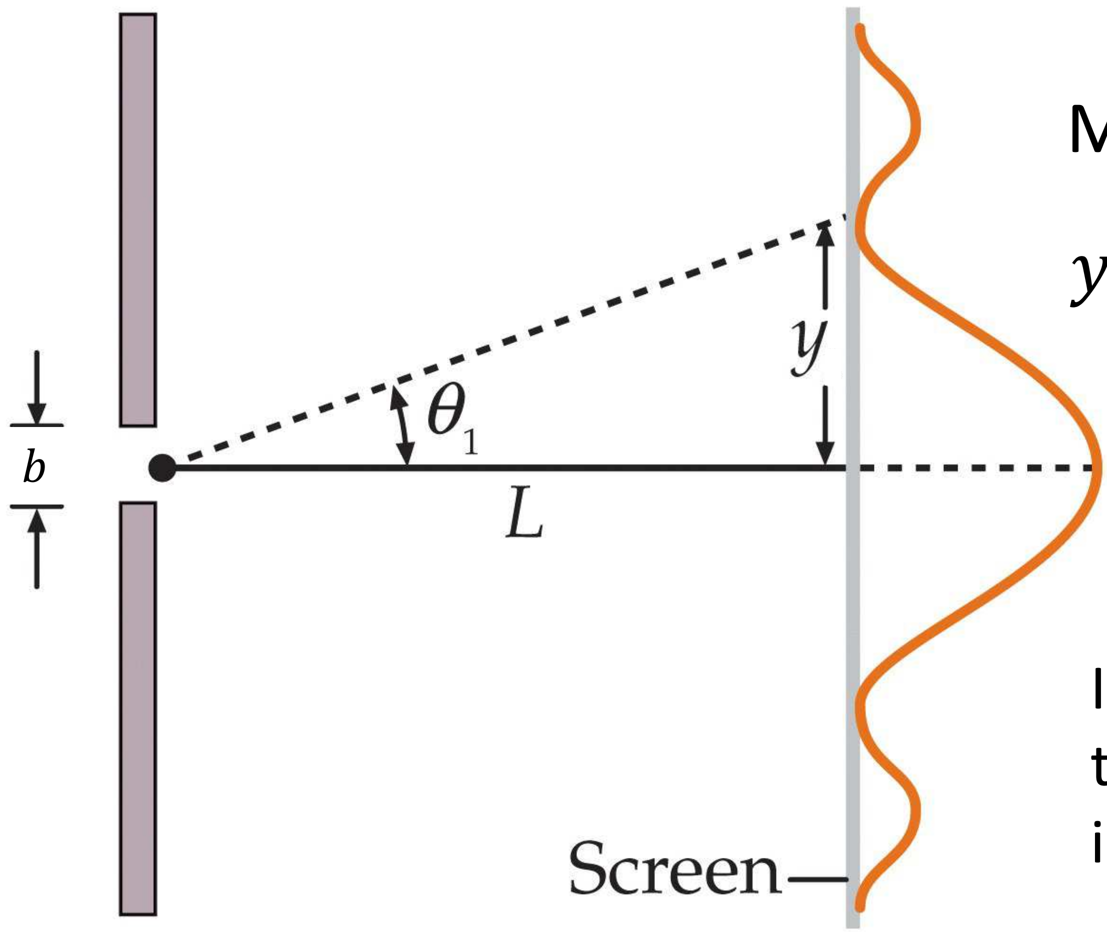
Destructive interference when

$$\frac{b}{2}\sin\theta = \frac{\lambda}{2}$$

Minima when

$$\sin\theta = {}^{\lambda}/_{b}$$

Single Slit Diffraction



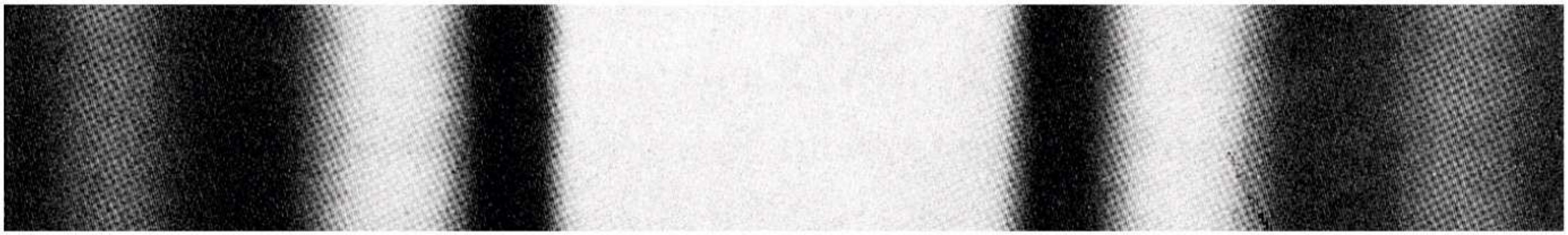
$$\sin\theta \approx \tan\theta = y/L$$

Minima located at

$$y = \frac{mL\lambda}{b}, m = 1,2,3,\ldots$$

In general, the “width” of the image on the screen is not even close to a.

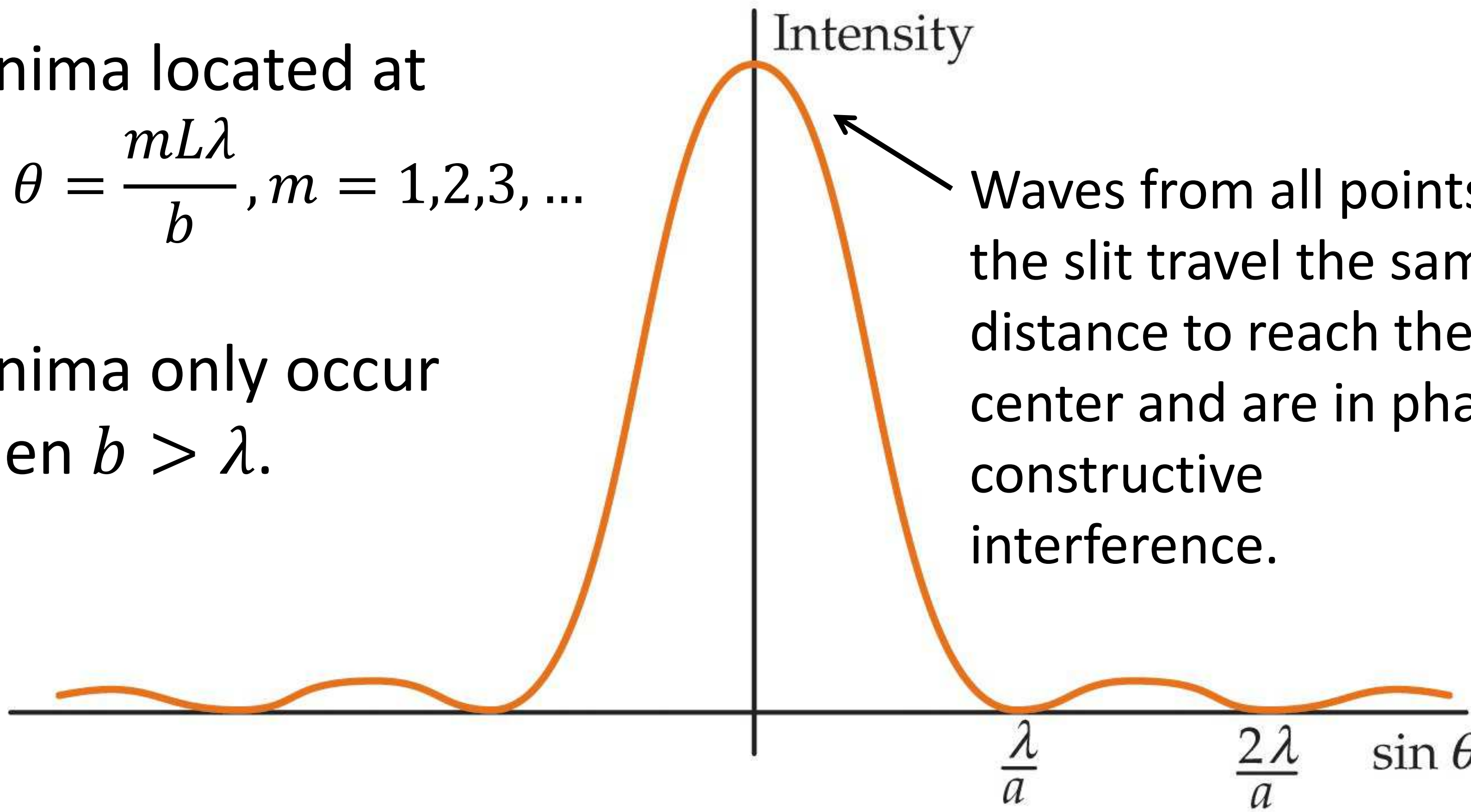
Single Slit Diffraction



Minima located at

$$\sin\theta = \frac{mL\lambda}{b}, m = 1,2,3, \dots$$

Minima only occur when $b > \lambda$.



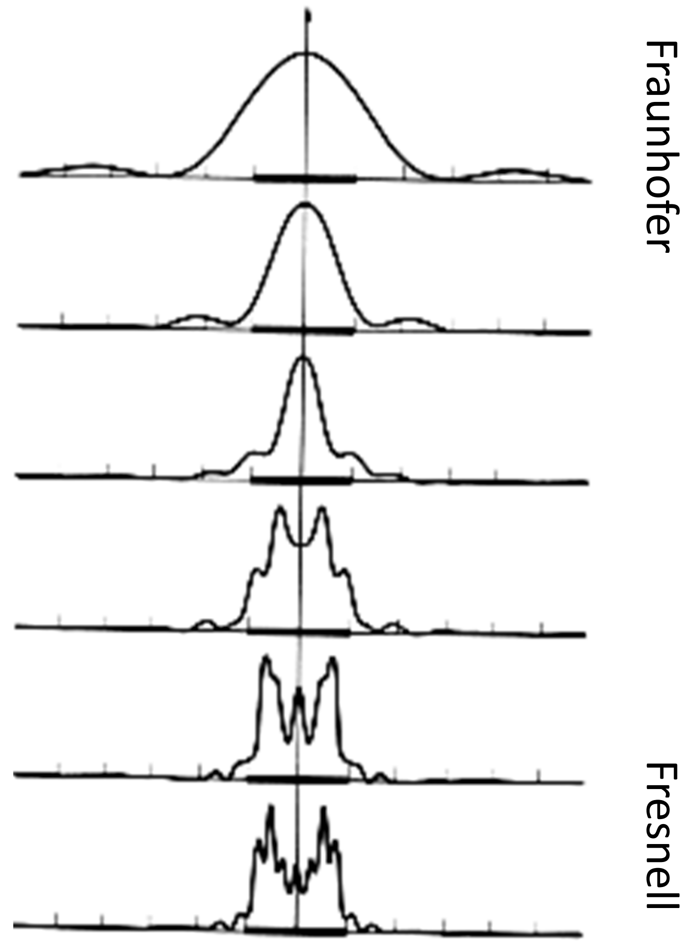
Waves from all points in the slit travel the same distance to reach the center and are in phase: constructive interference.

Fresnel and Fraunhofer Diffraction

Assumptions about the wave front that impinges on the slit:

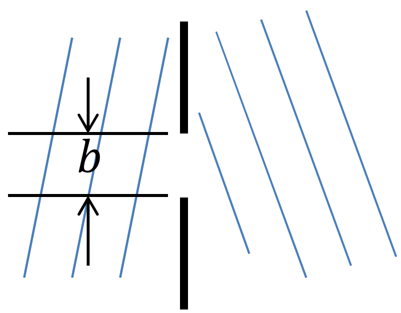
- When it's a plane, the phase varies linearly across the slit: Fraunhofer diffraction

- When the phase of the wave front has significant curvature: Fresnel diffraction



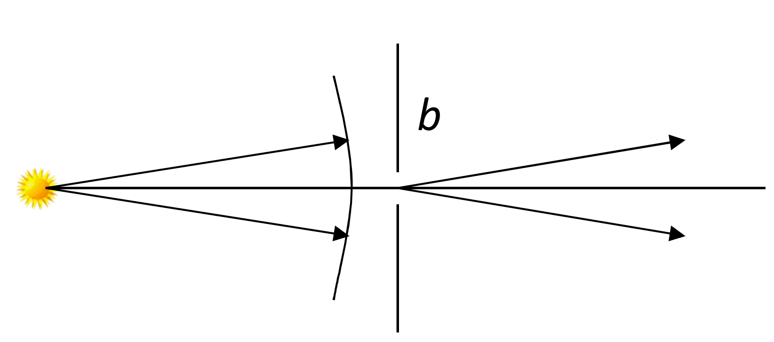
Fresnel and Fraunhofer Diffraction

- Fraunhofer diffraction
  - Far field: $R \gg b^2/\lambda$



  - R is the smaller of the distance to the source or to the screen

- Fresnel Diffraction:
  - Near field: wave front is not a plane at the aperture

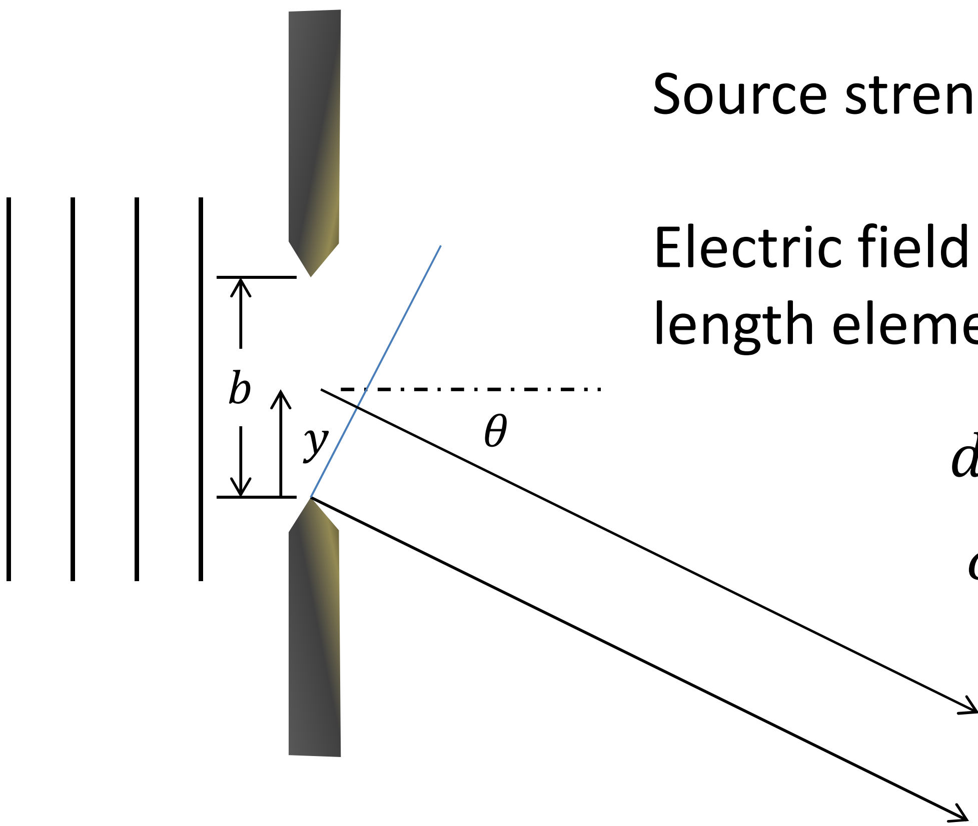


Single-Slit Fraunhofer Diffraction

Light with intensity I_0 impinges on a slit with width b

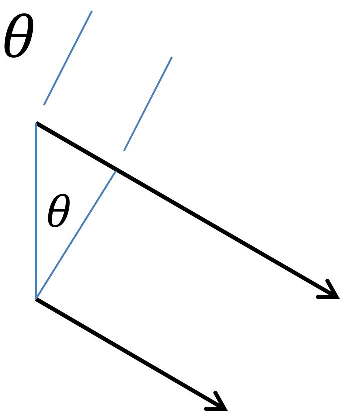
Source strength per unit length: $\mathcal{E}_L$

Electric field at a distance R due to the length element dy:

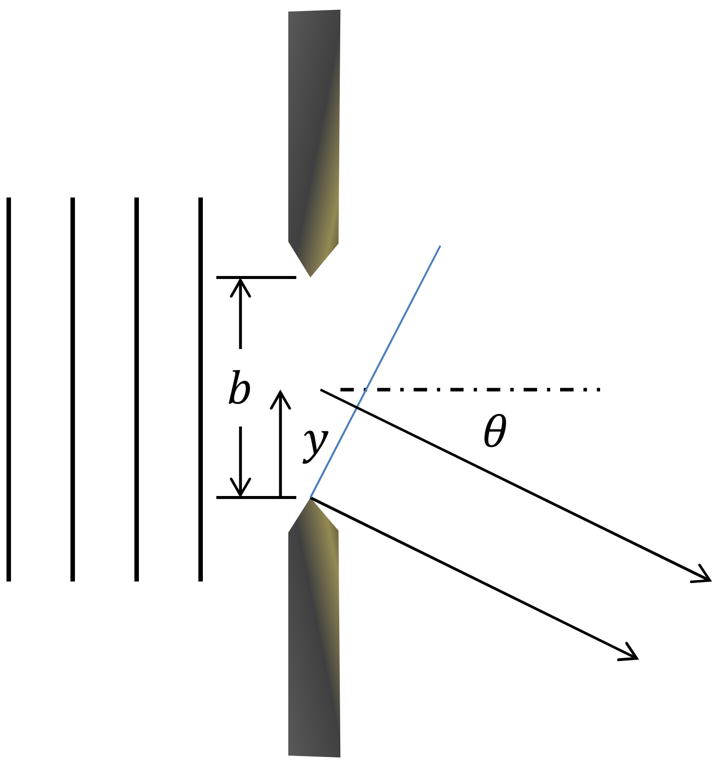


$$dE = \frac{\mathcal{E}_L e^{i\delta(y)} dy}{R}$$

$$\delta(y) = ky \sin\theta$$



Single-Slit Fraunhofer Diffraction



$$dE = \frac{\mathcal{E}_L e^{iky\sin\theta}\,dy}{R}$$

Let $y = 0$ be at the center of the slit.

Integrate from $-b/2$ to $+b/2$:

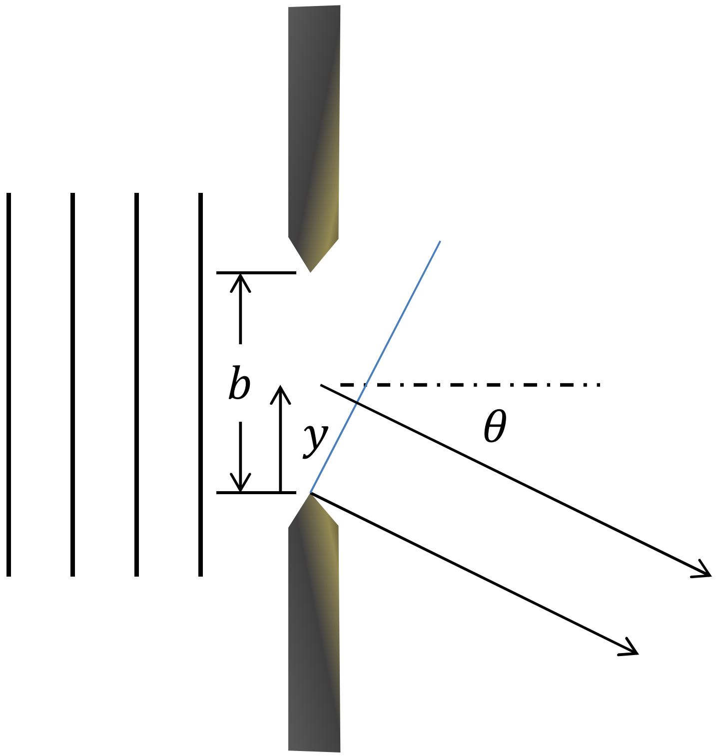
Total electric field:

$$E = \frac{\mathcal{E}_L}{R}\int_{-b/2}^{+b/2} e^{iky\sin\theta}\,dy$$

$$= \frac{\mathcal{E}_L}{R}\frac{e^{i(kb/2)\sin\theta} - e^{-i(kb/2)\sin\theta}}{ik\sin\theta}$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\left(\frac{1}{2}kb\sin\theta\right)}{\frac{1}{2}kb\sin\theta}$$

Single-Slit Fraunhofer Diffraction



$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin \beta}{\beta}$$

where

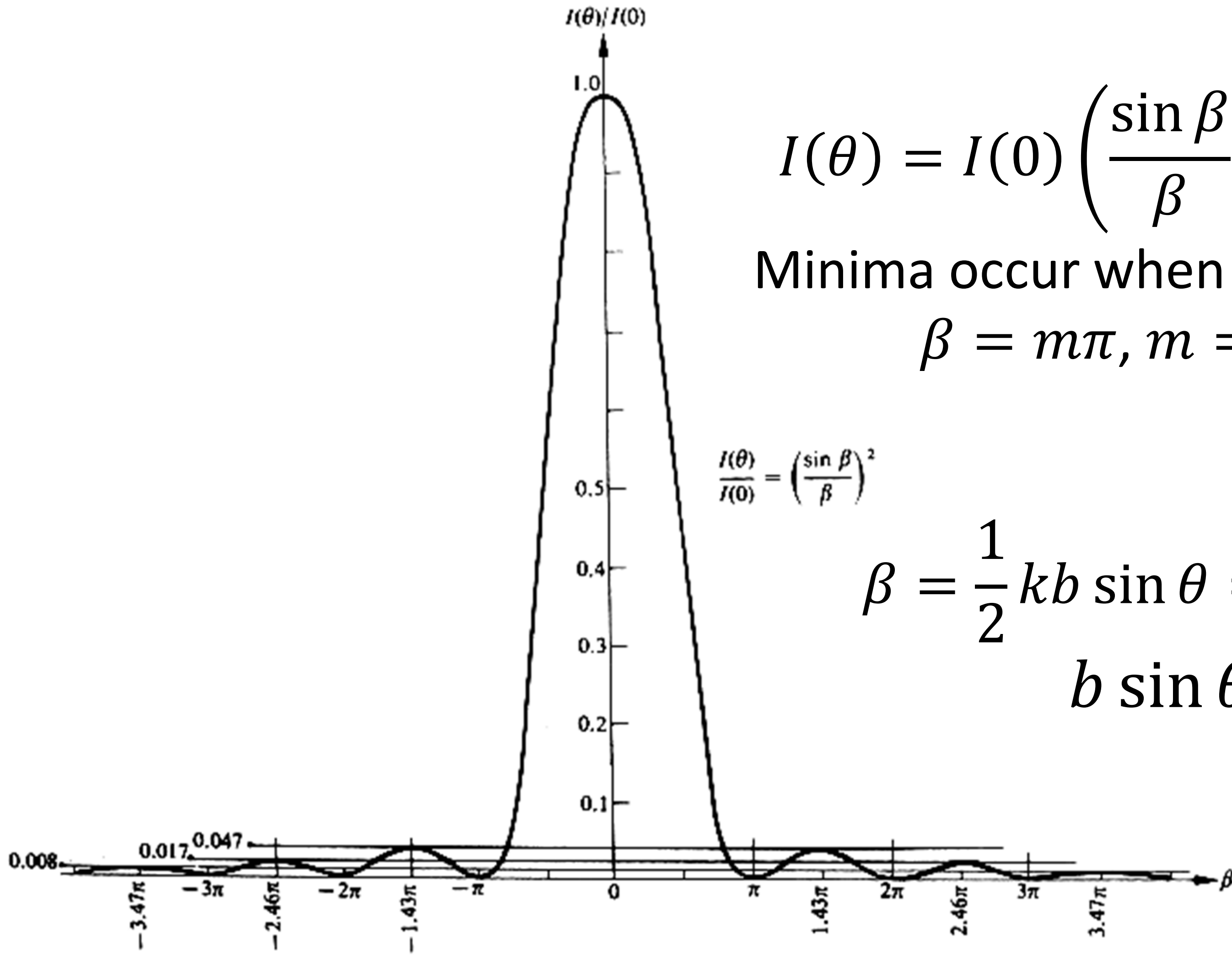
$$\beta = \frac{1}{2} kb \sin \theta$$

The intensity of the light will be

$$I(\theta) = I(0) \left(\frac{\sin \beta}{\beta} \right)^2$$

$$= I(0) \operatorname{sinc}^2 \beta$$

Single-Slit Fraunhofer Diffraction



$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2 = I(0)\,\text{sinc}^2\,\beta$$

Minima occur when

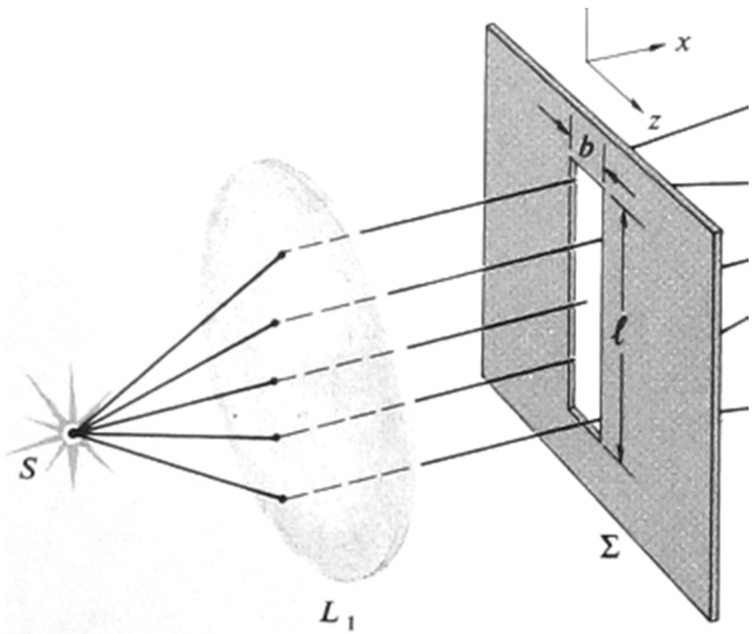
$$\beta = m\pi, m = \pm 1, \pm 2, \cdots$$

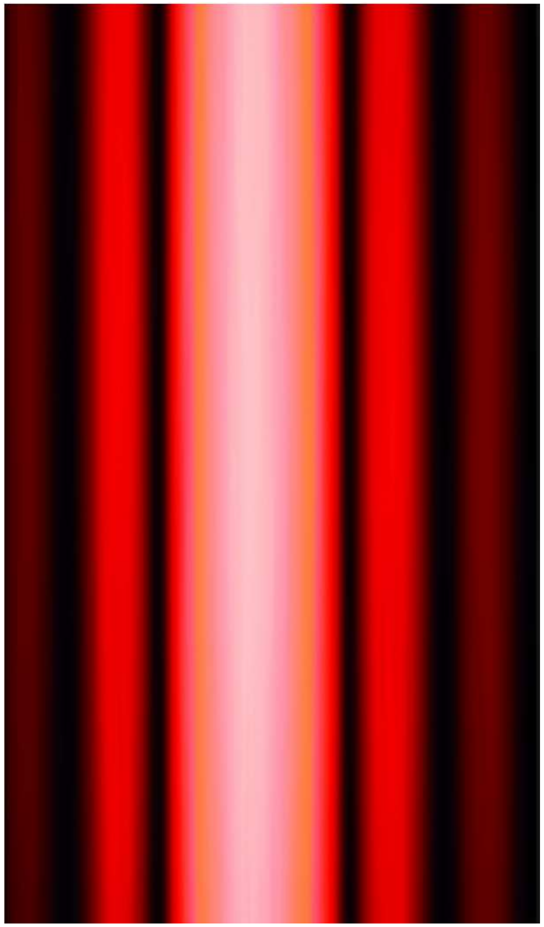
$$\beta = \frac{1}{2}kb\sin\theta = \frac{\pi b}{\lambda}\sin\theta = m\pi$$

$$b\sin\theta = m\lambda$$

Single slit: Fraunhofer diffraction

Adding dimension: long narrow slit
Diffraction most prominent in the narrow direction.





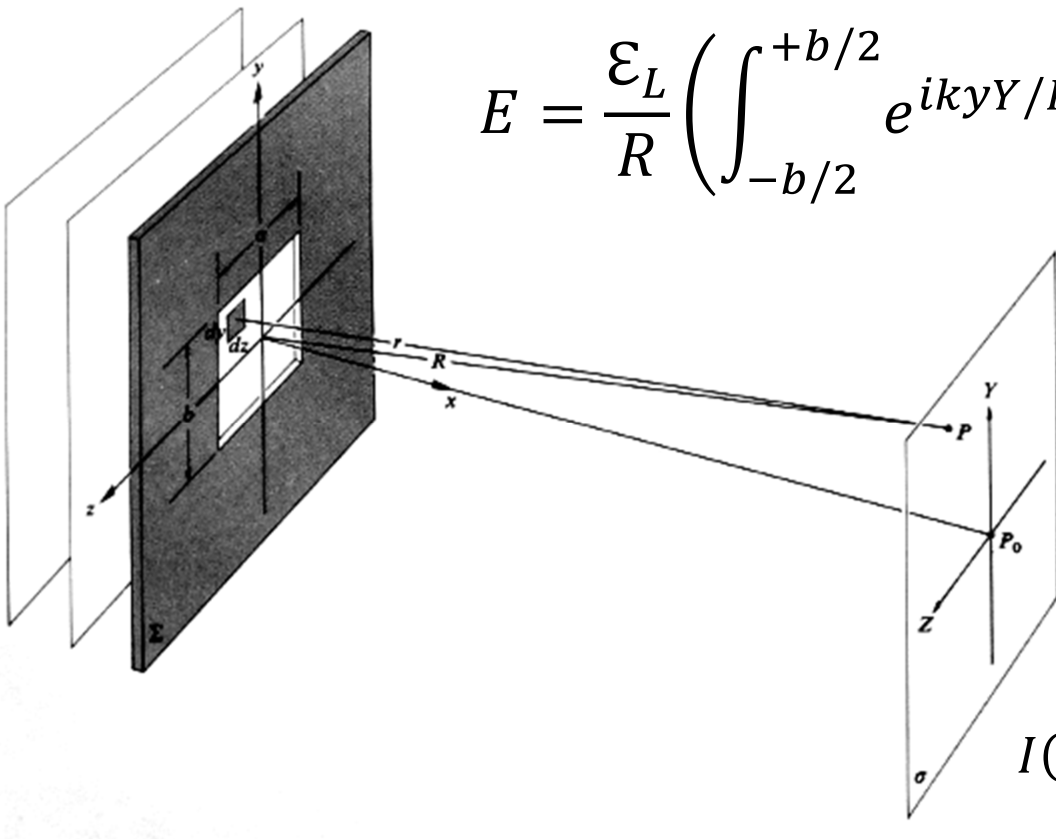
Emerging light has cylindrical symmetry

Rectangular Aperture Fraunhofer Diffraction

Source strength per unit area: $\mathcal{E}_A$

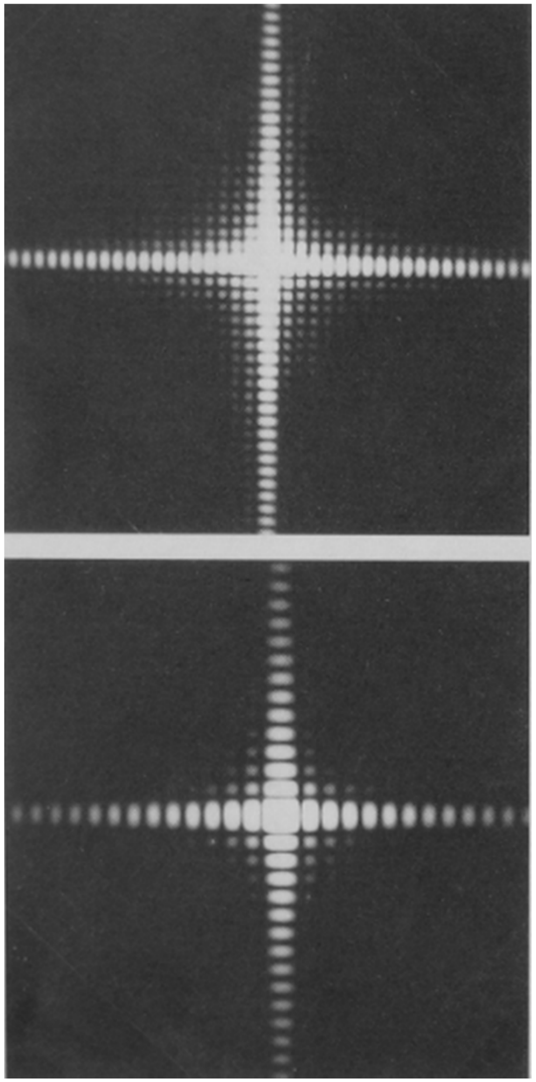
$$dE \approx \frac{\mathcal{E}_L e^{ikyY/R} e^{ikzZ/R} dydz}{R}$$

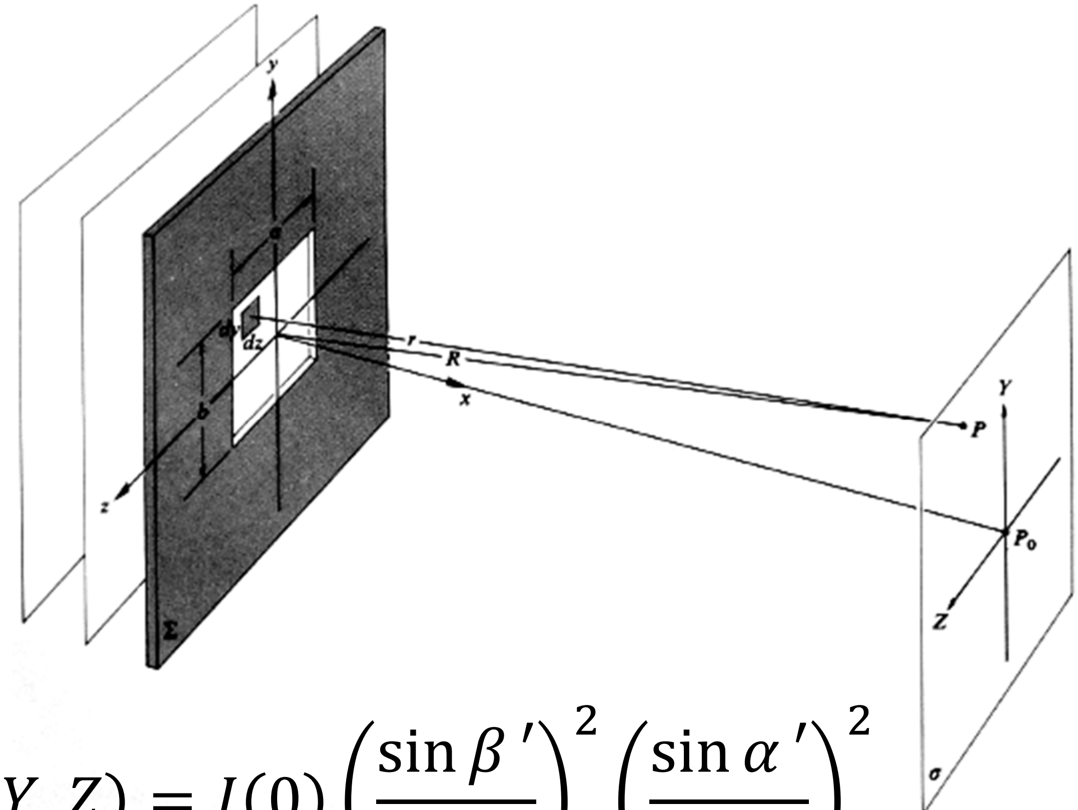
$$E = \frac{\mathcal{E}_L}{R}\left(\int_{-b/2}^{+b/2} e^{ikyY/R} dy\right)\left(\int_{-a/2}^{+a/2} e^{ikzZ/R} dz\right)$$



$$I(Y,Z) = I(0)\left(\frac{\sin\beta'}{\beta'}\right)^2\left(\frac{\sin\alpha'}{\alpha'}\right)^2$$

Rectangular Aperture



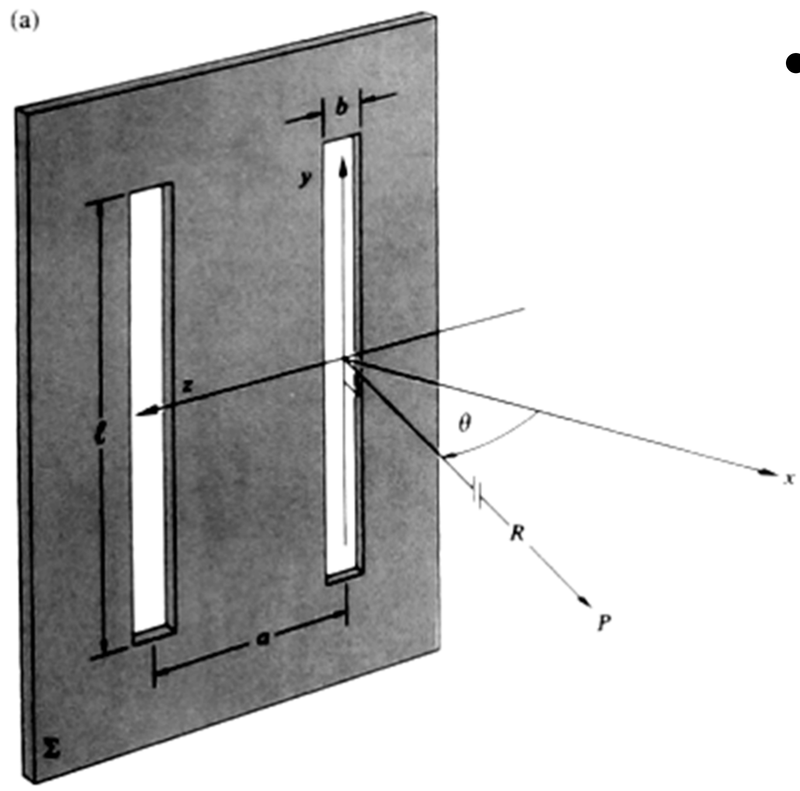


$$I(Y,Z) = I(0)\left(\frac{\sin\beta'}{\beta'}\right)^2\left(\frac{\sin\alpha'}{\alpha'}\right)^2$$

$$\beta' = \frac{1}{2}kbY/R$$

$$\alpha' = \frac{1}{2}kaZ/R$$

Double-Slit Fraunhofer Diffraction

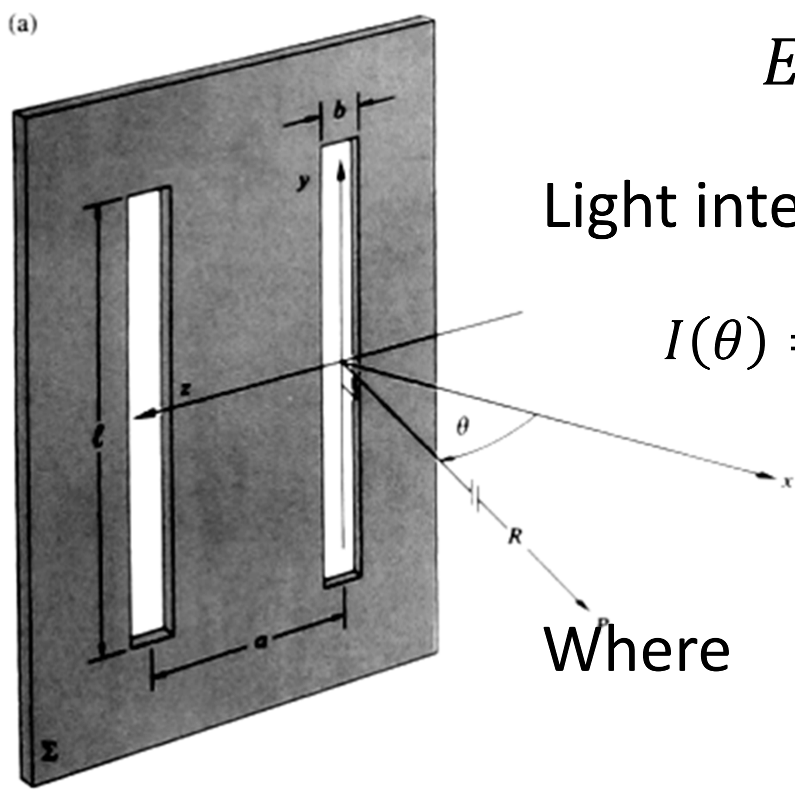


- Same idea, but this time we integrate over two slits:

$$E = \frac{\mathcal{E}_L}{R}\int_{-b/2}^{+b/2} e^{iky\sin\theta}\,dy + \frac{\mathcal{E}_L}{R}\int_{a-b/2}^{a+b/2} e^{iky\sin\theta}\,dy$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}\left(1 + e^{ika\,sin\,\theta}\right)$$

Double-Slit Fraunhofer Diffraction



$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} \left(1 + e^{ika\, sin\, \theta}\right)$$

Light intensity:

$$I(\theta) = 2I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \left(1 + \cos(ka\sin\theta)\right)$$

$$= 4I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha$$

Where

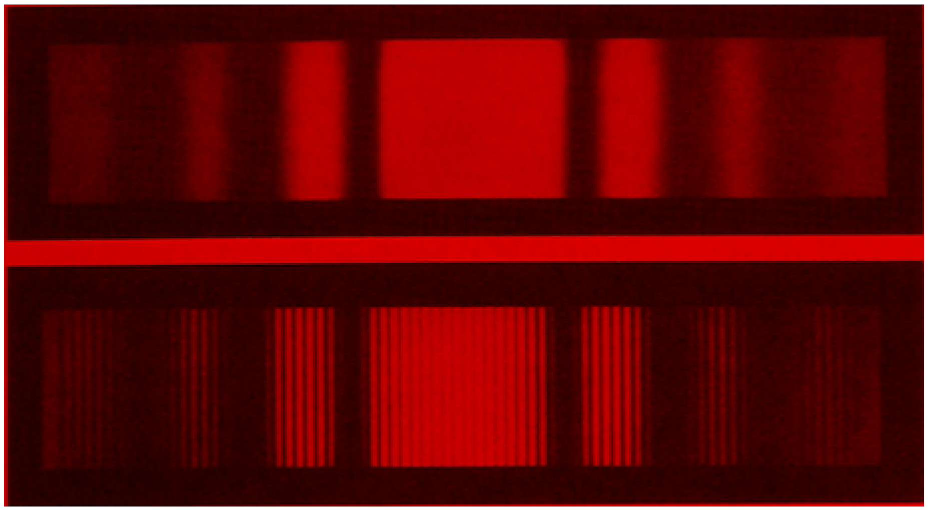
$$\alpha = \frac{1}{2} ka\sin\theta$$

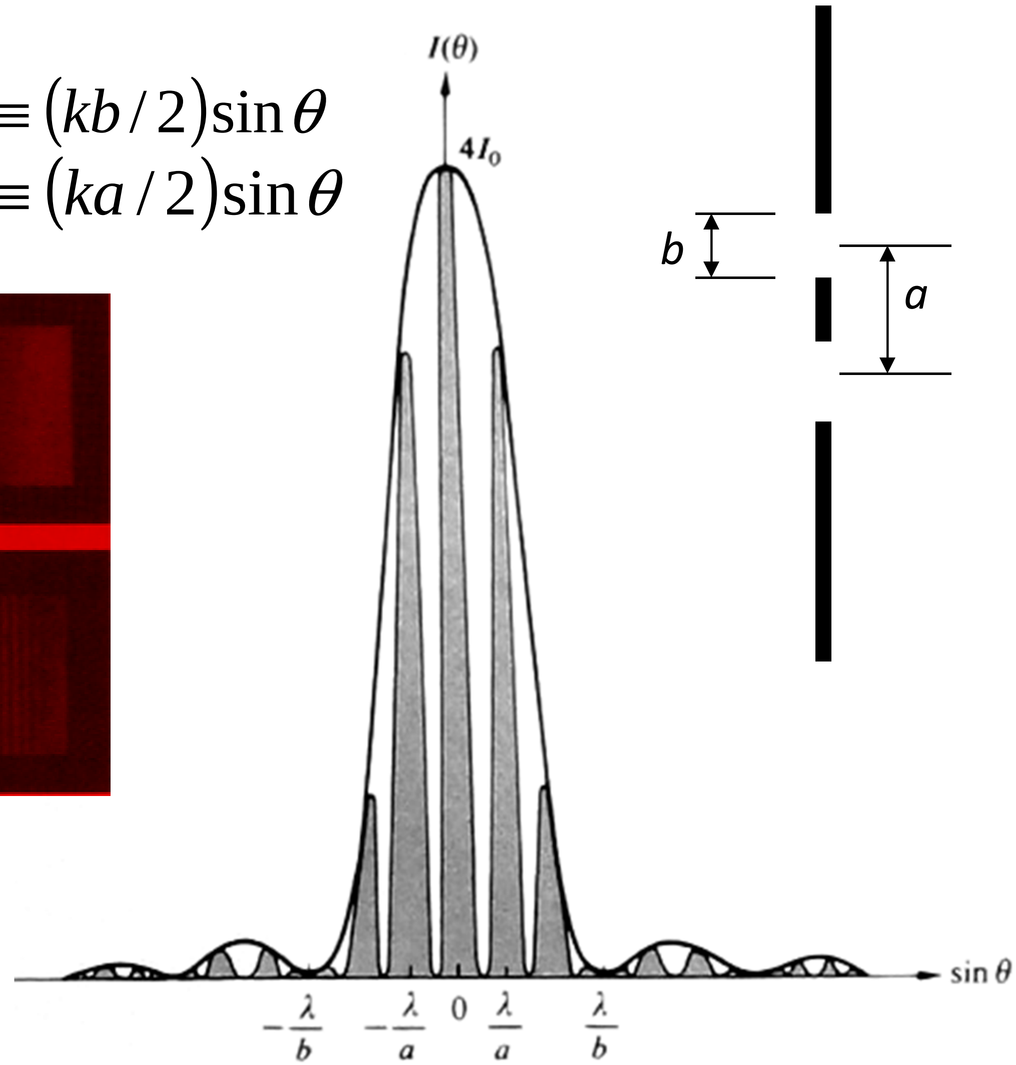
Since $a > b$, $\cos\alpha$ oscillates more rapidly than $\sin\beta$

Double Slit: Fraunhofer Diffraction

$$I(\theta) = 4I_0\left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha$$

$$\beta \equiv (kb/2)\sin\theta$$
$$\alpha \equiv (ka/2)\sin\theta$$





Minima: $\alpha = \pm\pi/2, \pm 3\pi/2$

$$a\sin\theta = (m+1/2)\lambda$$

m=0, ±1, ±2…

or: $\beta = m\pi$, where m=±1, ±2…

$$b\sin\theta = m\lambda$$

Three-Slit Fraunhofer Diffraction

$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} \frac{e^{3i\delta} - 1}{e^{i\delta} - 1}$$

$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} \frac{e^{3i\delta/2}}{e^{i\delta/2}} \frac{e^{3i\delta/2} - e^{-3i\delta/2}}{e^{i\delta/2} - e^{-i\delta/2}}$$

$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} e^{i\delta} \frac{\sin 3\delta/2}{\sin \delta/2}$$

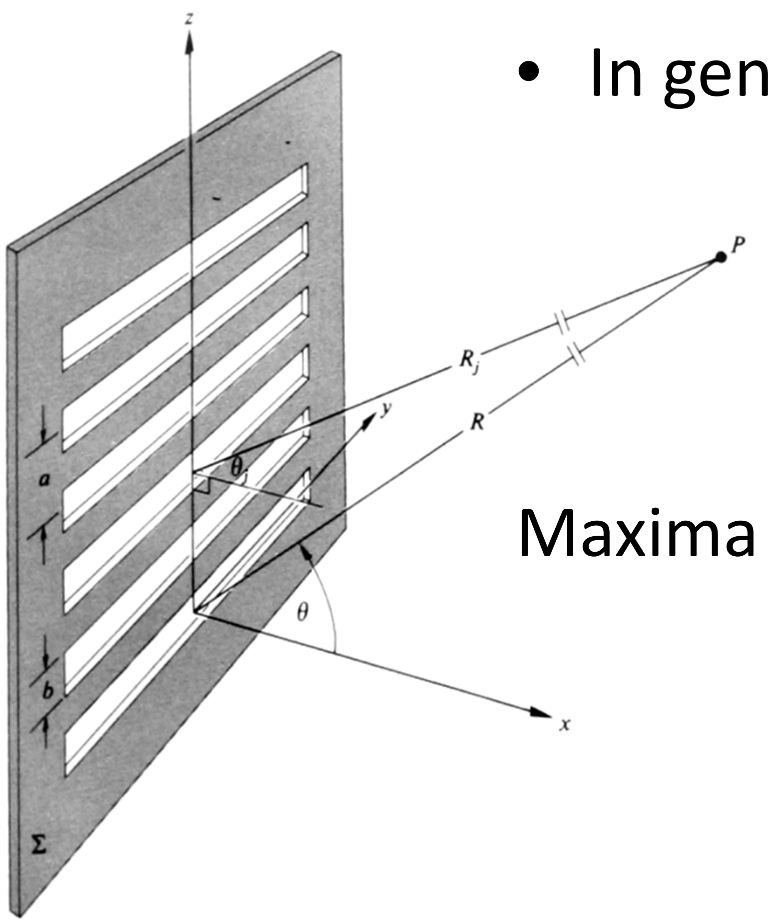
$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} e^{ika\sin\theta} \frac{\sin\left(\frac{3}{2} ka \sin\theta\right)}{\sin\left(\frac{1}{2} ka \sin\theta\right)}$$

$$= \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta} e^{2i\alpha} \frac{\sin 3\alpha}{\sin\alpha}$$

Light intensity: $I(\theta) = I(0) \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin 3\alpha}{\sin\alpha}\right)^2$

Three-Slit Fraunhofer Diffraction

- Light intensity: $I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin 3\alpha}{\alpha}\right)^2$



- In general, when there are N slits:

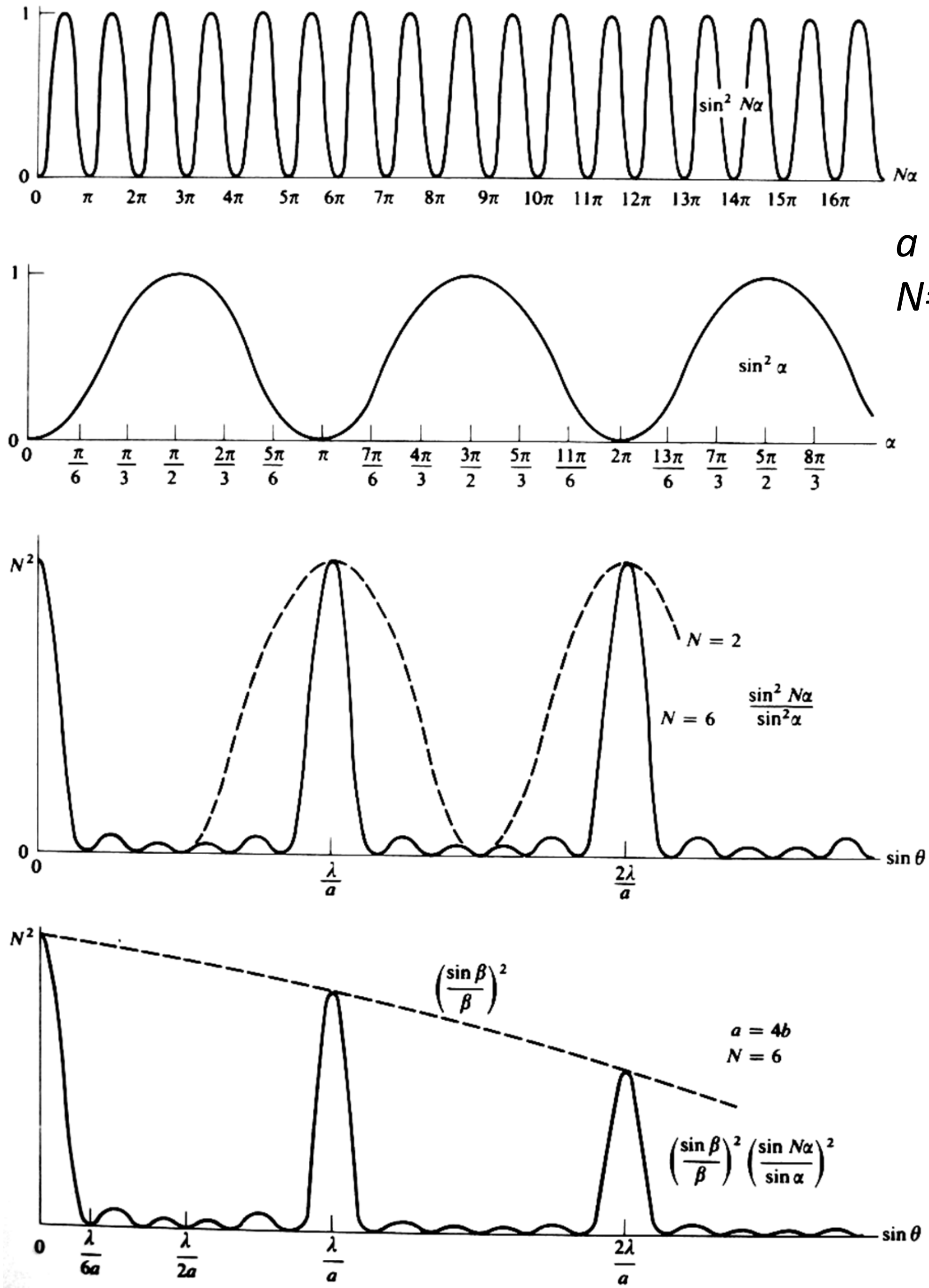
$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\alpha}\right)^2$$

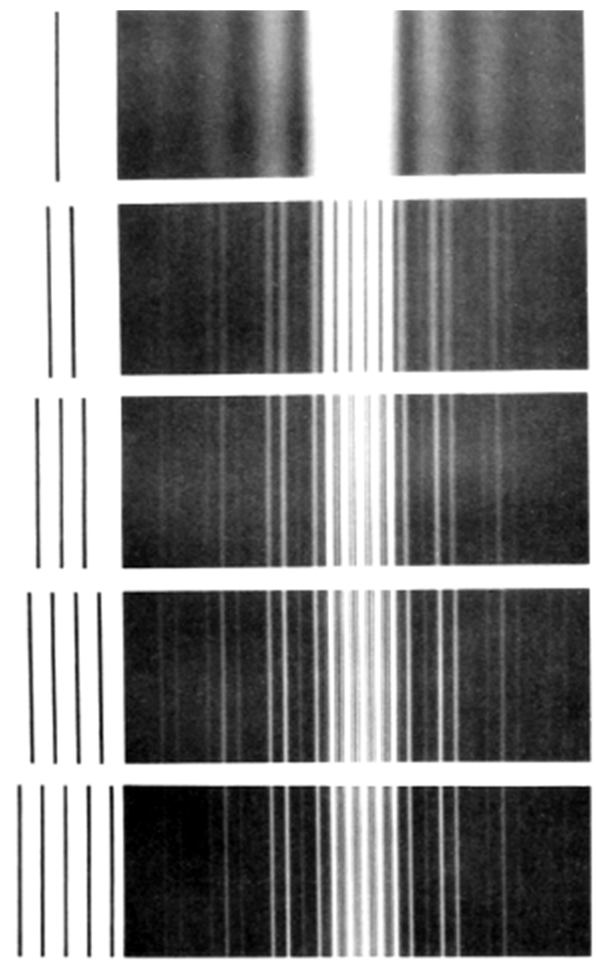
Maxima occur when $\alpha = \frac{1}{2}ka\sin\theta = m\pi$

$$a\sin\theta = m\lambda$$

$$I(\theta) = I_0 \left(\frac{\sin \beta}{\beta} \right)^2 \left(\frac{\sin N\alpha}{\sin \alpha} \right)^2$$

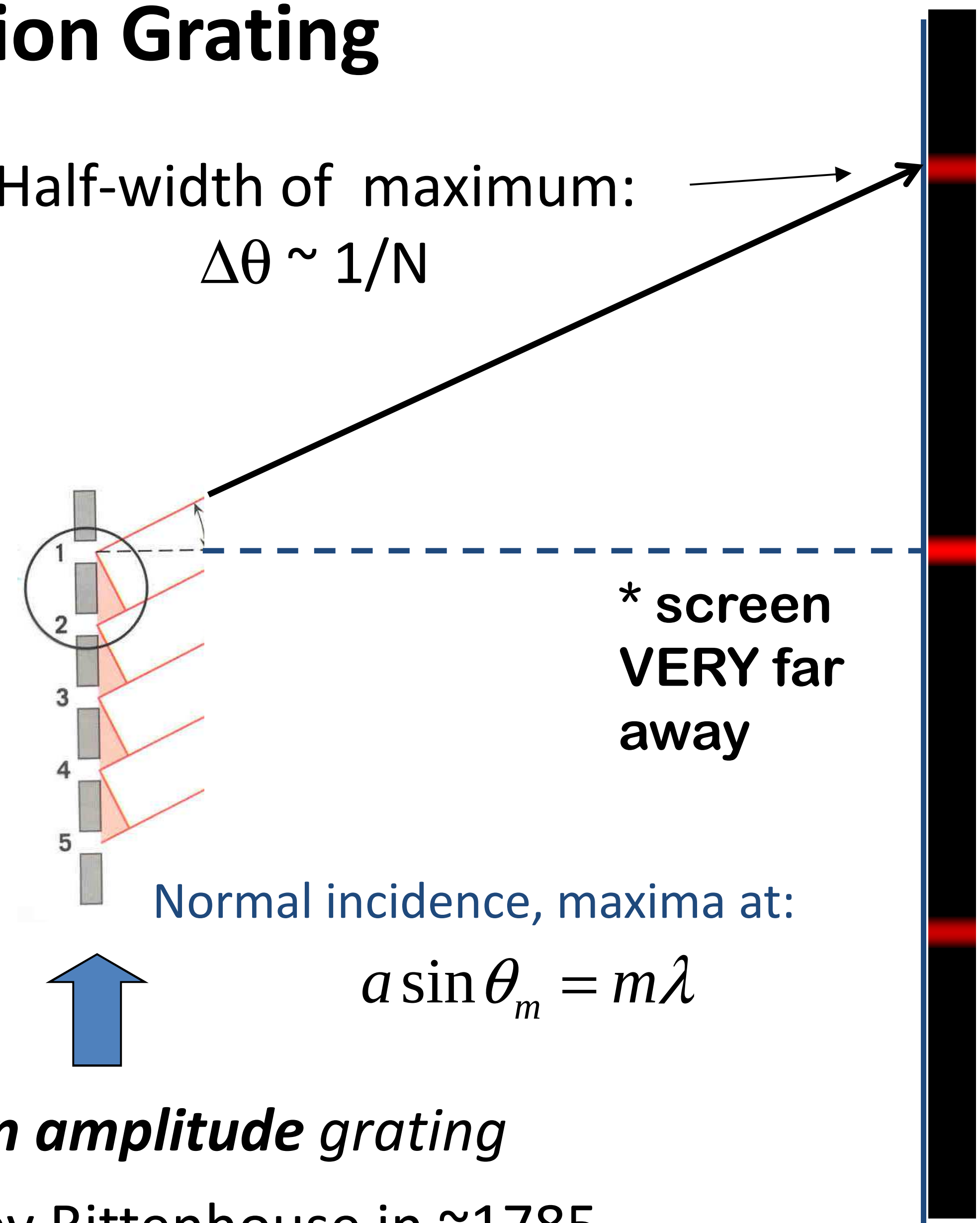
a = 4b
N=6





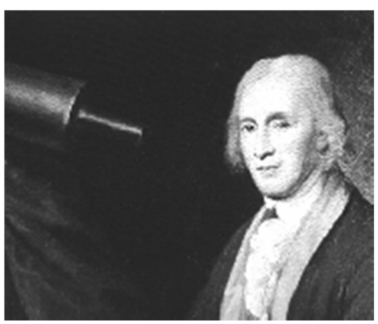
Diffraction Grating

Usually gratings have thousands of slits and are characterized by the number of slits per cm (for example: 6000 cm^{-1})



Transmission amplitude *grating*

Introduced by Rittenhouse in ~1785



David Rittenhouse
1732 - 1796