PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 33 – Polarization of Light and Interference

Spring 2014 Semester

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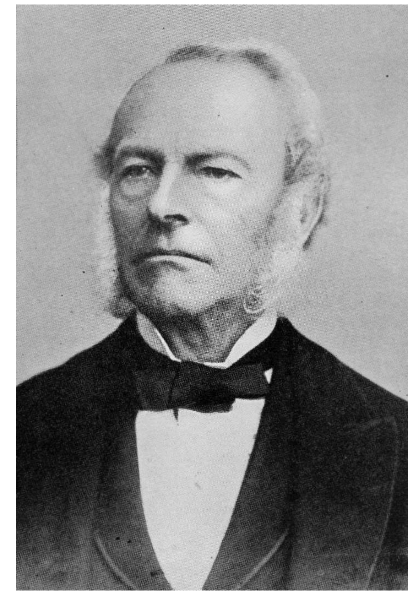
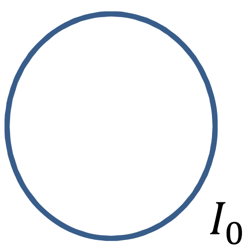
Stokes Parameters

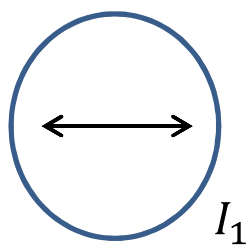
- Stokes considered a set of four polarizing filters
  - The choice is not unique…
- Each filter transmits exactly half the intensity of unpolarized light



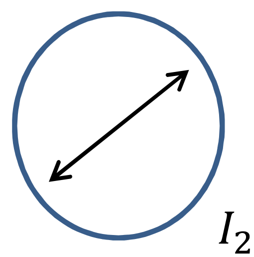
George Gabrial Stokes
1819-1903



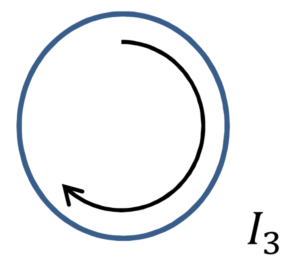
Unpolarized: filters out ½ the intensity of any incident light.



Linear: transmits only horizontal component



Linear: transmits only light polarized at 45°



Circular: transmits only R-polarized light

Stokes Parameters

- The Stokes parameters are defined as:

$$S_0 = 2I_0$$
$$S_1 = 2I_1 - 2I_0$$
$$S_2 = 2I_2 - 2I_0$$
$$S_3 = 2I_3 - 2I_0$$

- Usually normalize the incident intensity to 1.
- Unpolarized light:
  - half the light intensity is transmitted through each filter…

$S_0 = 1$ and $S_1 = S_2 = S_3 = 0$

The Jones Calculus

- Proposed by Richard Clark Jones in 1941
- Only applicable to beams of coherent light
- Electric field vectors:

$$\vec{E}_x(z,t) = E_{0x}\hat{\imath}\cos(kz - \omega t + \varphi_x)$$
$$\vec{E}_y(z,t) = E_{0y}\hat{\jmath}\cos(kz - \omega t + \varphi_y)$$

- Jones vector:

$$\tilde{E} = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix}$$

The Jones Calculus

- It is convenient to pick $\varphi_x = 0$ and normalize the Jones vector so that $\left|\tilde{E}\right| = 1$

$$\tilde{E} = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} \rightarrow \vec{E} = \frac{1}{\sqrt{E_{0x}^2 + E_{0y}^2}} \begin{bmatrix} E_{0x} \\ E_{0y} e^{i\xi} \end{bmatrix}$$

- Example:
  - Horizontal linear polarization: $\vec{E}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - Vertical linear polarization: $\vec{E}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - Linear polarization at $45°$: $\vec{E}_{45°} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The Jones Calculus

- Circular polarization:

$$\vec{E}_R = \frac{E_0}{2}[\hat{\imath}\cos(kz - \omega t) + \hat{\jmath}\sin(kz - \omega t)]$$

$$\vec{E}_L = \frac{E_0}{2}[\hat{\imath}\cos(kz - \omega t) - \hat{\jmath}\sin(kz - \omega t)]$$

- Linear representation:

$$\vec{E}_x(z,t) = E_{0x}\hat{\imath}\cos(kz - \omega t)$$

$$\vec{E}_y(z,t) = E_{0y}\hat{\jmath}\cos(kz - \omega t + \xi)$$

  – What value of ξ gives $\cos(kz - \omega t + \xi) = \sin(kz - \omega t)$?
  – That would be $\xi = -\pi/2$

The Jones Calculus

- Right circular polarization:

$$\vec{E}_R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{-i\pi/2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

- Left circular polarization:

$$\vec{E}_L = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{+i\pi/2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

- Adding the Jones vectors adds the electric fields, not the intensities:

$$\vec{E}_R + \vec{E}_L = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{2}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Horizontal linear polarization

The Jones Calculus

- When light propagates through an optical element, its polarization can change:

- $\overrightarrow{E'}$ and $\vec{E}$ are related by a 2x2 matrix (the Jones matrix):

$$\overrightarrow{E'} = A\,\vec{E}$$

- If light passes through several optical elements, then

$$\overrightarrow{E'} = A_n \cdots A_2 A_1 \vec{E}$$

(Remember to write the matrices in reverse order)

The Jones Calculus

Examples:

- Transmission through an optically inactive material:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Rotation of the plane of linear polarization (eg, propagation through a sugar solution)

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

  – When $\alpha = \frac{\pi}{2}$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

  – If $\vec{E}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $A\,\vec{E}_x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{E}_y$

The Jones Calculus

- Propagation through a quarter wave plate:

$$\vec{E} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \rightarrow \overrightarrow{E'} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

- What matrix achieves this?
  - The x-component is unchanged
  - The y-component is multiplied by $e^{-i\pi/2}$

$$A = \begin{bmatrix}1 & 0\\0 & e^{-i\pi/2}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}$$

- Note that an overall phase can be chosen for convenience and factored out
  - For example, in Hecht, Table 8.6: $A = e^{i\pi/4}\begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}$
  - Important not to mix inconsistent sets of definitions!

Mueller Matrices

- We can use the same approach to describe the change in the Stokes parameters as light propagates through different optical elements

$$S = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} \rightarrow S' = MS$$

M is a 4x4 matrix: “the Mueller matrix”

Mueller Matrices

- Example: horizontal linear polarizer
  - Incident unpolarized light

$$S_0 = 1, S_1 = 0, S_2 = 0, S_3 = 0$$

  - Emerging linear polarization

$$S_0 = \frac{1}{2}, S_1 = \frac{1}{2}, S_2 = 0, S_3 = 0$$

  - Mueller matrix:

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Mueller Matrices

- What is the polarization state of light that initially had right-circular polarization but passed through a horizontal polarizer?

$$S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S' = MS = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Mueller Matrices

- Example: linear polarizer with transmission axis at 45°:
  - Incident unpolarized light:
  $$S_0 = 1, S_1 = 0, S_2 = 0, S_3 = 0$$
  - Emerging linear polarization:
  $$S_0 = 1, S_1 = 0, S_2 = 1, S_3 = 0$$
  - Mueller matrix:
  $$M = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- This works in this specific case. You would need to check that it also works for other types of incident polarized light.

Jones Calculus/Mueller Matrices

- Some similarities:
  - Polarization state represented as a vector
  - Optical elements represented by matrices
- Differences:
  - Jones calculus applies only to coherent light
  - Jones calculus quantifies the phase evolution of the electric field components
  - Can be used to analyze interference
  - Stokes parameters only describe the irradiance (intensity) of light
  - Stokes parameters/Mueller matrices only apply to incoherent light – they do not take into account phase information or interference effects

Interference

- Electric field:

$$\vec{E}(\vec{x},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{x} - \omega t)$$

- Light intensity:

$$I = c\epsilon \left\langle \left|\vec{E}\right|^2 \right\rangle_T$$

- Two electric fields:

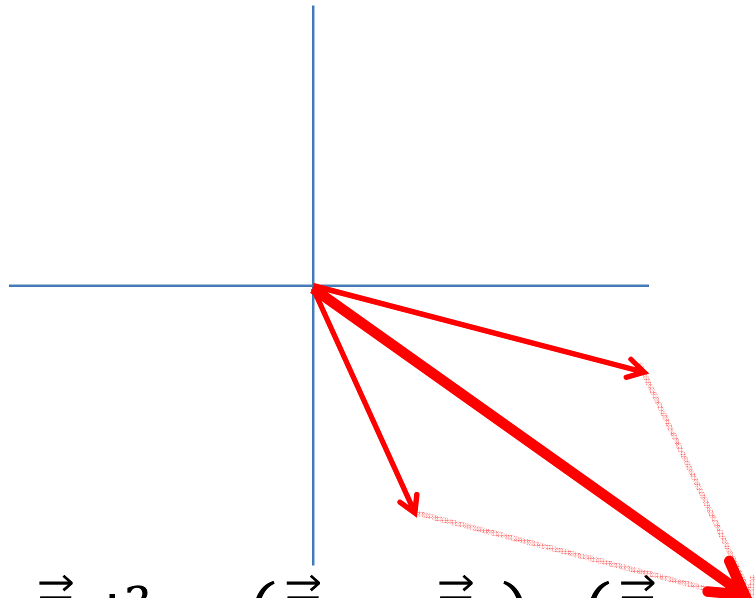
$$\vec{E}_1(\vec{x},t) = \vec{E}_{10} \cos(\vec{k}\cdot\vec{x} - \omega t + \xi_1)$$
$$\vec{E}_2(\vec{x},t) = \vec{E}_{20} \cos(\vec{k}\cdot\vec{x} - \omega t + \xi_2)$$

- Light intensity:

$$I = v\epsilon \langle |\vec{E}_1 + \vec{E}_2|^2 \rangle_T$$

Interference

$$I = v\epsilon\left\langle|\vec{E}_1 + \vec{E}_2|^2\right\rangle_T$$



$$|\vec{E}_1 + \vec{E}_2|^2 = \left(\vec{E}_1 + \vec{E}_2\right)\cdot\left(\vec{E}_1 + \vec{E}_2\right)$$

$$= \left|\vec{E}_1\right|^2 + \left|\vec{E}_2\right|^2 + 2\vec{E}_1 \cdot \vec{E}_2$$

$$I = v\epsilon\left\langle\left|\vec{E}_1\right|^2\right\rangle_T + v\epsilon\left\langle\left|\vec{E}_2\right|^2\right\rangle_T + 2v\epsilon\left\langle\vec{E}_1 \cdot \vec{E}_2\right\rangle_T$$

Interference

$$I = v\epsilon\left\langle\left|\vec{E}_1\right|^2\right\rangle_T + v\epsilon\left\langle\left|\vec{E}_2\right|^2\right\rangle_T + 2v\epsilon\langle\vec{E}_1\cdot\vec{E}_2\rangle_T$$

$$= I_1 + I_2 + I_{12}$$

$$I_{12} = v\epsilon\vec{E}_{01}\cdot\vec{E}_{02}\cos\delta$$

Phase difference: $\delta = \vec{k}_1\cdot\vec{x} - \vec{k}_2\cdot\vec{x} + \xi_1 - \xi_2$

- Why didn't we care about I_{12} when discussing geometric optics?
  - Incoherent light: $\langle I_{12}\rangle = 0$
  - Random polarizations
  - Path lengths long compared with λ: $\langle I_{12}\rangle = 0$
  - Many possible paths for light to propagate along

Interference

- Another way to have $I_{12} = 0$ is when the electric fields are orthogonal:

$$I = I_1 + I_2 + I_{12}$$

$$I_{12} = v\epsilon \vec{E}_{01} \cdot \vec{E}_{02} \cos\delta$$

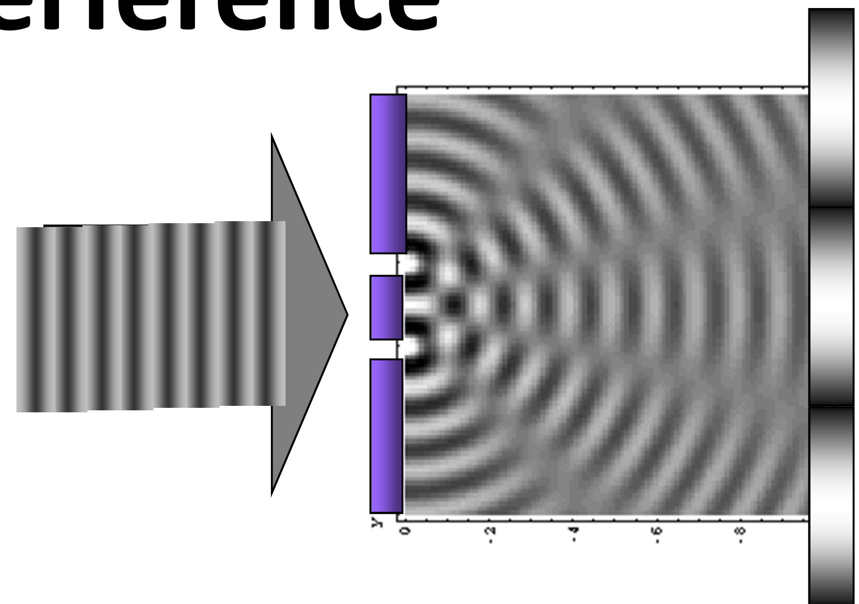
- If the two electric fields correspond to opposite polarization, then there is no interference
- If the two electric fields are parallel (same polarization), then

$$I_{12} = 2\sqrt{I_1 I_2} \cos\delta$$

- Interference depends on the phase difference

Interference

- Two point sources:



$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

- Constructive interference: $\cos\delta > 0$
- Total constructive interference: $\cos\delta = 0, \pm 2\pi, \ldots$
- Destructive interference: $\cos\delta < 0$
- Total destructive interference: $\cos\delta = \pm\pi, \pm 3\pi, \ldots$
- Special case when $\vec{E}_{01} = \vec{E}_{02}$:

$$I = 2I_0(1 + \cos\delta) = 4I_0\cos^2\frac{\delta}{2}$$

Conservation of Energy

- Energy should be conserved…
- The intensity is greater than the incoherent sum in some places, but less than the incoherent sum in other places:

$$I = I_1 + I_2 + I_{12}$$
$$I_{12} = v\epsilon \vec{E}_{01} \cdot \vec{E}_{02} \cos\delta$$

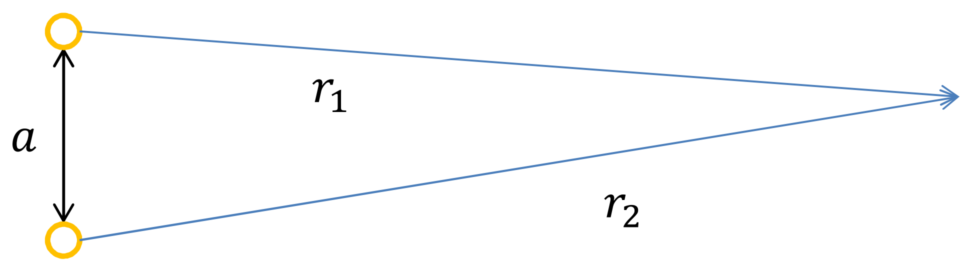
- Positive definite: I_1 and I_2
- Positive and negative: I_{12}
- Spatial average of I_{12} is zero

Interference Maxima and Minima

$$I = 2I_0(1 + \cos\delta) = 4I_0 \cos^2\frac{\delta}{2}$$

(when $\vec{E}_{01} = \vec{E}_{02}$)

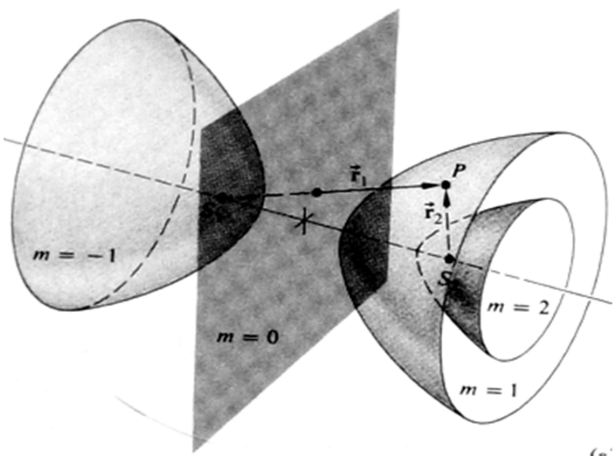
- Recall that $\delta = \vec{k}_1 \cdot \vec{x} - \vec{k}_2 \cdot \vec{x} + \xi_1 - \xi_2$
- Consider the following case:
  - the sources are at different positions
  - $|\vec{k}_1| = |\vec{k}_2| = k$
  - the sources are in phase, $\xi_1 - \xi_2 = 0$

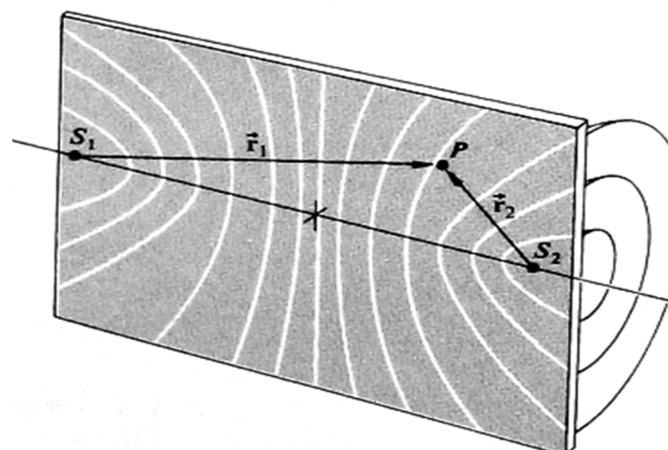


Interference Maxima and Minima

$$\delta = \vec{k}_1 \cdot \vec{x} - \vec{k}_2 \cdot \vec{x} + \xi_1 - \xi_2$$
$$= k(r_1 - r_2)$$
$$I = 4I_0 \cos^2\frac{\delta}{2} = 4I_0 \cos^2\frac{1}{2}k(r_1 - r_2)$$

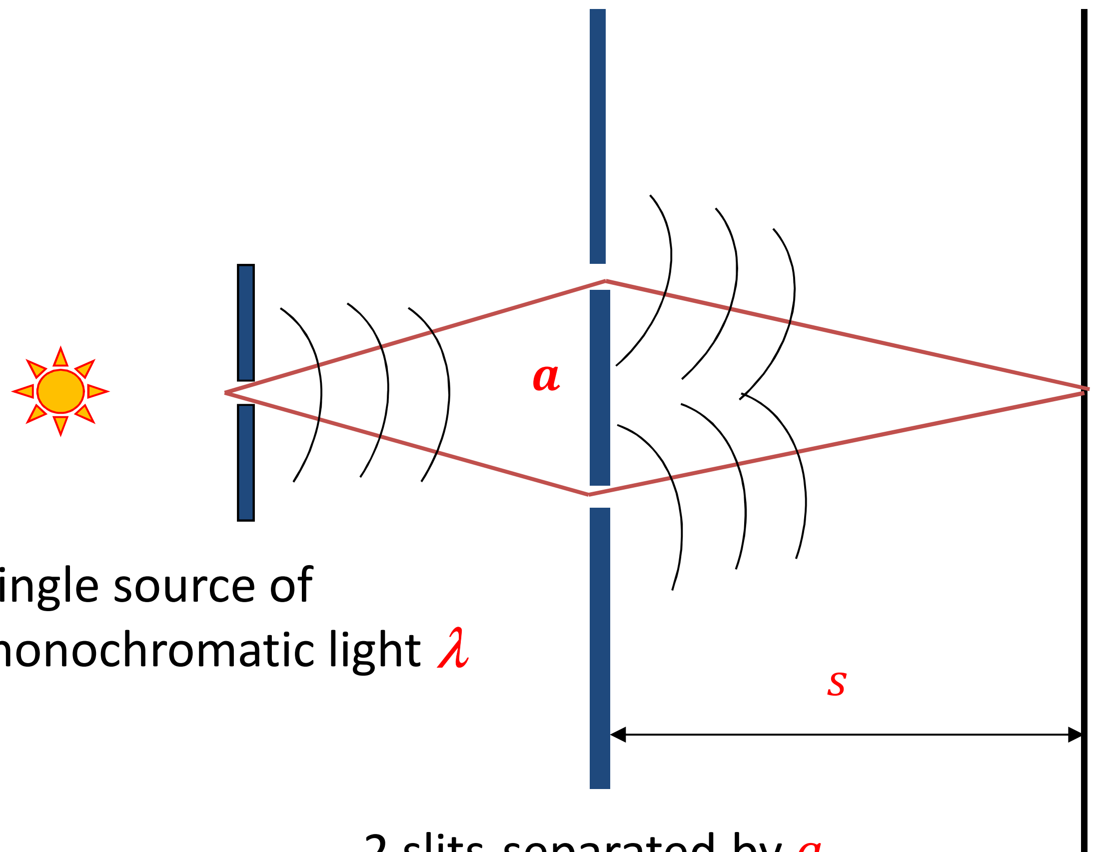
- Maximum when $(r_1 - r_2) = \frac{2\pi m}{k} = m\lambda,\ m = 0, \pm 1, \pm 2, \ldots$
- Minimum when $(r_1 - r_2) = \frac{\pi m'}{k} = \frac{m'}{2}\lambda,\ m' = \pm 1, \pm 3, \ldots$





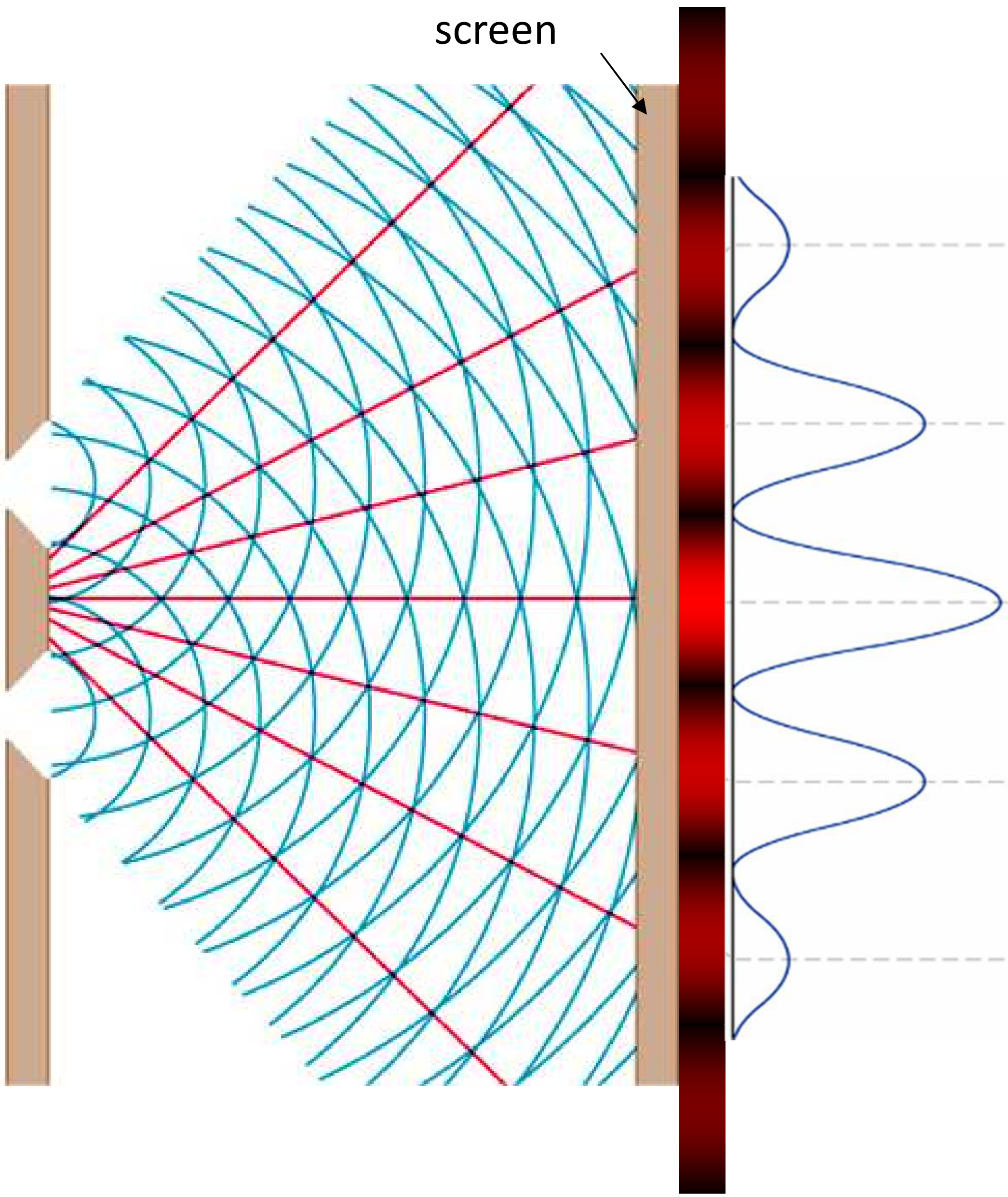
hyperboloid of revolution

Young’s Double-Slit Experiment



The rays start in phase, and travel the same distance, so they will arrive in phase.

Young's Double-Slit Experiment: Screen



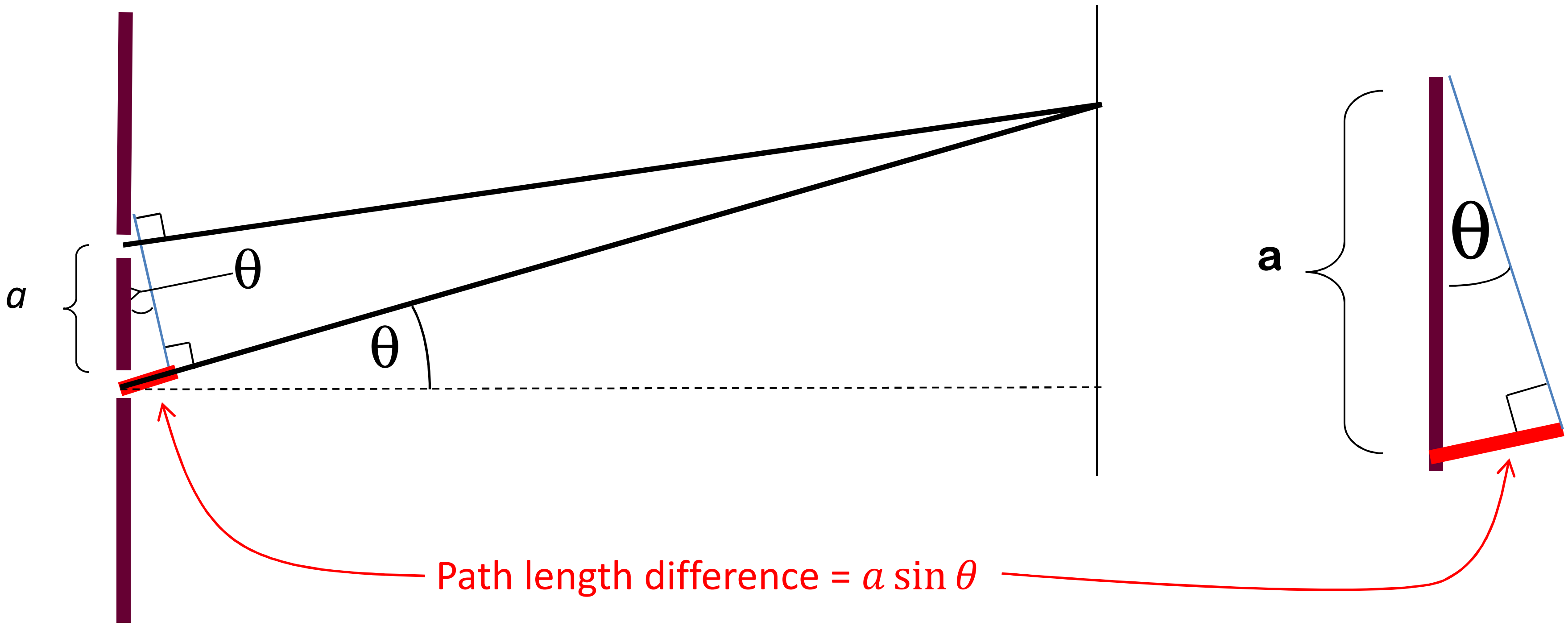
Path difference changes across the screen:

Sequence of minima and maxima

At points where the difference in path length is **0, ±λ, ± 2λ, …,** the screen is bright, (constructive).

At points where the difference in path length is $\pm\frac{\lambda}{2}, \pm\frac{3\lambda}{2}, \ldots$, the screen is dark, (destructive).

Young's Double-Slit Experiment



Constructive interference $a\sin\theta = m\lambda$

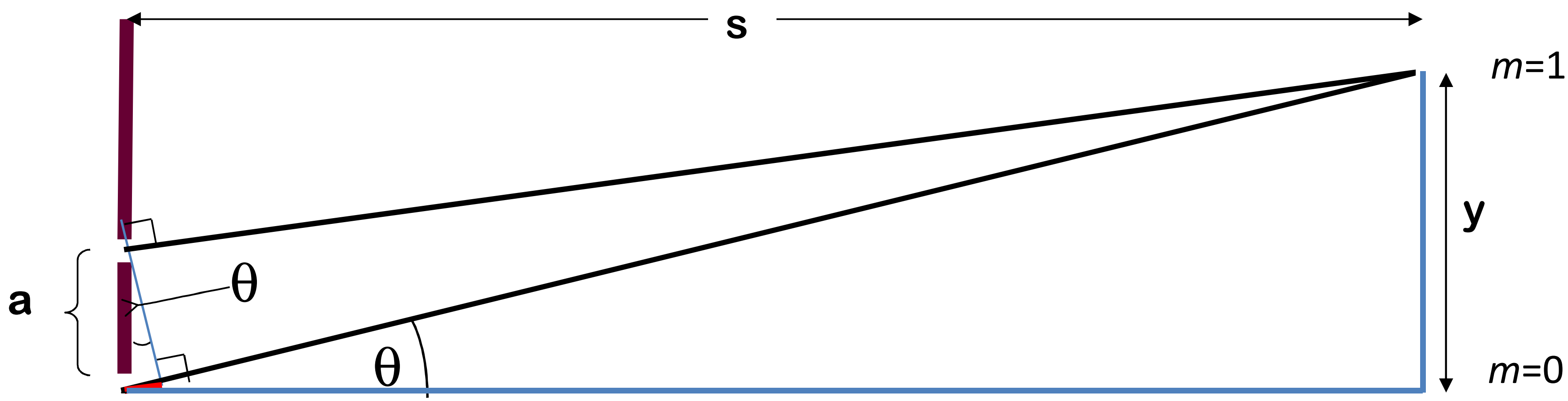
Destructive interference $a\sin\theta = \left(m + \frac{1}{2}\right)\lambda$

where m = 0, ±1, ± 2, …

Need $\lambda < a$ for distinct maxima

Example

Two slits 0.46 mm apart are 500 cm away from the screen. What would be the distance between the zero'th and first maximum for light with λ=532 nm?



$$a \sin\theta = m\lambda$$

$$\downarrow$$

$$ay/s = m\lambda$$

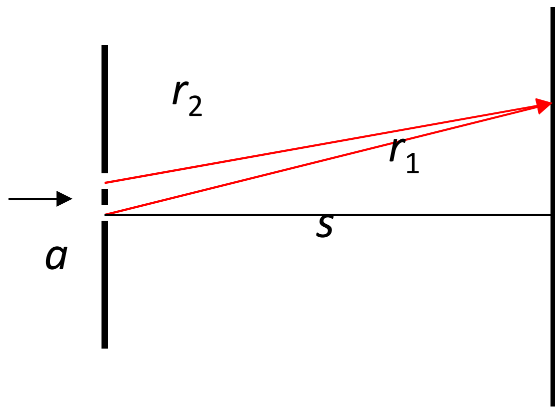
$$\downarrow$$

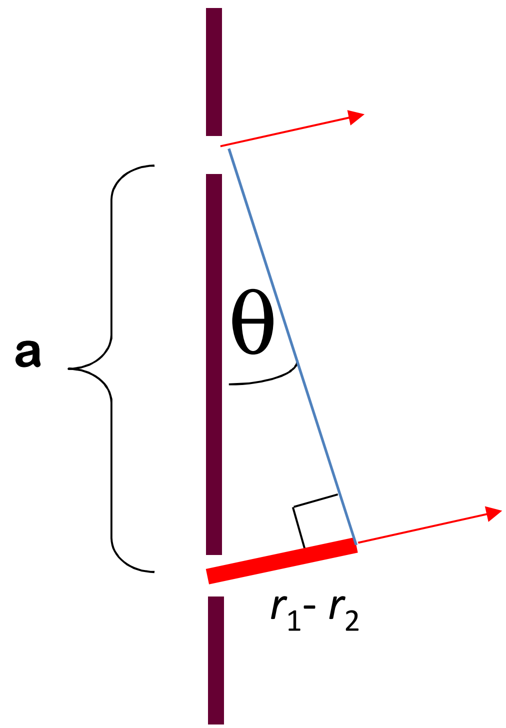
$$y = \frac{m\lambda s}{a}$$

Geometry: $\sin\theta \approx \tan\theta = y/s$ (for small θ)

$$\longrightarrow \quad y = \frac{(1)(532 \times 10^{-7}\ cm)(500\ cm)}{(0.046\ cm)} = 0.58\ \text{cm}$$

Young's Double Slit Experiment





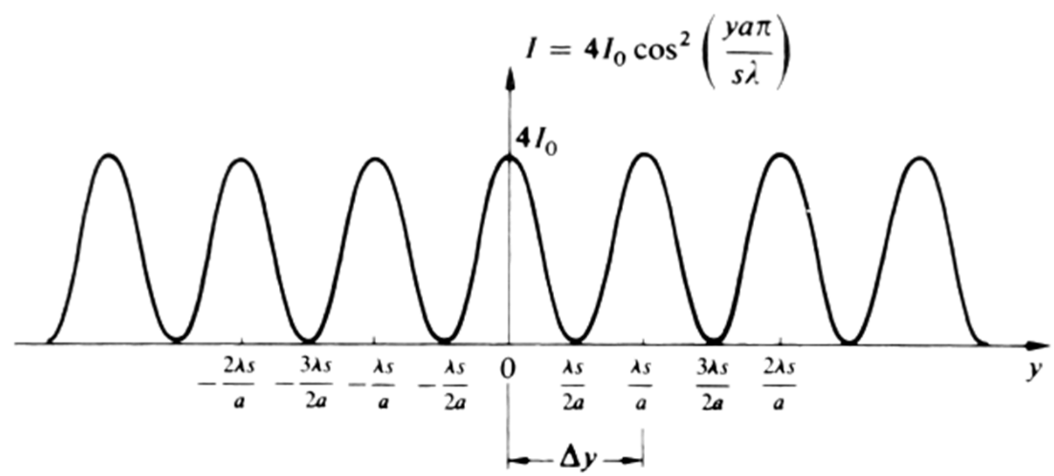
Far from the source, $s \gg a$,

$$I = 4I_0 \cos^2 \frac{\delta}{2}$$

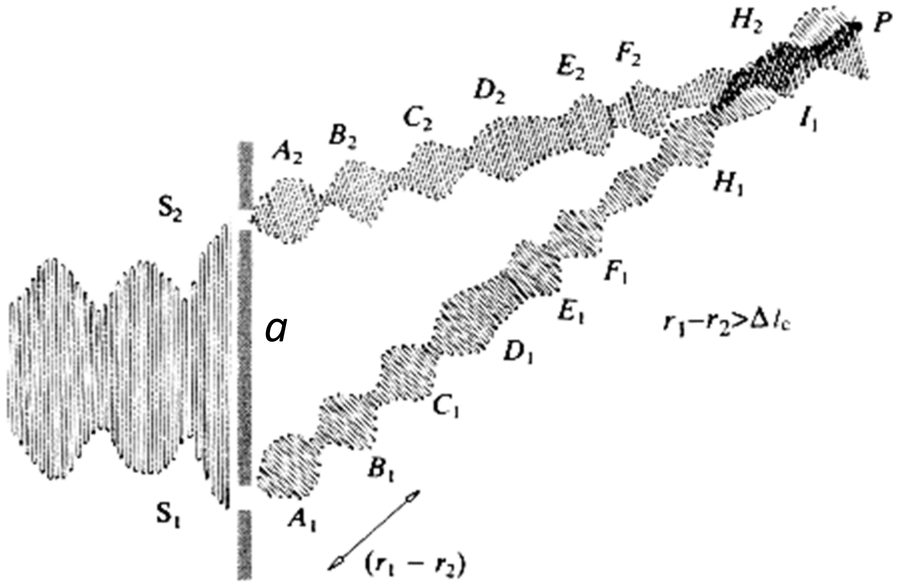
$$= 4I_0 \cos^2 \left(\frac{k(r_1 - r_2)}{2}\right)$$

$$r_1 - r_2 = a \sin\theta \approx a \tan\theta \approx \frac{ay}{s}$$

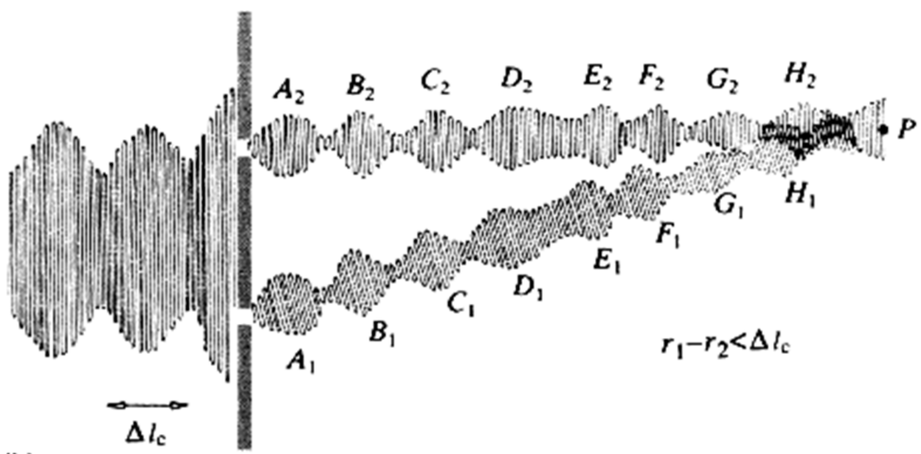
$$I \approx 4I_0 \cos^2 \frac{kay}{2s} = 4I_0 \cos^2 \frac{\pi a y}{s\lambda}$$



Coherence Length



(a)



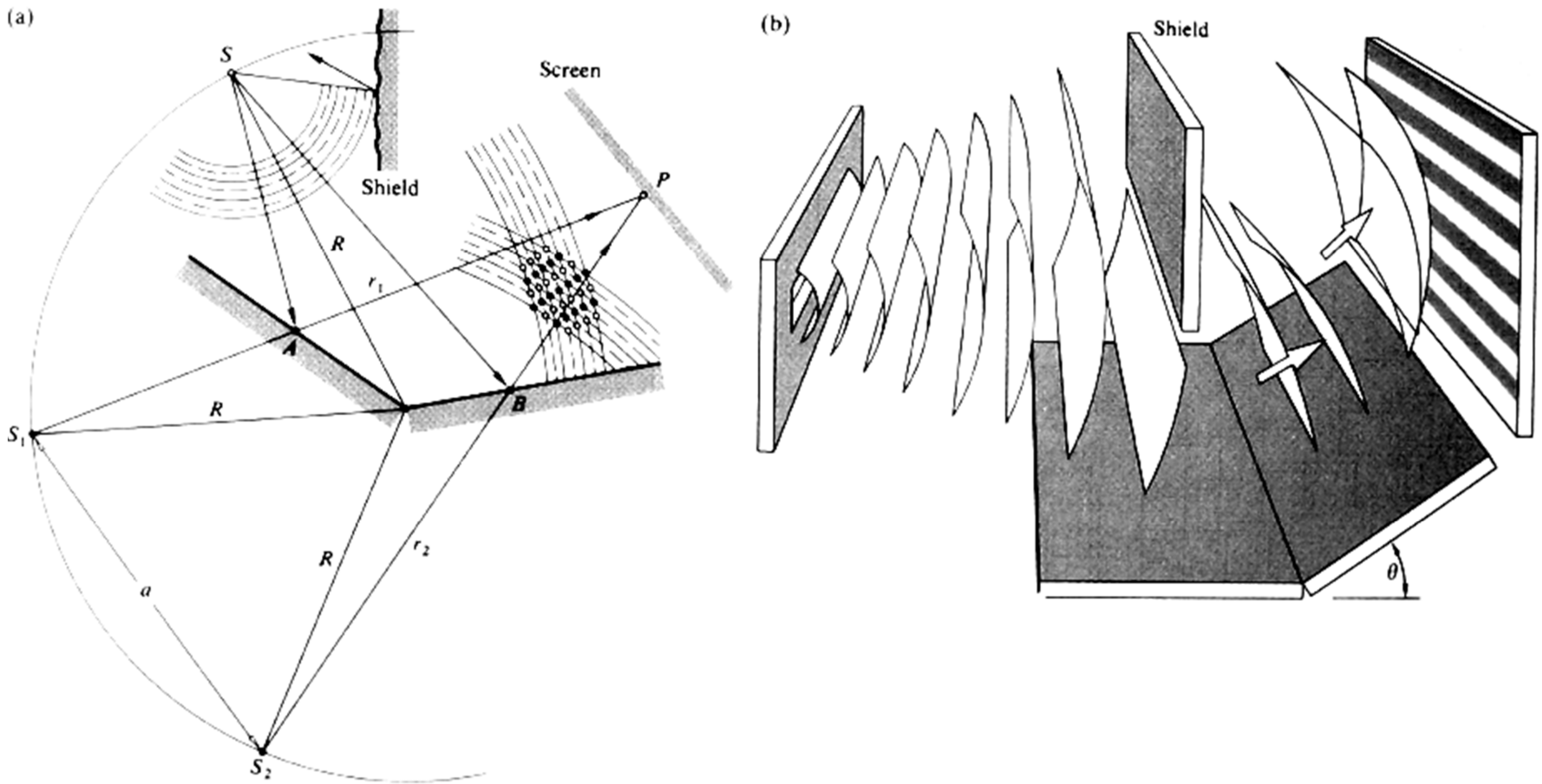
(b)

1. Spatial coherence: wave front should be coherent over distance *a*
2. Spatial coherence:
   $$r_1 - r_2 < l_c$$
3. Waves should not be orthogonally polarized

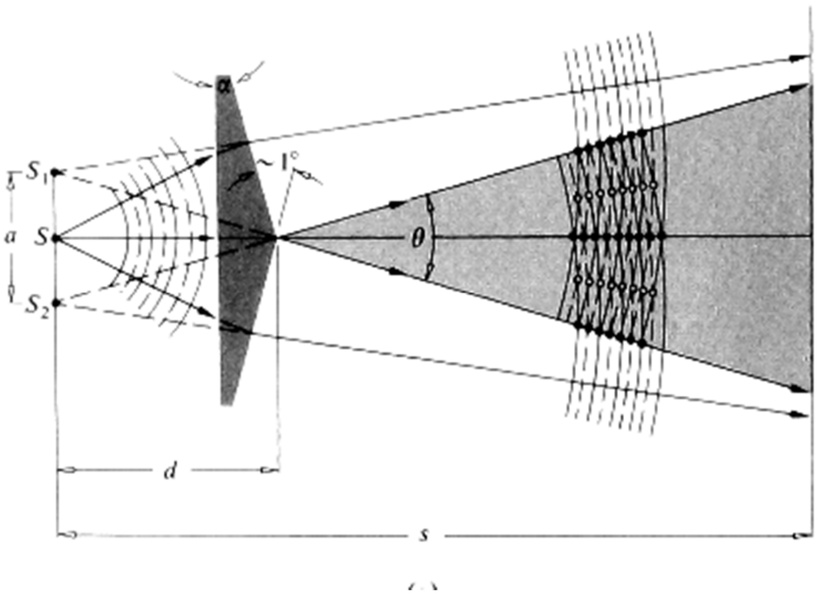
Lasers have very long coherence lengths

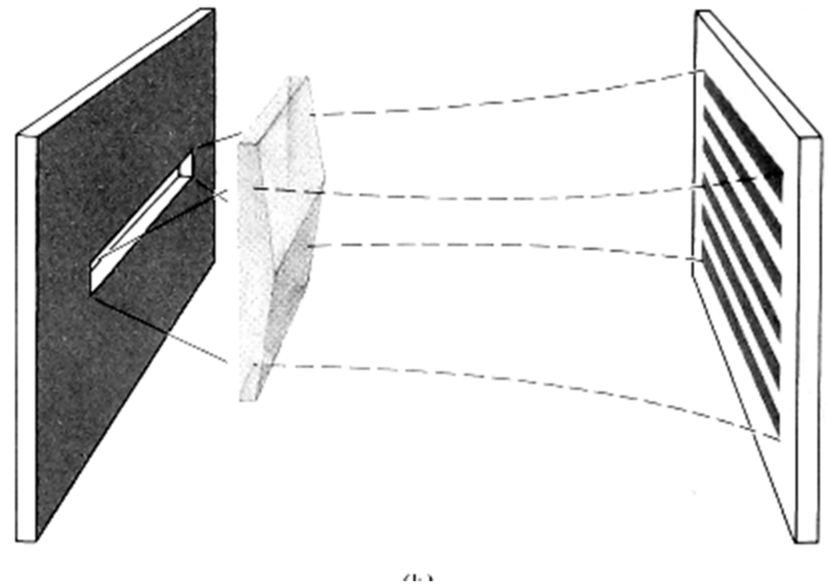
White light is coherent only over short distances: $l_c \sim 3\lambda$

Other Interference Experiments: Fresnel's Double Mirror Interferometer



Other Interference Experiments: Fresnel's Double Prism Interferometer





Other Interference Experiments: Lloyd's Mirror Interferometer