Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 30 – Electromagnetic Waves

Spring 2014 Semester

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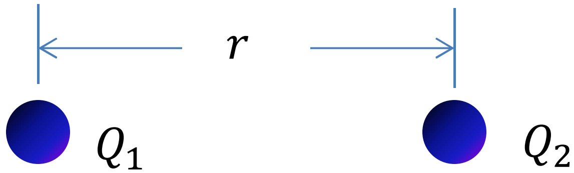
Electromagnetism

- Geometric optics overlooks the wave nature of light.
  - Light inconsistent with longitudinal waves in an ethereal medium
  - Still an excellent approximation when feature sizes are large compared with the wavelength of light
- But geometric optics could not explain
  - Polarization
  - Diffraction
  - Interference
- A unified picture was provided by Maxwell c. 1864

Forces on Charges

- Coulomb's law of electrostatic force:



- The magnitude of the attractive/repulsive force is

$$\vec{F} = k \frac{|Q_1||Q_2|}{r^2} \hat{r}$$

where

$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \; N \cdot m^2 \cdot C^{-2}$$

and therefore

$$\epsilon_0 = 8.85 \times 10^{-12} \; C^2 \cdot N^{-1} \cdot m^{-2}$$

(This constant is called the "permittivity of free space")

Electric Field

- An electric charge changes the properties of the space around it.
  - It is the source of an "electric field".
  - It could be defined as the "force per unit charge":

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} \quad \Rightarrow \quad \vec{F} = q\vec{E}$$

  - Quantum field theory provides a deeper description…
- Gauss's Law:

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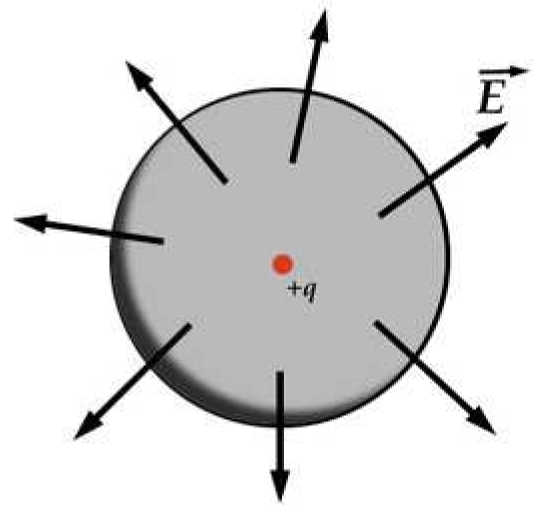
$$\underbrace{\int_S \hat{n}\cdot\vec{E}\,dA}_{\text{Electric "flux" through surface S}} = \frac{Q_{inside}}{\epsilon_0}$$

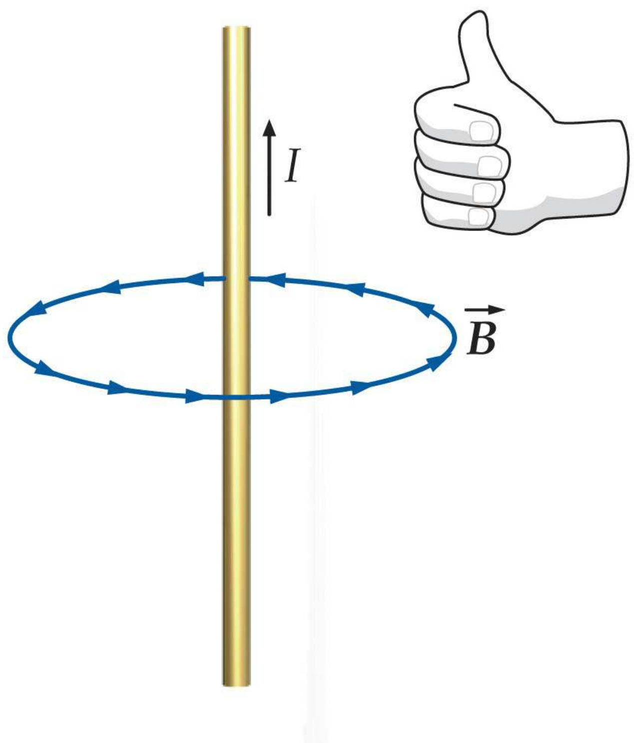
Electric "flux" through surface S



Magnetic Field

- Moving charges (ie, electric current) produce a magnetic field:

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{\ell} \times \hat{r}}{r^2}$$



A moving charge in a magnetic field experiences a force:

$$\vec{F} = q\, \vec{v} \times \vec{B}$$

Lorentz force law:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Gauss's Law for Magnetism

- Electric charges produce electric fields:

$$\int_S \hat{n} \cdot \vec{E}\, dA = \frac{Q_{inside}}{\epsilon_0}$$

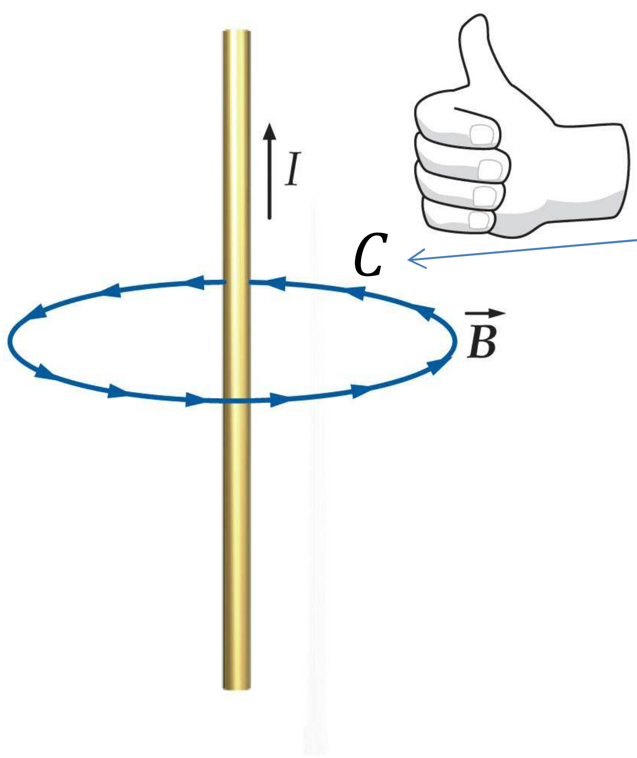
- But there are no "magnetic charges":

2

$$\underbrace{\int_S \hat{n} \cdot \vec{B}\, dA}_{\text{Magnetic "flux" through surface S}} = 0$$

Ampere's Law

- An electric current produces a magnetic field that curls around it:

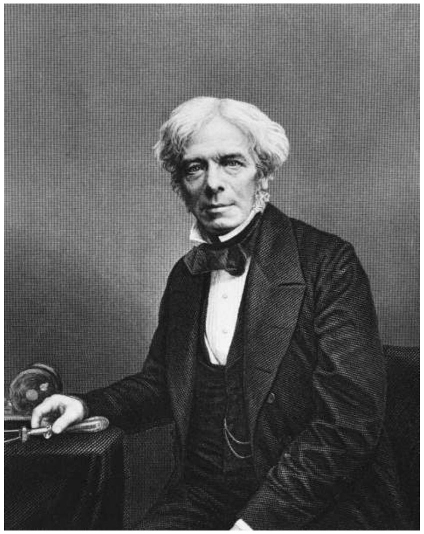


$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$$

(Close, but not quite the whole story)

Faraday's Law of Magnetic Induction

- Magnetic flux: $\phi_m = \int_S \vec{B} \cdot \hat{n}\, dA$
- Faraday observed that a changing magnetic flux through a wire loop induced a current
  - It transferred energy to the charge carriers in the wire



$$\mathcal{E} = -\frac{d\phi_m}{dt}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt}\int_S \hat{n} \cdot \vec{B}\, dA$$

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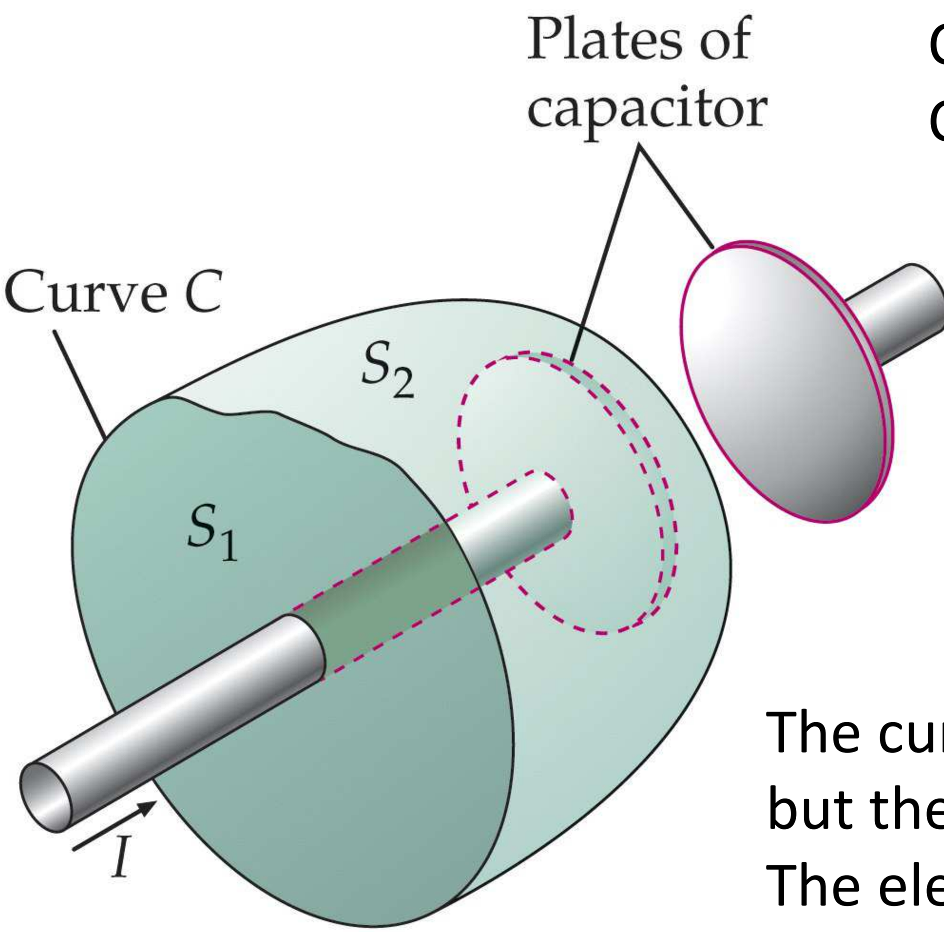
The Problem with Ampere's Law



Current through S_1 is I
Current through S_2 is 0!

The current through S_2 is zero, but the electric flux is not zero. The electric flux changes as charge flows onto the capacitor.

Maxwell's Displacement Current

- We can think of the changing electric flux through S_2 as if it were a current:

$$I_d = \epsilon_0 \frac{d\phi_e}{dt} = \epsilon_0 \frac{d}{dt} \int_{S_2} \vec{E} \cdot \hat{n}\, dA$$

$$\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \boldsymbol{\mu_0}(\boldsymbol{I} + \boldsymbol{I_d}) = \boldsymbol{\mu_0}\mathbf{I} + \boldsymbol{\mu_0}\boldsymbol{\epsilon_0} \frac{\boldsymbol{d}}{\boldsymbol{dt}} \int_{S_2} \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}}\, \boldsymbol{dA}$$

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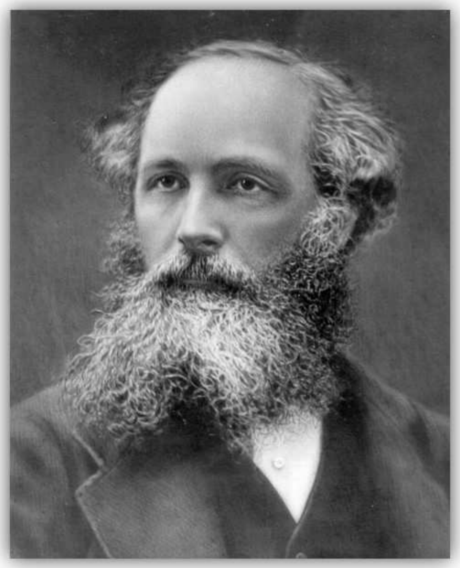
Maxwell's Equations (1864)



$$\oint_S \hat{n} \cdot \vec{E}\, dA = \frac{Q_{inside}}{\epsilon_0} \qquad (1)$$

$$\oint_S \hat{n} \cdot \vec{B}\, dA = 0 \qquad (2)$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt} \qquad (3)$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\phi_e}{dt} \qquad (4)$$

Maxwell's Equations in Free Space

In "free space" where there are no electric charges or sources of current, Maxwell's equations are quite symmetric:

$$\oint_S \hat{n} \cdot \vec{E}\, dA = 0$$

$$\oint_S \hat{n} \cdot \vec{B}\, dA = 0$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \varepsilon_0 \frac{d\phi_e}{dt}$$

Maxwell's Equations in Free Space

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \varepsilon_0 \frac{d\phi_e}{dt}$$

A changing magnetic flux induces an electric field.

A changing electric flux induces a magnetic field.

Will this process continue indefinitely?

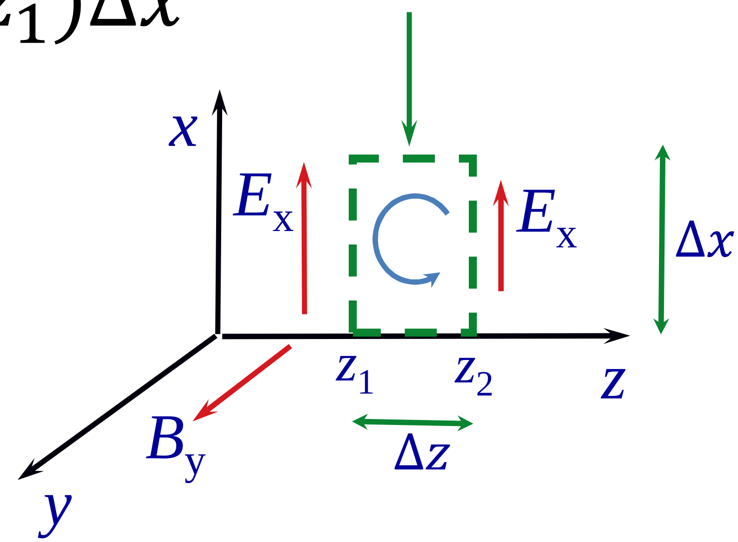
Light is an Electromagnetic Wave

- Faraday's Law:

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_m}{dt} = -\frac{\partial B_y}{\partial t} \Delta x \Delta z$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = E_x(z_2)\Delta x - E_x(z_1)\Delta x$$

$$\approx \frac{\partial E_x}{\partial z} \Delta z \Delta x$$



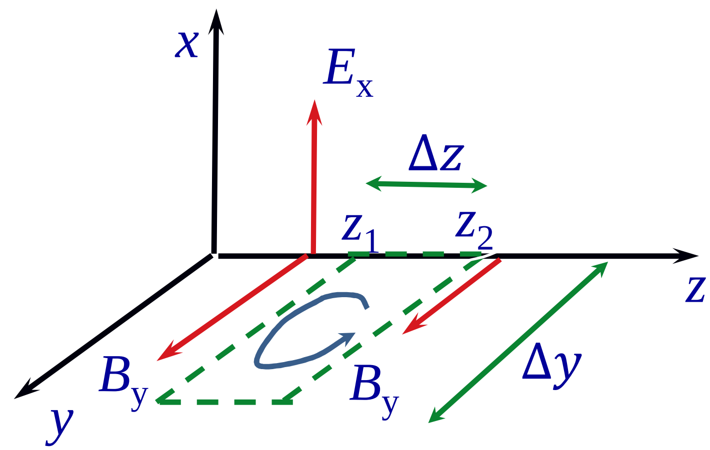
Light is an Electromagnetic Wave

- Ampere's law:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\phi_e}{dt} = \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \Delta y \Delta z$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = B_y(z_1)\Delta y - B_y(z_2)\Delta y$$

$$\approx -\frac{\partial B_y}{\partial z} \Delta z \Delta y$$



Putting these together…

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$-\frac{\partial B_y}{\partial z} = \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t}$$

Differentiate the first with respect to z:

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial^2 B_y}{\partial z \partial t}$$

Differentiate the second with respect to t:

$$-\frac{\partial^2 B_y}{\partial z \partial t} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\boldsymbol{\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}}$$

Velocity of Electromagnetic Waves

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\boldsymbol{\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}}$$

Speed of wave propagation is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$= \frac{1}{\sqrt{(4\pi \times 10^{-7} N/A^2)(8.854 \times 10^{-12}\ C^2/N \cdot m)}}$$

$$= \mathbf{2.998 \times 10^8\ m/s}$$

(speed of light)

Light is an Electromagnetic Wave

Speed of light was measured by Fizeau in 1849:

$$v = 315{,}300\ \text{km/s}$$

Maxwell wrote:

This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.

Electromagnetic Waves

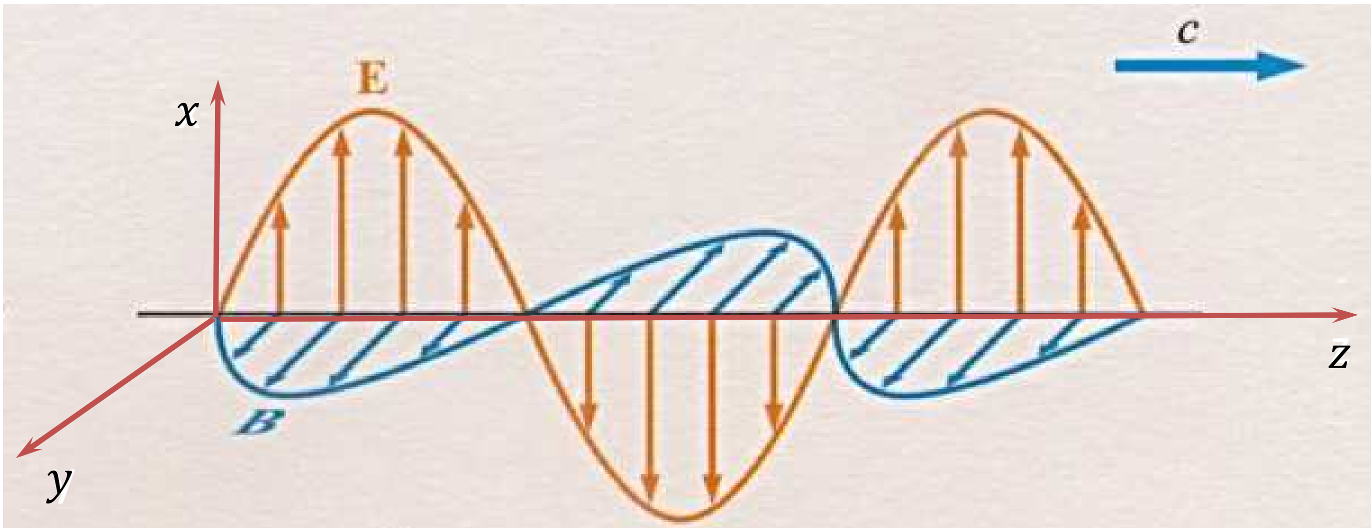
$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

- A solution is $E_x(z,t) = E_0 \sin(kz - \omega t)$ where $\omega = kc = 2\pi c/\lambda$
- What is the magnetic field?

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} = -kE_0 \cos(kz - \omega t)$$

$$B_y(x,t) = \frac{k}{\omega} E_0 \sin(kx - \omega t)$$

Electromagnetic Waves



- $\vec{E}$, $\vec{B}$ and $\vec{v}$ are mutually perpendicular.
- In general, the direction is

$$\hat{s} = \hat{E} \times \hat{B}$$

Energy in Electromagnetic Waves

Energy density of electric and magnetic fields:

$$u_e = \frac{1}{2}\epsilon_0 E^2 \qquad u_m = \frac{1}{2\mu_0}B^2$$

For an electromagnetic wave, $B = E/c = E\sqrt{\mu_0 \epsilon_0}$

$$u_m = \frac{1}{2\mu_0}B^2 = \frac{1}{2}\epsilon_0 E^2 = u_e$$

The total energy density is

$$u = u_m + u_e = \epsilon_0 E^2$$

Intensity of Electromagnetic Waves

- Intensity is defined as the average power transmitted per unit area.

Intensity = Energy density × wave velocity

$$I = \epsilon_0 c \langle E^2 \rangle = \frac{\langle E^2 \rangle}{\mu_0 c}$$

$$\mu_0 c = 377\ \Omega \equiv Z_0$$

(Impedance of free space)

Polarization

- Light is an oscillating electromagnetic field
- The electric field has a direction

$$\vec{E}(x,t) = \vec{E}_0 \cos\left(\vec{k} \cdot \vec{r} - \omega t\right)$$

- No need to specify the magnetic field direction:

$$\vec{S} = \vec{E} \times \vec{H}$$

$$\vec{H} = (\hat{k} \times \vec{E})/Z \qquad \text{where } Z = \sqrt{\mu/\varepsilon}$$

- $\vec{H}$ refers to the magnetic field due to the light, not including any induced magnetic fields in the presence of matter.
- *Coherent* light has the same phase over macroscopic distances and time
- *Polarized* light has the electric field aligned over macroscopic distances and time

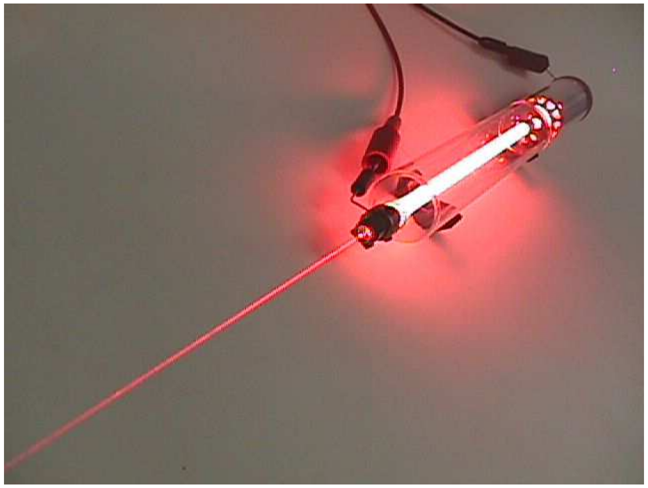
Polarization

Sources of un-polarized light

- Hot atoms transfer kinetic energy to electrons randomly
- Electrons randomly de-excite, emitting incoherent light – uncorrelated in phase and polarization

Polarization

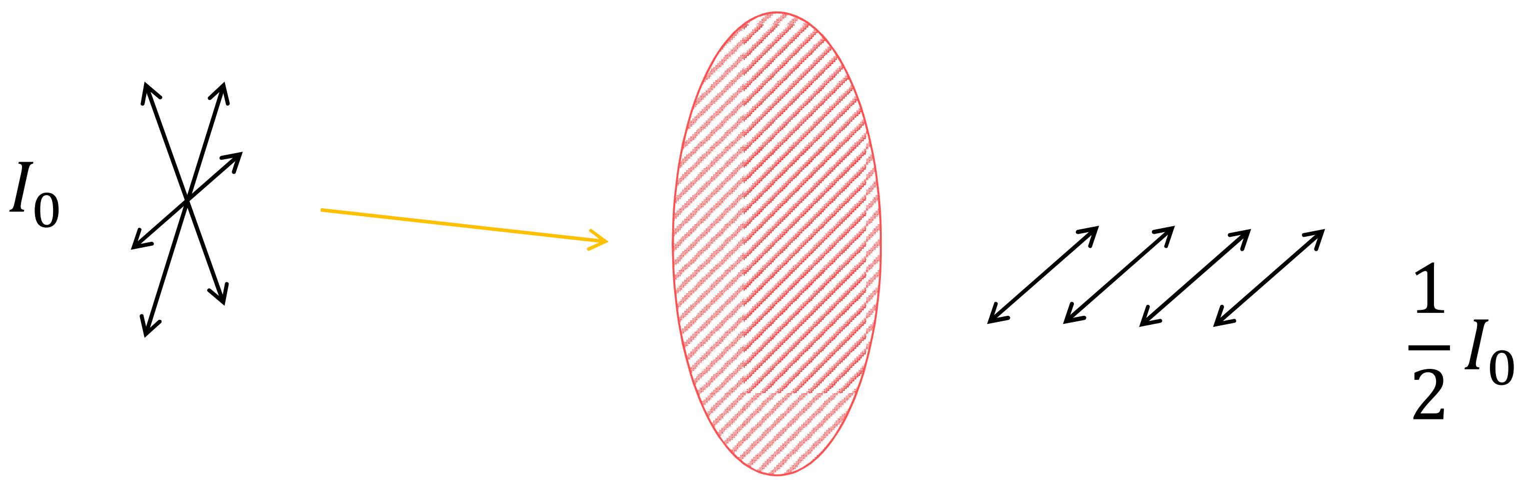
Sources of polarized light



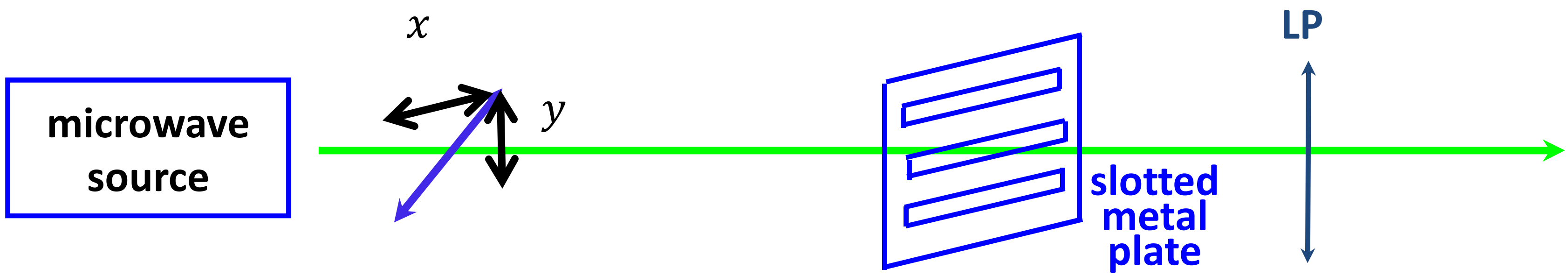
- Lasers produce light by stimulated emission
  - A photon causes an excited atom to emit another photon
  - The photon is emitted in phase and with the same polarization
- The resulting beam is highly coherent and polarized

Polarization by Absorption

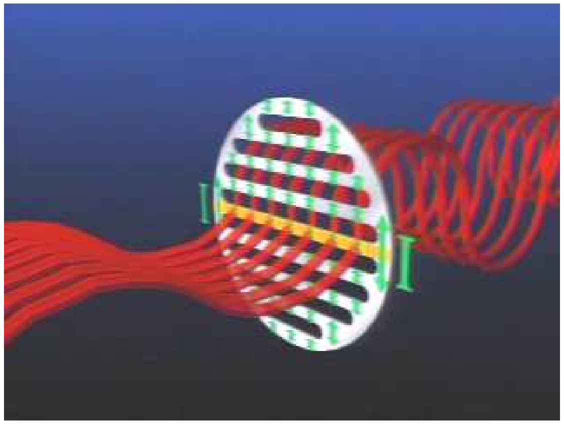
- A polarizer absorbs the component with $\vec{E}$ oriented along a particular axis.
- The light that emerges is linearly polarized along the perpendicular axis.
- If the light is initially un-polarized, half the light is absorbed.



Example with Microwaves



- The electric field in the x-direction induces currents in the metal plate and loses energy:
  - Horizontally polarized microwaves are absorbed
- No current can flow in the y-direction because of the slots
  - Vertically polarized microwaves are transmitted



Polarization by Reflection

- Reflected light is preferentially polarized
- The other component must be transmitted
- Transmission and reflection coefficients must depend on the polarization

Boundary Conditions

- In the presence of matter, the components electric and magnetic fields perpendicular to a surface change abruptly
- The component parallel to a surface is the same on both sides.
- Summary:
  - Perpendicular to surface

$$\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$$
$$\mu_1 H_{1\perp} = \mu_2 H_{2\perp}$$

  - Parallel to surface

$$E_{1\parallel} = E_{2\parallel}$$
$$H_{1\parallel} = H_{2\parallel}$$

Reflection From a Surface

First case:

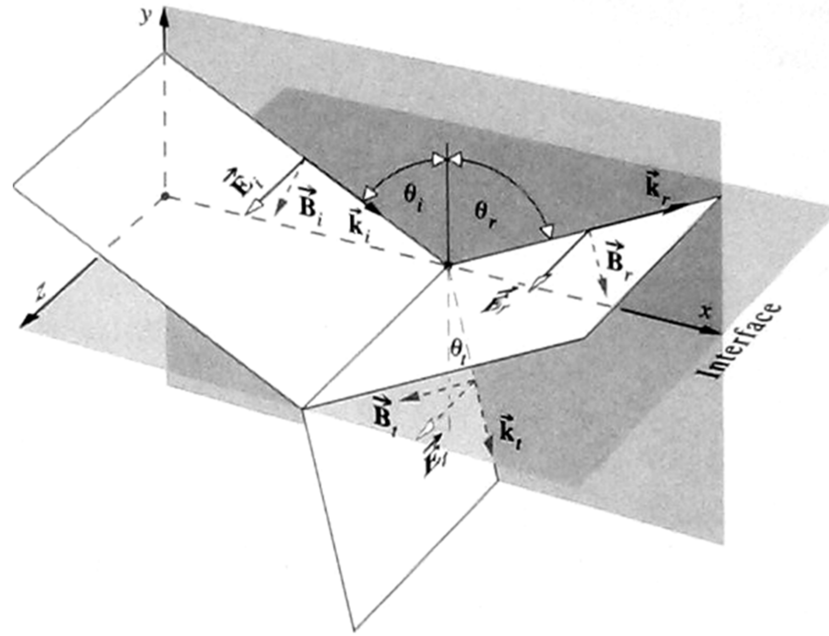
$\vec{E}$ is parallel to the surface…

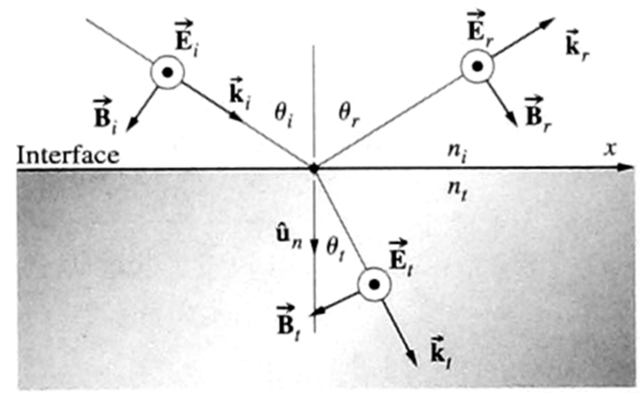
$$E_i + E_r = E_t$$

$\vec{H}$ has components parallel and perpendicular to the surface

$$H_{\parallel i} + H_{\parallel r} = H_{\parallel t}$$

But $H_{\parallel} = \vec{H} \cdot \hat{\tau}$…





Reflection From a Surface

$\vec{E}$ is perpendicular to $\hat{n}$

$$\hat{\tau} = \frac{\vec{n} \times \hat{E}}{\left|\vec{E}\right|}$$

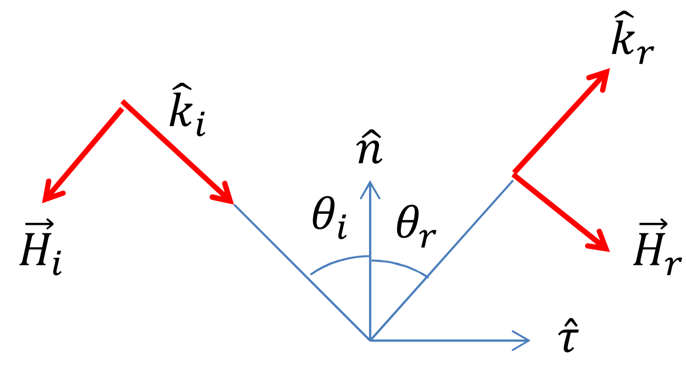
$\vec{H}$ can be written

$$\vec{H} = \frac{\hat{k} \times \vec{E}}{Z}$$

So we can write

$$\vec{H}_i \cdot \hat{\tau} = \frac{1}{ZE_i}\left(\hat{k} \times \vec{E}_i\right) \cdot \left(\vec{n} \times \hat{E}_i\right)$$

$$= \frac{E_i}{Z}\left(\hat{k} \cdot \hat{n}\right) = -\frac{E_i}{Z}\cos\theta_i$$



$\vec{E}$ is out of the page

Likewise,

$$\vec{H}_r \cdot \hat{\tau} = \frac{E_r}{Z}\cos\theta_r$$

$\vec{E}$ perpendicular to $\hat{n}$

- Boundary condition for $\vec{H}$:

$$H_{\parallel i} + H_{\parallel r} = H_{\parallel t}$$

- Previous results:

$$H_{i\parallel} = -\frac{E_i}{Z_1}\cos\theta_i \qquad H_{r\parallel} = \frac{E_r}{Z_1}\cos\theta_r = \frac{E_r}{Z_1}\cos\theta_i$$

$$\frac{-E_i\cos\theta_i + E_r\cos\theta_i}{Z_1} = \frac{-E_t\cos\theta_t}{Z_2}$$

$$E_i + E_r = E_t$$

- Two equations in two unknowns…

$\vec{E}$ perpendicular to $\hat{n}$

$$\frac{-E_i \cos\theta_i + E_r \cos\theta_i}{Z_1} = \frac{-E_t \cos\theta_t}{Z_2}$$

$$\frac{E_i \cos\theta_t + E_r \cos\theta_t}{Z_2} = \frac{E_t \cos\theta_t}{Z_2}$$

- Solve for E_r/E_i:

$$\boldsymbol{\frac{E_r}{E_t} = \frac{Z_2 \cos\theta_i - Z_1 \cos\theta_t}{Z_2 \cos\theta_i + Z_1 \cos\theta_t}}$$

- Solve for E_t/E_i:

$$\boldsymbol{\frac{E_t}{E_t} = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_i + Z_1 \cos\theta_t}}$$

Reflection From A Surface

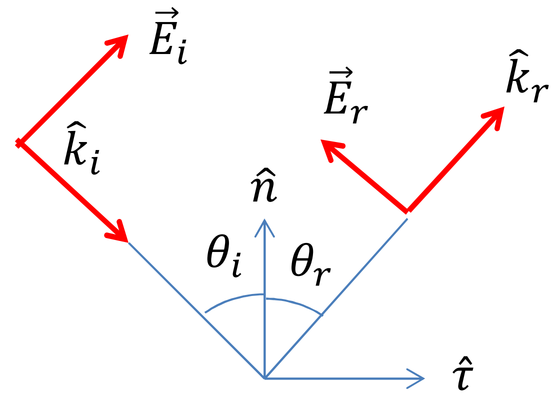
$\vec{H}$ parallel to surface

$$H_i + H_r = H_t$$

$$E_{\parallel i} + E_{\parallel r} = E_{\parallel t}$$

$$E_i \cos\theta_i - E_r \cos\theta_r = E_t \cos\theta_t$$

$$\frac{E_i}{Z_1} + \frac{E_r}{Z_1} = \frac{E_t}{Z_2}$$



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- Two equations in two unknowns…

$\vec{\boldsymbol{H}}$ perpendicular to $\hat{\boldsymbol{n}}$

$$\frac{E_i \cos\theta_i - E_r \cos\theta_i}{Z_2} = \frac{E_t \cos\theta_t}{Z_2}$$

$$\frac{E_i \cos\theta_t + E_r \cos\theta_t}{Z_1} = \frac{E_t \cos\theta_t}{Z_2}$$

- Solve for E_r/E_i:

$$\boldsymbol{\frac{E_r}{E_t} = \frac{Z_1 \cos\theta_i - Z_2 \cos\theta_t}{Z_1\, cos\,\theta_i + Z_2\, cos\,\theta_t}}$$

- Solve for E_t/E_i:

$$\boldsymbol{\frac{E_t}{E_t} = \frac{2Z_1\, cos\,\theta_i}{Z_1\, cos\,\theta_i + Z_2\, cos\,\theta_t}}$$

Fresnel's Equations

- In most dielectric media, $\mu_1 = \mu_2$ and therefore

$$\frac{Z_1}{Z_2} = \sqrt{\frac{\mu_1 \varepsilon_2}{\mu_2 \varepsilon_1}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} = \frac{\sin\theta_i}{\sin\theta_t}$$

- After some trigonometry…

$$\left(\frac{E_r}{E_i}\right)_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \qquad \left(\frac{E_r}{E_i}\right)_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$\left(\frac{E_t}{E_i}\right)_\perp = \frac{(n_1/n_2)\sin(2\theta_i)}{\sin(\theta_i+\theta_t)} \qquad \left(\frac{E_t}{E_i}\right)_\parallel = \frac{2\cos(\theta_i)\sin(\theta_t)}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}$$

For $\vec{E}$ perpendicular and parallel to ***plane of incidence***.
(not the same as perpendicular or parallel to the surface)

Normal Incidence

- At normal incidence, $\cos\theta = 1$ there really is no component parallel to the plane of incidence.
- In this case

$$\left(\frac{E_r}{E_i}\right)_\perp = \frac{n_1 - n_2}{n_1 + n_2}$$

$$\left(\frac{E_t}{E_i}\right)_\perp = \frac{2n_2}{n_1 + n_2}$$

- This what we arrived at previously using only the difference in the speed of light in the two materials.