Midterm Exam:

Date: Thursday, March 13th
Time: 8:00 – 10:00 pm
Room: PHYS 112
Material: French, chapters 1-8
Review

1. Simple harmonic motion (one degree of freedom)
   – mass/spring, pendulum, floating objects, RLC circuits
   – damped harmonic motion

2. Forced harmonic oscillators
   – amplitude/phase of steady state oscillations
   – transient phenomena

3. Coupled harmonic oscillators
   – masses/springs, coupled pendula, RLC circuits
   – forced oscillations

4. Uniformly distributed discrete systems
   – masses on string fixed at both ends
   – lots of masses/springs
5. Continuously distributed systems (standing waves)
   – string fixed at both ends
   – sound waves in pipes (open end/closed end)
   – transmission lines
   – Fourier analysis

6. Progressive waves in continuous systems
   – reflection/transmission coefficients
Simple Harmonic Motion

- Any system in which the force is opposite the displacement will oscillate about a point of stable equilibrium.
- If the force is proportional to the displacement it will undergo simple harmonic motion.
- Examples:
  - Mass/massless spring
  - Elastic rod (characterized by Young’s modulus)
  - Floating objects
  - Torsion pendulum (shear modulus)
  - Simple pendulum
  - Physical pendulum
  - LC circuit
Simple Harmonic Motion

• You should be able to draw a free-body diagram and express the force in terms of the displacement.

• Use Newton’s law:  \( m\ddot{x} = F \) or \( I\ddot{\theta} = N \)

• Write it in standard form:
  \[ \ddot{x} + \omega^2 x = 0 \]

• Solutions are of the form:
  \[ x(t) = A \cos(\omega t - \delta) \]
  \[ x(t) = A \cos \omega t + B \sin \omega t \]

• **You must be able to use the initial conditions to solve for the constants of integration**
Examples

\[ m\ddot{x} = -mgx/\ell \]

\[ m\ddot{x} = -kx \]
Examples

\[ m\ddot{x} = ? \]
Damped Harmonic Motion

- Damping forces remove energy from the system
- We will only consider cases where the force is proportional to the velocity: $F = -bv$
- You should be able to construct a free-body diagram and write the resulting equation of motion:
  $$m\ddot{x} + b\dot{x} + kx = 0$$
  - You should be able to write it in the standard form:
    $$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$
- **You must be able to solve this differential equation!**
Damped Harmonic Motion

\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \]

Let \( x(t) = A e^{\alpha t} \)

- Characteristic polynomial:
  \[ \alpha^2 + \gamma \alpha + \omega_0^2 = 0 \]

- Roots (use the quadratic formula):
  \[ \alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2} \]

- Classification of solutions:
  - Over-damped: \( \gamma^2 / 4 - (\omega_0)^2 > 0 \) (distinct real roots)
  - Critically damped: \( \gamma^2 / 4 = (\omega_0)^2 \) (one root)
  - Under-damped: \( \gamma^2 / 4 - (\omega_0)^2 < 0 \) (complex roots)
Damped Harmonic Motion

• Over-damped motion: \( \gamma^2 / 4 - (\omega_0)^2 > 0 \)

\[
x(t) = Ae^{-\frac{\gamma}{2} t} e^{t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}} + Be^{-\frac{\gamma}{2} t} e^{-t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}}
\]

• Under-damped motion: \( \gamma^2 / 4 - (\omega_0)^2 < 0 \)

\[
x(t) = Ae^{-\frac{\gamma}{2} t} e^{it\sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}} + Be^{-\frac{\gamma}{2} t} e^{-it\sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}}
\]

• Critically damped motion:

\[
x(t) = (A + Bt)e^{-\frac{\gamma}{2} t}
\]

• You must be able to use the initial conditions to solve for the constants of integration
Example

Sum of potential differences:

\[-L \frac{di}{dt} - i(t)R - \frac{1}{C} \left( Q_0 + \int_0^t i(t)dt \right) = 0\]

Initial charge, \(Q_0\), defines the initial conditions.
Differentiate once with respect to time:

\[ L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0 \]

Remember, the solution is \( i(t) \) but the initial conditions might be in terms of \( Q(t) = Q_0 + \int i(t) dt \)

(See examples from the lecture notes...)
Forced Harmonic Motion

• Now the differential equation is
  \[ m\ddot{x} + b\dot{x} + kx = F(\omega) = F_0 \cos \omega t \]

• Driving function is not always given in terms of a real force... remember Assignment #3:
  \[ \ddot{y} + \gamma \dot{y} + \omega_0^2 y = -\frac{d^2\eta}{dt^2} = C\omega^2 \cos \omega t \]

• General properties:
  – Steady state properties: \( t \gg 1/\gamma \)
  – Solution is \( y(t) = A \cos(\omega t - \delta) \)
  – Amplitude, \( A \), and phase, \( \delta \), depend on \( \omega \)
Forced Harmonic Motion

“Q” quantifies the amount of damping:

\[ Q = \frac{\omega_0}{\gamma} \]

(large Q means small damping force)

\[ A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left(\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right)^{1/2}} \]

\[ \delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right) \]

But watch out when \( F_0 = C \omega^2 \)
Resonance

- Qualitative features: amplitude

$$\frac{\omega_{\text{free}}}{\omega_0} = 1$$

$$\omega_{\text{free}} = \sqrt{\frac{\omega_0^2 - \gamma^2}{4}}$$
Average Power

- The rate at which the oscillator absorbs energy is:

\[
\bar{P}(\omega) = \frac{(F_0)^2 \omega_0}{2kQ} \left( \frac{1}{\left( \frac{\omega}{\omega_0} \right)^2 + \frac{1}{Q^2}} \right)
\]

Full-Width-at-Half-Max:

\[
FWHM = \frac{\omega_0}{Q} = \gamma
\]
Resonance

• Qualitative features: phase shift

\[ \delta = \tan^{-1}\left( \frac{1/Q}{\omega_0 - \frac{\omega}{\omega_0}} \right) \]

\[ \delta \rightarrow 0 \text{ at low frequencies} \]
\[ \delta \rightarrow \pi \text{ at high frequencies} \]
\[ \delta = \frac{\pi}{2} \text{ when } \omega = \omega_0 \]
Coupled Oscillators

- Restoring force on pendulum A:
  \[ F_A = -k(x_A - x_B) \]
- Restoring force on pendulum B:
  \[ F_B = k(x_A - x_B) \]

\[
\begin{align*}
  m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) &= 0 \\
  m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) &= 0
\end{align*}
\]
Coupled Oscillators

• You must be able to draw the free-body diagram and set up the system of equations.

\[ m\ddot{x}_A + \frac{mg}{\ell} x_A + k(x_A - x_B) = 0 \]
\[ m\ddot{x}_B + \frac{mg}{\ell} x_B - k(x_A - x_B) = 0 \]

• You must be able to write this system as a matrix equation.

\[
\begin{pmatrix}
\ddot{x}_A \\
\ddot{x}_B
\end{pmatrix} +
\begin{pmatrix}
(\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\
-(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2
\end{pmatrix}
\begin{pmatrix}
x_A(t) \\
x_B(t)
\end{pmatrix} = 0
\]
Coupled Oscillators

- Assume solutions are of the form
  \[
  \begin{pmatrix}
  x_A(t) \\
  x_B(t)
  \end{pmatrix} =
  \begin{pmatrix}
  x_A \\
  x_B
  \end{pmatrix}
  \cos(\omega t - \delta)
  \]

- Then,
  \[
  \begin{pmatrix}
  (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\
  -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2
  \end{pmatrix}
  \begin{pmatrix}
  x_A \\
  x_B
  \end{pmatrix} = 0
  \]

- **You must be able to calculate the eigenvalues of a 2x2 or 3x3 matrix.**
  - Calculate the determinant
  - Calculate the roots by factoring the determinant or using the quadratic formula.

- These are the frequencies of the normal modes of oscillation.
Coupled Oscillators

- You must be able to calculate the eigenvectors of a 2x2 or 3x3 matrix

- General solution:
  \[ \ddot{x}(t) = A\dot{x}_1 \cos(\omega_1 t - \alpha) + B\dot{x}_2 \cos(\omega_2 t - \beta) + \cdots \]

- You must be able to solve for the constants of integration using the initial conditions.
Coupled Discrete Systems

- The general method of calculating eigenvalues will always work, but for simple systems you should be able to decouple the equations by a change of variables.

\[
\begin{align*}
\ell c + b & = 0 \\
\ell b - c & = 0 \\
b + \#d & = 0 \\
c + \#d & = 0
\end{align*}
\]

\[
\begin{align*}
M & = b + c \\
M & = b - c \\
M + \#k & = 0 \\
\#k & = 0
\end{align*}
\]

\[
\begin{align*}
m\ddot{x}_A + \frac{mg}{\ell} x_A + k(x_A - x_B) &= 0 \\
m\ddot{x}_B + \frac{mg}{\ell} x_B - k(x_A - x_B) &= 0 \\
\ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B &= 0 \\
\ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A &= 0
\end{align*}
\]

\[
\begin{align*}
\omega_0 &= \sqrt{g/\ell}, \quad \omega_c = \sqrt{k/m} \\
q_1 &= x_A + x_B \\
q_2 &= x_A - x_B \\
\dot{q}_1 + (\omega_0)^2 q_1 &= 0 \\
\dot{q}_2 + (\omega')^2 q_2 &= 0
\end{align*}
\]
Forced Oscillations

• We mainly considered the qualitative aspects
  – We did not analyze the behavior when damping forces were significant

• Main features:
  – Resonance occurs at each normal mode frequency
  – Phase difference is $\delta = \pi/2$ at resonance

• Example: $x_A$ driven by the force $F(\omega) = F_0 \cos \omega t$
  – Calculate force term applied to normal coordinates
    $F_1(\omega) = F_2(\omega) = F_0 \cos \omega t$
  – Reduced to two one-dimensional forced oscillators:
    $\ddot{q}_1 + (\omega_0)^2 q_1 = F_0/m \cos \omega t$
    $\ddot{q}_2 + (\omega')^2 q_2 = F_0/m \cos \omega t$
Uniformly Distributed Discrete Systems

Equations of motion for masses in the middle:

\[ \ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0 \]
\[ (\omega_0)^2 = k/m \]

\[ \ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0 \]
\[ (\omega_0)^2 = T/ml \]
Uniformly Distributed Discrete Masses

• Proposed solution:

\[ x_n(t) = A_n \cos \omega t \]

\[ \frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2} \]

• We solved this to determine \( A_n \) and \( \omega_k \):

\[ A_{n,k} = C \sin \left( \frac{nk\pi}{N + 1} \right) \]

\[ \omega_k = 2\omega_0 \sin \left( \frac{k\pi}{2(N + 1)} \right) \]

• General solution:

\[ x_n(t) = \sum_{k=1}^{N} a_k \sin \left( \frac{nk\pi}{N + 1} \right) \cos(\omega_k t - \delta_k) \]
Vibrations of Continuous Systems

- Amplitude of mass $n$ for normal mode $k$:
  \[ A_{n,k} = C \sin \left( \frac{n k \pi}{N + 1} \right) \]

- Frequency of normal mode $k$:
  \[ \omega_k = 2 \omega_0 \sin \left( \frac{k \pi}{2(N + 1)} \right) \]

- Solution for normal modes:
  \[ x_n(t) = A_{n,k} \cos \omega_k t \]

- General solution:
  \[ x_n(t) = \sum_{k=1}^{N} a_k \sin \left( \frac{n k \pi}{N + 1} \right) \cos(\omega_k t - \delta_k) \]
Masses on a String

First normal mode

Second normal mode
This is the exact same problem as the previous two examples.
Forced Coupled Oscillators

• Qualitative features are the same:
  – Motion can be decoupled into a set of $N$ independent oscillator equations (normal modes)
  – Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
  – Phase difference approaches $\pi/2$ at resonance

• You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.
Continuous Distributions

Limit as \( N \to \infty \) and \( m/\ell \to \mu \):

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

Boundary conditions specified at \( x = 0 \) and \( x = L \):

- Fixed ends: \( y(0) = y(L) = 0 \)
- Maximal motion at ends: \( \dot{y}(0) = \dot{y}(L) = 0 \)
- Mixed boundary conditions

Normal modes will be of the form

\[
y_n(x, t) = a_n \sin(k_n x) \cos(\omega_n t - \alpha_n)
\]

or

\[
y_n(x, t) = a_n \cos(k_n x) \cos(\omega_n t - \alpha_n)
\]
Properties of the Solutions

\[ y(L, t) \sim \sin k_n L = 0 \quad \Rightarrow \quad k_n L = n\pi \]

<table>
<thead>
<tr>
<th>mode</th>
<th>wavelength</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>first</td>
<td>(2L)</td>
<td>(\frac{v}{2L})</td>
</tr>
<tr>
<td>second</td>
<td>(L)</td>
<td>(\frac{v}{L})</td>
</tr>
<tr>
<td>third</td>
<td>(\frac{2L}{3})</td>
<td>(\frac{3v}{2L})</td>
</tr>
<tr>
<td>fourth</td>
<td>(\frac{L}{2})</td>
<td>(\frac{2v}{L})</td>
</tr>
</tbody>
</table>

\[ \lambda_n = \frac{2L}{n} \]
\[ \omega_n = \frac{n\pi v}{L} \]
\[ f_n = \frac{nv}{2L} \]
Boundary Conditions

• Examples:
  – String fixed at both ends: \( y(0) = y(L) = 0 \)
  – Organ pipe open at one end: \( \dot{y}(0) = \dot{y}(L) = 0 \)
    • Driving end has maximal pressure amplitude
  – Organ pipe closed at one end: \( \dot{y}(0) = 0, y(L) = 0 \)
  – Transmission line open at one end: \( i(L) = 0 \)
  – Transmission line shorted at one end: \( v(L) \propto \frac{di(L)}{dt} = 0 \)
Fourier Analysis

- Normal modes satisfying $y(0) = y(L) = 0$
  \[ y_n(x, t) = a_n \sin \left( \frac{n\pi x}{L} \right) \cos(\omega_n t - \alpha_n) \]

- General solution:
  \[ y(x, t) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L} \right) \cos(\omega_n t - \alpha_n) \]

- Initial conditions:
  \[ y(x, 0) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L} \right) \cos(\alpha_n) = \sum_{n=1}^{\infty} a'_n \sin \left( \frac{n\pi x}{L} \right) \]
  \[ \dot{y}(x, 0) = -\sum_{n=1}^{\infty} a_n \omega_n \sin \left( \frac{n\pi x}{L} \right) \sin(\alpha_n) = \sum_{n=1}^{\infty} b'_n \sin \left( \frac{n\pi x}{L} \right) \]
Fourier Analysis

• Fourier sine transform:

\[ u(x) = \sum_{n=1}^{\infty} a'_n \sin \left( \frac{n\pi x}{L} \right) \]

\[ a'_n = \frac{2}{L} \int_{0}^{L} u(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \]

• Fourier cosine transform:

\[ b'_n = \frac{2}{L} \int_{0}^{L} v(x) \cos \left( \frac{n\pi x}{L} \right) \, dx \]
Fourier Analysis

\[ a'_n = a_n \cos \alpha_n \]
\[ b'_n = a_n \omega_n \sin \alpha_n \]

Solve for amplitudes:

\[ a_n = \sqrt{a'_n^2 + \frac{b'_n^2}{\omega_n^2}} \]

Solve for phase:

\[ \tan \alpha_n = \frac{b'_n}{a'_n \omega_n} \]
Fourier Analysis

- **Suggestion:** don’t simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.

- **Example:**
  - If you are given $\dot{y}(x, 0) = 0$ and $y(0) = y(L) = 0$ then you know that solutions are of the form
    \[
    y(x, t) = \sum a_n \sin \left( \frac{n\pi x}{L} \right) \cos \omega_n t
    \]
  - If you are given $y(x, 0) = 0$ and $y(0) = y(L) = 0$ then solutions are of the form
    \[
    y(x, t) = \sum_{\text{odd } n} a_n \sin \left( \frac{n\pi x}{L} \right) \sin \omega_n t
    \]
Progressive Waves

• Far from the boundaries, other descriptions are more transparent:
  \[ y(x, t) = f(x \pm vt) \]

• The Fourier transform gives the frequency components:
  \[
  A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx) \, dx
  \]

  \[
  g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx) \, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx) \, dk B(k)
  \]

  \[
  = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx) \, dx
  \]

• Narrow pulse in space \(\Rightarrow\) wide range of frequencies
• Pulse spread out in space \(\Rightarrow\) narrow range of frequencies
Properties of Progressive Waves

• Power carried by a wave:
  – String with tension $T$ and mass per unit length $\mu$
    $$P = \frac{1}{2} \mu \omega^2 A^2 v = \frac{1}{2} Z \omega^2 A^2$$

• Impedance of the medium:
  $$Z = \mu v = \frac{T}{v}$$

• Important properties:
  – Impedance is a property of the medium, not the wave
  – Energy and power are proportional to the square of the amplitude
Reflections

• Wave energy is reflected by discontinuities in the impedance of a system

• Reflection and transmission coefficients:
  – The wave is incident and reflected in medium 1
  – The wave is transmitted into medium 2

\[
\rho = \frac{Z_2 - Z_1}{Z_1 + Z_2}
\]

\[
\tau = \frac{2Z_2}{Z_1 + Z_2}
\]

• Wave amplitudes:

\[
A_r = \rho A_i
\]

\[
A_t = \tau A_i
\]
Reflected and Transmitted Power

• Power is proportional to the square of the amplitude.
  – Reflected power: \( P_r = \rho^2 P_i \)
  – Transmitted power: \( P_t = \tau^2 P_i \)

• You should be able to demonstrate that energy is conserved:
  \[ \text{ie, show that } P_i = P_r + P_t \]
That’s all for now...

• Study these topics – make sure you understand the examples and assignment questions.

• Next topics: waves applied to optics.