

Physics 42200

Waves & Oscillations

Lecture 2 – French, Chapter 1

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Simple Harmonic Motion

- Mass-spring system:
 - Force given by Hooke's law:

$$F = -kx$$

- Newton's second law:

$$F = ma = m\ddot{x} = m \frac{d^2x}{dt^2}$$

- Equation of motion:

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0$$

$$\text{where } \omega = \sqrt{k/m}$$

Simple Harmonic Motion

- Differential equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

- Solutions can be written in various ways:

$$x(t) = \mathbf{A} \cos(\omega t + \boldsymbol{\varphi})$$

$$x(t) = \mathbf{A} \sin \omega t + \mathbf{B} \cos \omega t$$

(and many others...)

- Two *constants of integration* need to be determined from initial conditions or other information.

Simple Harmonic Motion

- How do we know that these are solutions?
- Compute the derivatives:

$$x(t) = A \cos(\omega t + \varphi)$$

$$\dot{x}(t) = -A\omega \sin(\omega t + \varphi)$$

$$\ddot{x}(t) = -A\omega^2 \cos(\omega t + \varphi) = -\omega^2 x(t)$$

- Substitute into the differential equation:

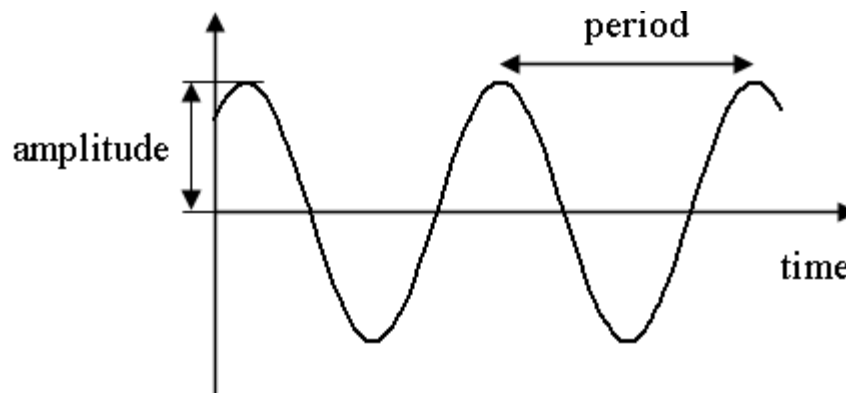
$$\ddot{x} + \omega^2 x = (-\omega^2 x(t)) + \omega^2 x(t) = 0$$

- Mathematical details:
 - How could we deduce that this was a solution if we didn't already know it was?

Simple Harmonic Motion

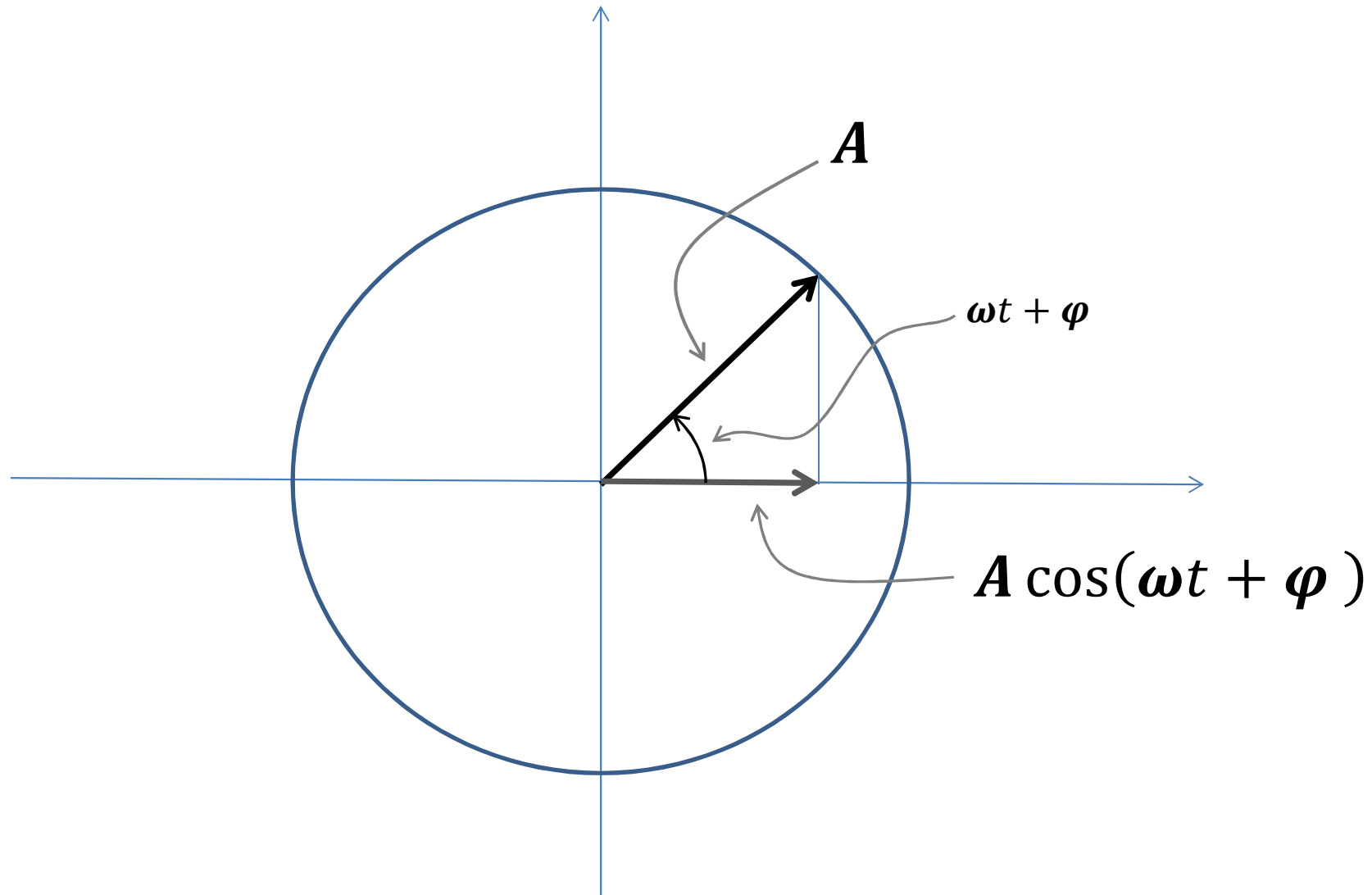
- Properties of the solution:

$$x(t) = A \cos(\omega t + \varphi)$$



- Notation:
 - Amplitude: A
 - Initial phase: φ
 - Angular frequency: ω
 - Frequency: $f = \omega/2\pi$
 - Period: $T = 1/f = 2\pi/\omega$

Descriptions of Harmonic Motion

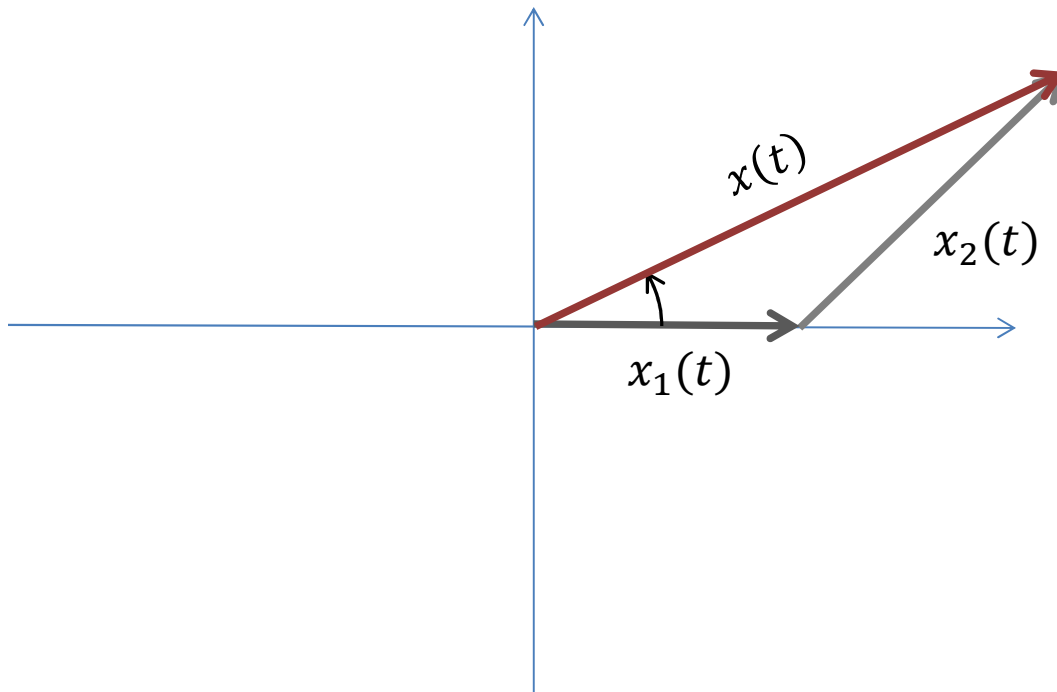


Uniform Circular Motion

- The graph of $x(t) = A \cos(\omega t + \varphi)$ is the same as the projection onto the x -axis of a vector of length A , rotating with angular frequency ω .
- This is a useful geometric description of the motion.
 - Two-component vectors are introduced only for convenience (we call them “phasors”)
 - The solution we are interested in is just the projection onto the x -axis.
- Example...

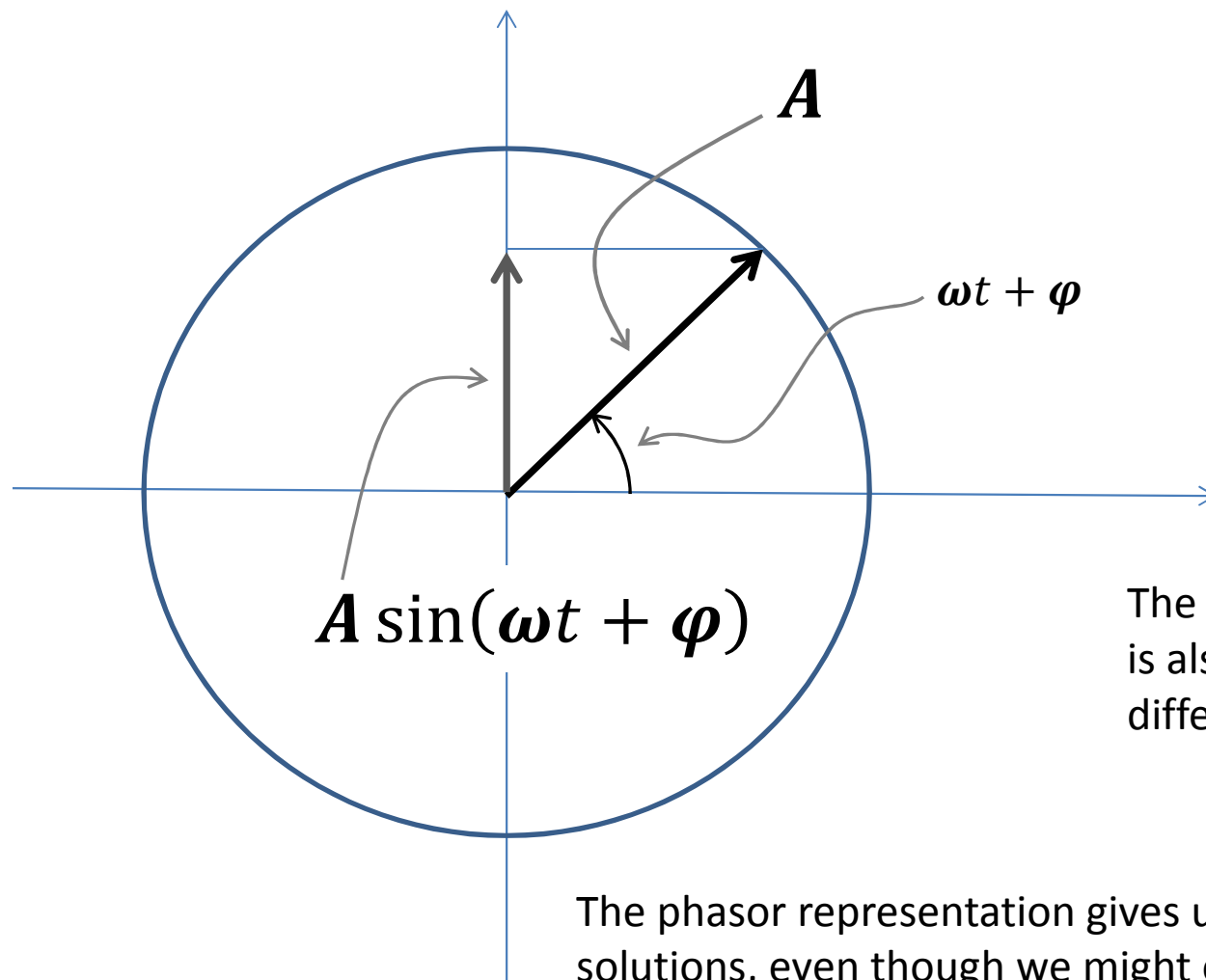
Uniform Circular Motion

- The differential equation $\ddot{x} + \omega^2 x = 0$ is linear:
 - Suppose $x_1(t)$ and $x_2(t)$ are both solutions
 - Then the function $x(t) = a x_1(t) + b x_2(t)$ is also a solution for any real numbers a and b .



Actually, the *functions* $x_1(t)$, $x_2(t)$ and $x(t)$ are the projections of these vectors onto the x-axis.

Uniform Circular Motion



The projection onto the y-axis is also a solution to the differential equation.

The phasor representation gives us two independent solutions, even though we might only want to use only one of them to describe the motion.

Phasor Representation

- The phasor provides all the information we need to describe the motion
 - If we just knew the value of x at one time t , we still don't know what A and φ are.
 - But if we know x and y at time t then we have enough information to calculate both A and φ .
- The more general description of the motion can be useful for analyzing problems even if the “physical” solution to the equations of motion is just one of its projections.

Complex Representation

- Basic definitions:

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$z = x + iy$$

$$z^* = x - iy$$

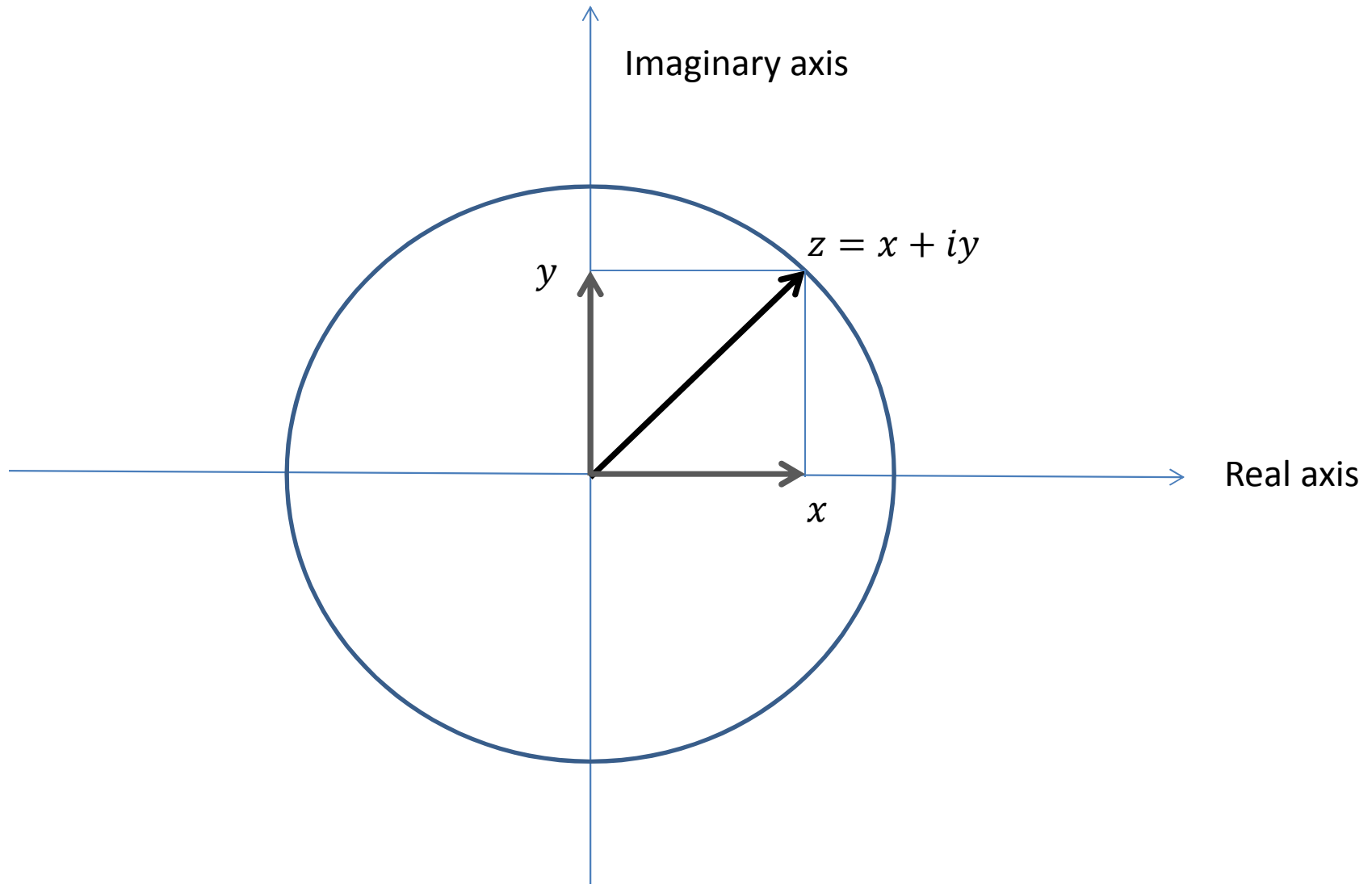
$$|z| = \sqrt{z^* z} = \sqrt{x^2 + y^2}$$

$$\operatorname{Re}(z) = x = (z + z^*)/2$$

$$\operatorname{Im}(z) = y = (z - z^*)/2i$$

(where x and y are real numbers)

Complex Representation



Complex Representation

- But complex numbers are way better...

- Euler's identity:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

- Complex numbers in this form satisfy:

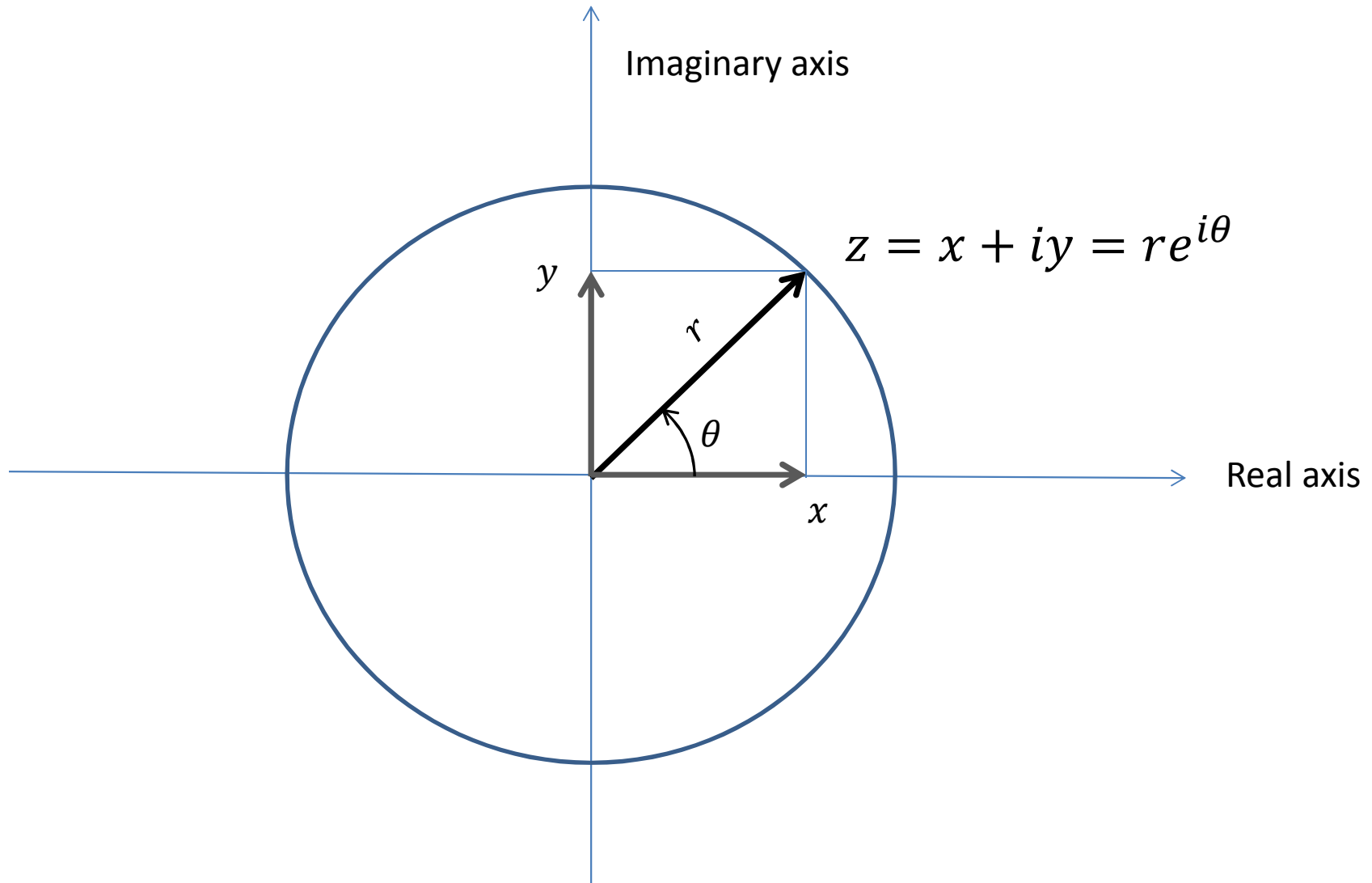
$$(e^{i\theta})^* = e^{-i\theta}$$

$$\begin{aligned} e^{i\theta} e^{-i\theta} &= (\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta) \\ &= \cos^2 \theta + \sin^2 \theta = 1 \end{aligned}$$

- In general, we can always write

$$z = x + iy = r e^{i\theta}$$

Complex Representation



Simple Harmonic Motion

- The other way we will describe solutions to

$$\ddot{x} + \omega^2 x = 0$$

will be using complex numbers...

- Let $x(t) = \textcolor{red}{r}e^{i(\omega t + \textcolor{red}{\varphi})} = \underbrace{(\textcolor{red}{r}e^{i\textcolor{red}{\varphi}})}_{\text{This part is just a constant.}} e^{i\omega t} = \textcolor{red}{c}e^{i\omega t}$

This part is just a constant.

- Derivatives are:

$$\dot{x}(t) = i\omega \textcolor{red}{c}e^{i\omega t}$$

$$\ddot{x}(t) = (i\omega)^2 \textcolor{red}{c}e^{i\omega t} = -\omega^2 x(t)$$

- It is a solution:

$$\ddot{x} + \omega^2 x = (-\omega^2 x(t)) + \omega^2 x(t) = 0$$

Simple Harmonic Motion

- The *physical* displacement of the mass must be a real number.
- The displacement as a function of time is given by the real component, $Re[x(t)]$.
- The complex representation contains more information than is present in just the function describing the physical displacement.
 - It provides *both* amplitude *and* phase information
- We will probably use this a lot in this course...