PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 19 – French, Chapter 6

Spring 2014 Semester

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Midterm Exam:

Date:	Thursday, March 13th
Time:	8:00 – 10:00 pm
Room:	Probably PHYS 112
Material:	French, chapters 1-8

Fourier Analysis

- Wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- Boundary conditions at $t = 0$:

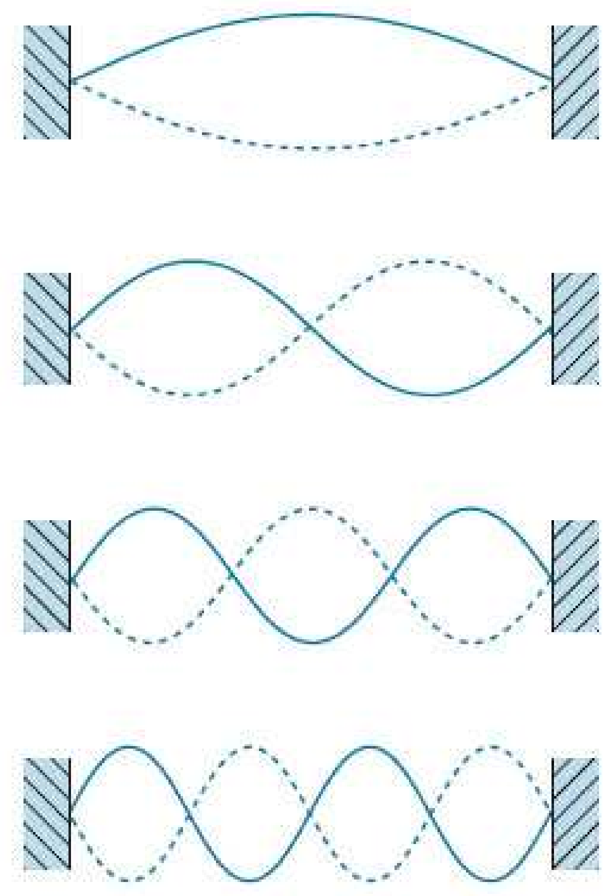
$$y(0,t) = y(L,t) = 0$$

- Normal modes of oscillation:

$$\omega_n = \frac{n\pi v}{L}$$

- Wavelengths of normal modes:

$$\lambda_n = \frac{2L}{n}$$



Fourier Analysis

- Normal modes of oscillation:

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

- General solution:

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \alpha_n)$$

- The parameters a_n and α_n are determined from initial conditions. At $t = 0$, we can write

$$y(x,0) = u(x) = \sum_{n=1}^{\infty} a'_n \sin\left(\frac{n\pi x}{L}\right)$$

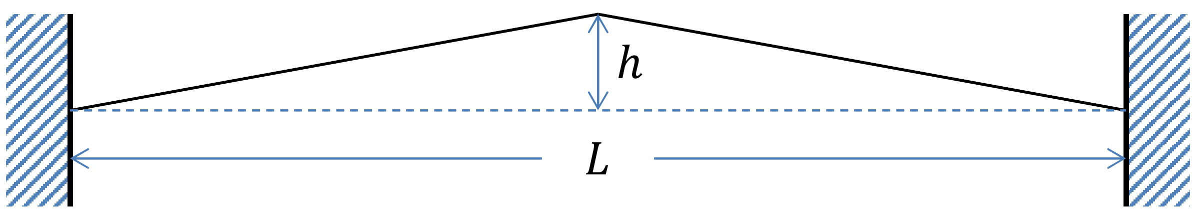
where $a_n' = a_n \cos\alpha_n$.

Fourier Analysis

- Fourier's method:

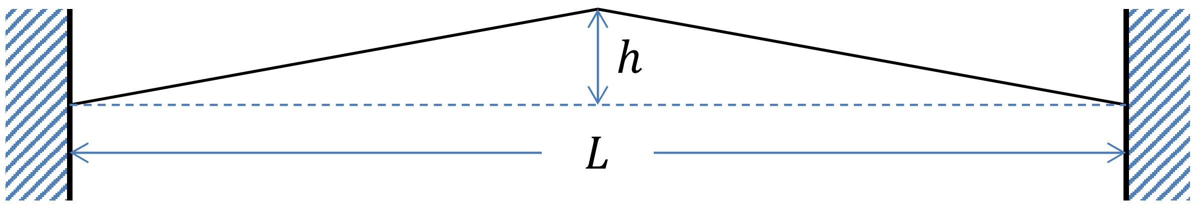
$$a'_k = \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) u(x)dx$$

- Example: (*French 6-12*):



  – The string has tension T and mass per unit length μ
  – If the string is released from rest, what is $y(x,t)$?

Example



- Initial displacement is described by the function

$$u(x) = \begin{cases} \dfrac{2hx}{L} & 0 < x < L/2 \\ 2h\left(1 - \dfrac{x}{L}\right) & L/2 < x < L \end{cases}$$

- The string is released from rest, so $v(x) = 0$.
- That must mean that $\alpha_n = 0$ and so $a_n' = a_n$.

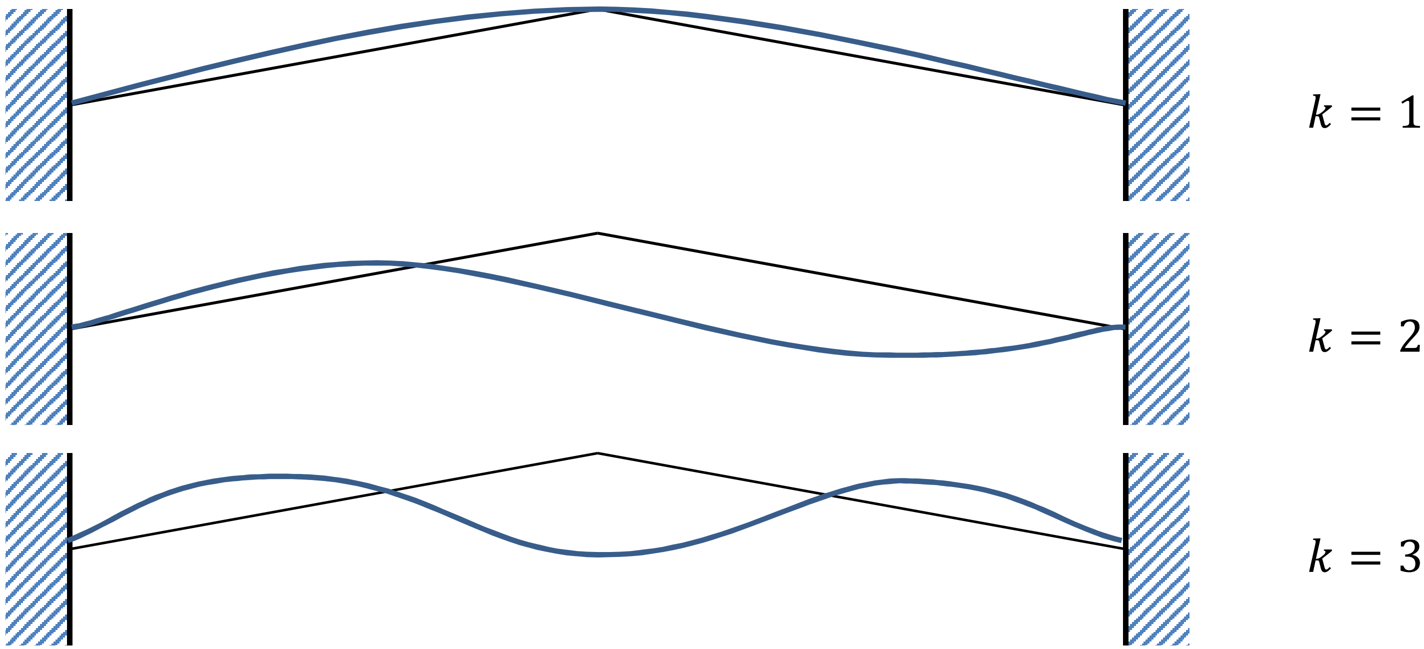
Example

- Fourier coefficients:

$$a_k = \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) u(x)dx$$
$$= \frac{2}{L}\int_0^{L/2} \sin\left(\frac{k\pi x}{L}\right)\left(\frac{2hx}{L}\right) dx$$
$$+ \frac{2}{L}\int_{L/2}^{L} \sin\left(\frac{k\pi x}{L}\right)(2h(1-x/L))\, dx$$
$$= \frac{4h}{L^2}\int_0^{L/2} x\sin\left(\frac{k\pi x}{L}\right) dx - \frac{4h}{L^2}\int_{L/2}^{L} x\sin\left(\frac{k\pi x}{L}\right) dx + \frac{4h}{L}\int_{L/2}^{L} \sin\left(\frac{k\pi x}{L}\right) dx$$

- *This is getting messy…*
- *Don't get your table of integrals just yet…*
- *Think about the symmetry of the problem.*

Example



- Integrals with even values of k will be zero.
  - This is because $\sin(k\pi x/L)$ is odd but $u(x)$ is even when reflected about the point $x = L/2$.
- When k is odd, we can just double the value of the integral from 0 to $L/2$.

Example

- When k is odd,

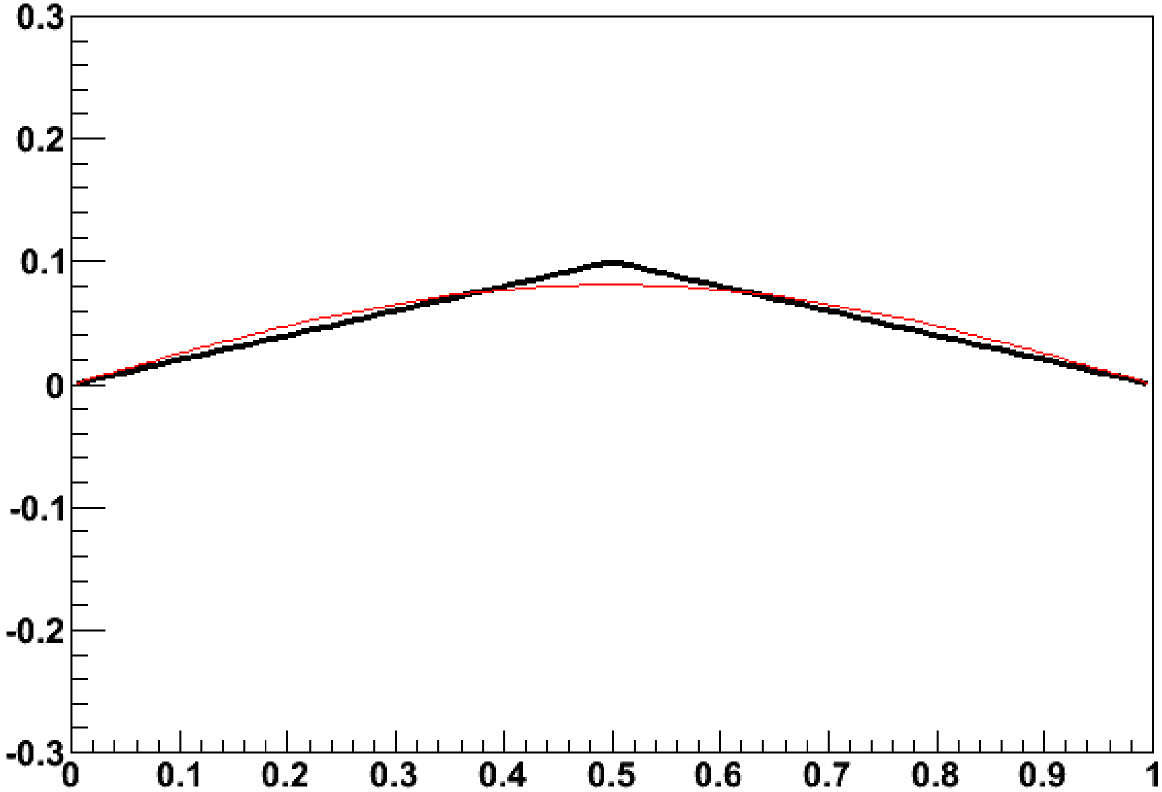
$$a_k = \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right)u(x)dx = \frac{4}{L}\int_0^{L/2} \sin\left(\frac{k\pi x}{L}\right)\left(\frac{2hx}{L}\right)dx$$
$$= \frac{8h}{L^2}\int_0^{L/2} x\sin\left(\frac{k\pi x}{L}\right)dx$$

- Now get your table of integrals:

$$(91) \quad \int x\sin(ax)dx = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$$

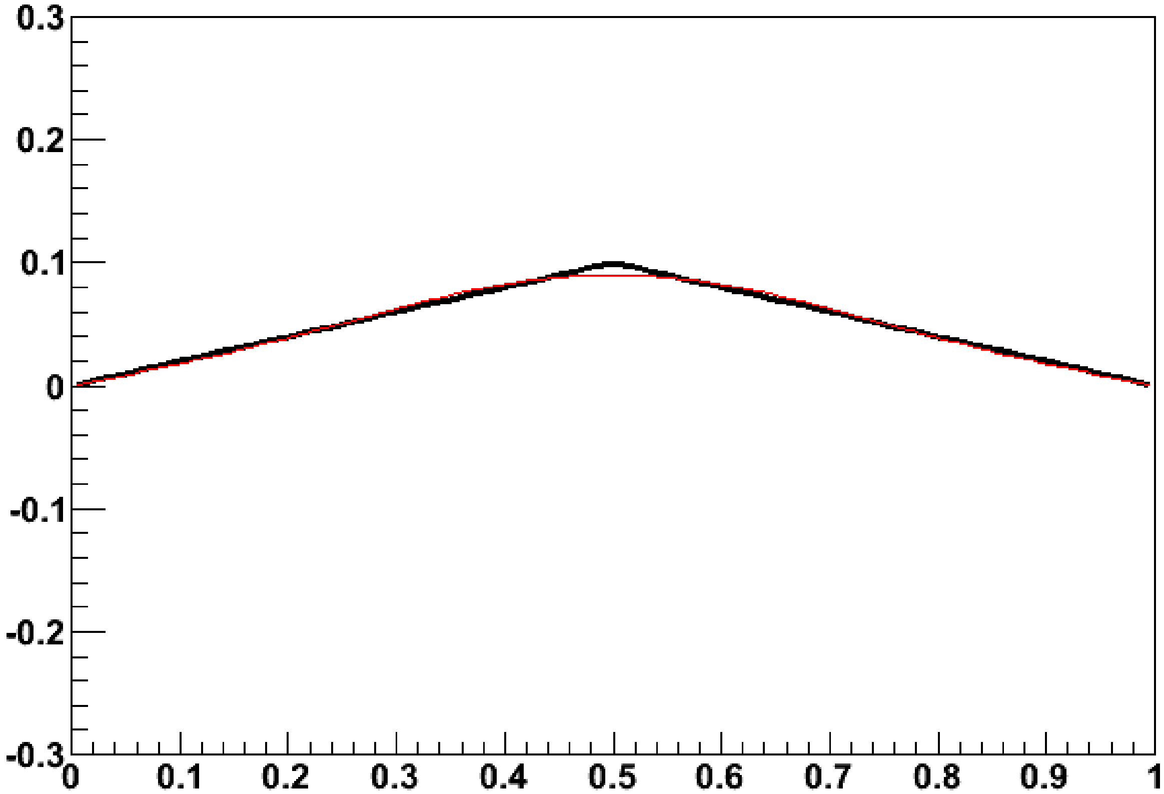
$$a_k = -\frac{8h}{L^2}\frac{Lx}{k\pi}\cos\left(\frac{k\pi x}{L}\right) + \frac{8h}{L^2}\frac{L^2}{k^2\pi^2}\sin\left(\frac{k\pi x}{L}\right)\Bigg|_0^{L/2}$$
$$= \pm\frac{8h}{k^2\pi^2}$$

Example



$$a_1 = \frac{8h}{\pi^2} = 0.8106\ h$$

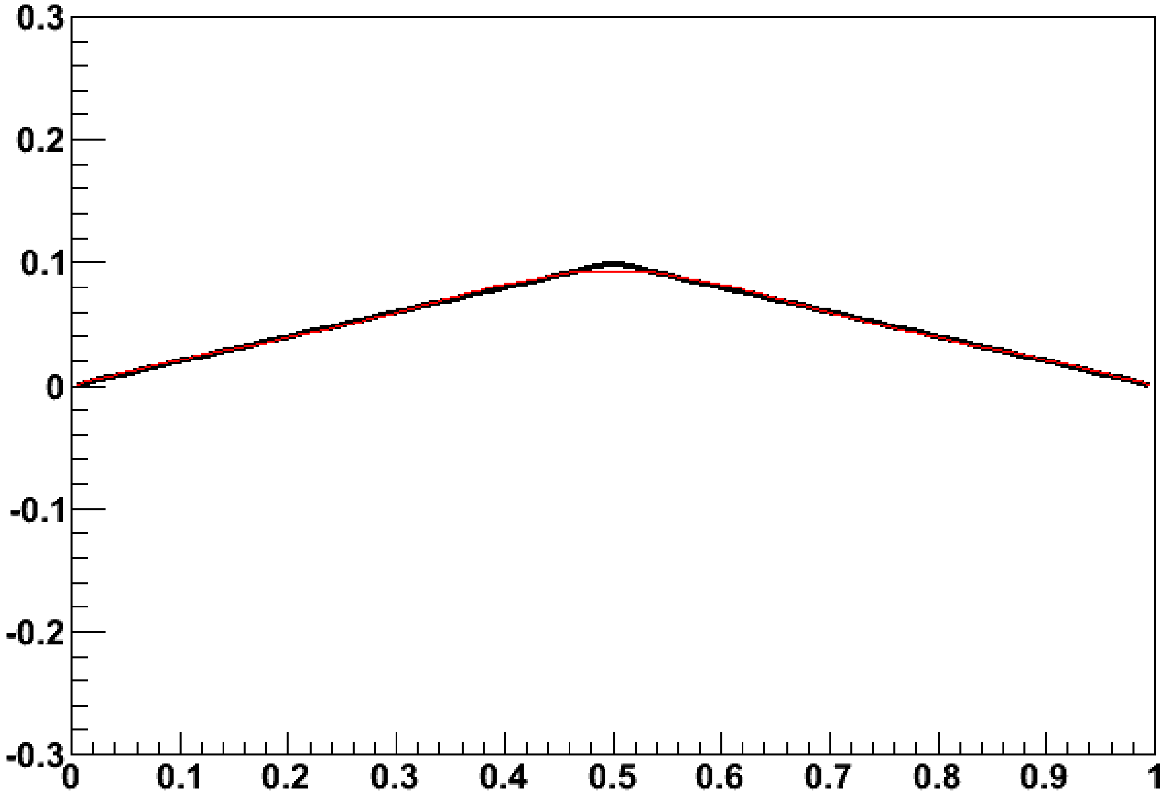
Example



$$a_1 = \frac{8h}{\pi^2} = 0.8106\ h$$

$$a_3 = -\frac{8h}{9\pi^2} = -0.0901\ h$$

Example

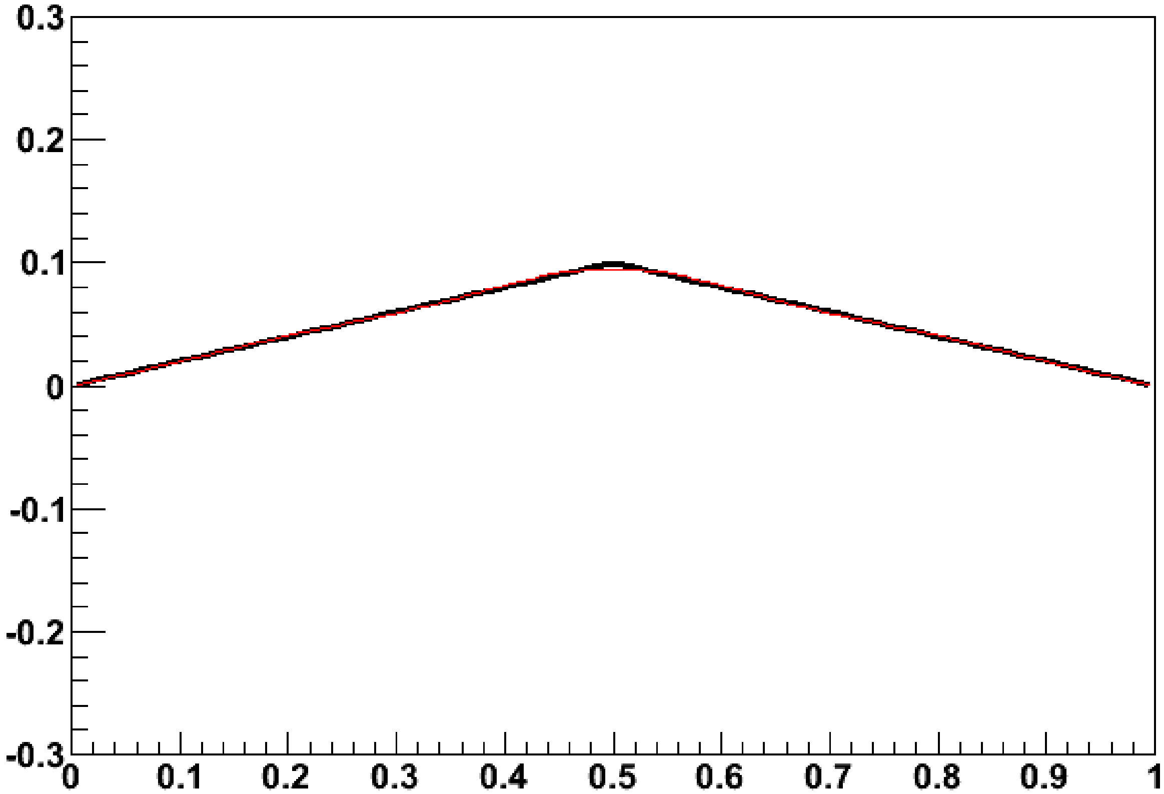


$$a_1 = \frac{8h}{\pi^2} = 0.8106\ h$$

$$a_3 = -\frac{8h}{9\pi^2} = -0.0901\ h$$

$$a_5 = \frac{8h}{25\pi^2} = 0.0324\ h$$

Example



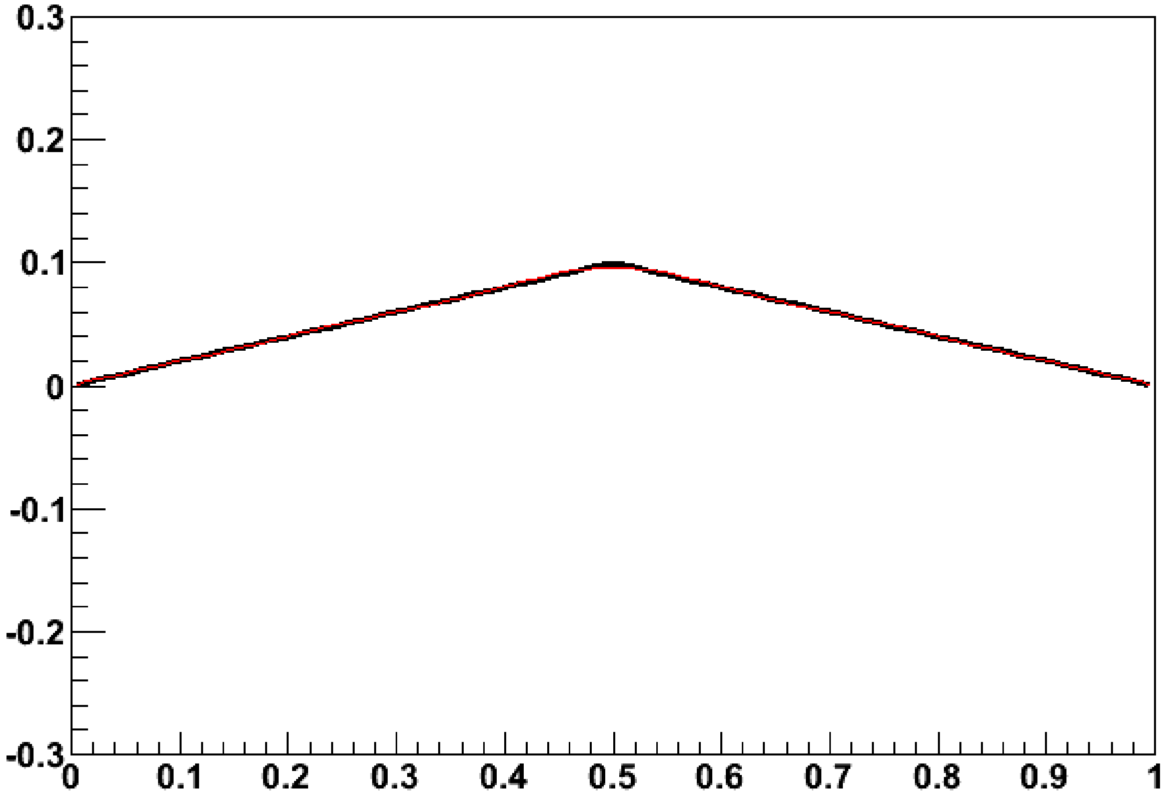
$$a_1 = \frac{8h}{\pi^2} = 0.8106\ h$$

$$a_3 = -\frac{8h}{9\pi^2} = -0.0901\ h$$

$$a_5 = \frac{8h}{25\pi^2} = 0.0324\ h$$

$$a_7 = -\frac{8h}{49\pi^2} = -0.0165\ h$$

Example



$$a_1 = \frac{8h}{\pi^2} = 0.8106\ h$$

$$a_3 = -\frac{8h}{9\pi^2} = -0.0901\ h$$

$$a_5 = \frac{8h}{25\pi^2} = 0.0324\ h$$

$$a_7 = -\frac{8h}{49\pi^2} = -0.0165\ h$$

$$a_9 = \frac{8h}{81\pi^2} = 0.0100\ h$$

Example

$$y(x,t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_k t)$$

$$\omega_k = \frac{k\pi v}{L}$$

- Period of lowest frequency mode:

$$T = \frac{1}{\nu_1} = \frac{2L}{v}$$

- After time $t = T/2 = L/v$,

$$y(x,t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{n\pi x}{L}\right)\cos(\pi k)$$

$$= -u(x)$$

-1 because $a_k \neq 0$ only when k is odd.

- The string maintains its triangular shape! However, the motion is more complex than just $u(x)\cos\left(\frac{2\pi t}{T}\right)$…

Wave Propagation

- The wave equation:
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$
- We worked out solutions that satisfied specific boundary conditions.
- A general solution is any function that is of the form
$$y(x,t) = f(x \pm vt)$$
- Are these two pictures compatible?

Wave Propagation

- Solutions for normal modes:

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

$$\omega_n = \frac{\pi n v}{L}$$

- Trigonometric identity:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

- This gives,

$$y_n(x,t) = \frac{1}{2}\left[\sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right]$$

Wave Propagation

$$y_n(x,t) = \frac{1}{2}\left[\sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right]$$

- Write this as

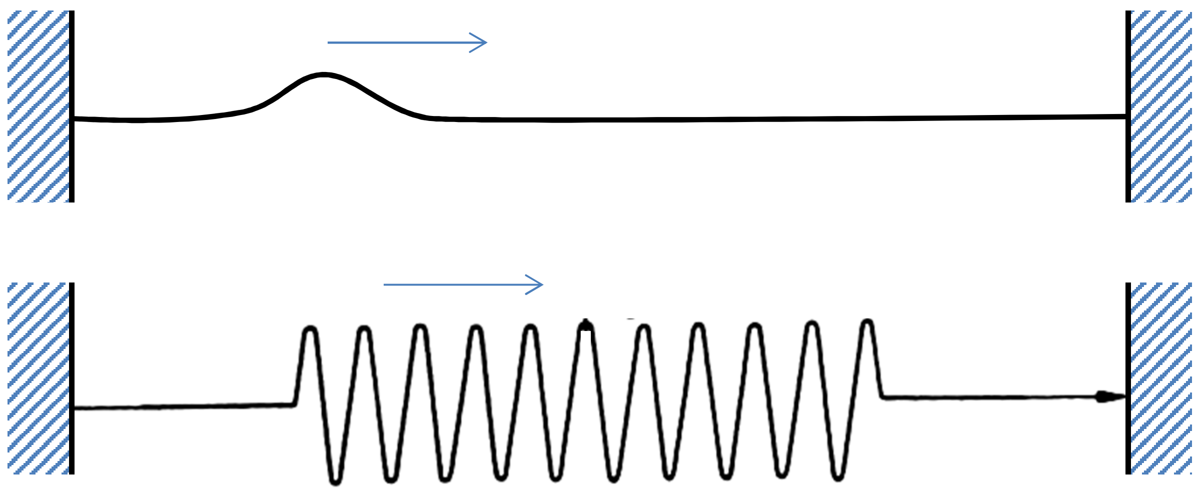
$$y_n(x,t) = \frac{1}{2}[\sin(k(x+vt)) + \sin(k(x-vt))]$$

$$k = \frac{n\pi}{L}$$

- This is the equation for two sine-waves moving in opposite directions.
- The text refers to these as "progressive waves".
- The "standing waves" that satisfy the boundary conditions are the superposition of "progressive waves" that move in opposite directions.

Wave Propagation

- Waves can propagate in either direction.
- Easiest to visualize in terms of a pulse, or wave packet:



- If this disturbance is far from the ends, the effect is the same as letting $L \to \infty$

Wave Propagation

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

$y(0,t) = y(L,t) = 0$

- In general, we could write

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \alpha_n)$$

- In the limit where the disturbance is very far from either boundary, the Fourier sine transform is:

$$B(k) = \int_{-\infty}^{\infty} u(x) \sin(kx)\, dx$$

- Similarly, we can define the Fourier cosine transform:

$$A(k) = \int_{-\infty}^{\infty} u(x) \cos(kx)\, dx$$

Wave Propagation

$$B(k) = \int_{-\infty}^{\infty} u(x)\sin(kx)\,dx$$

- Similarly, we can define the Fourier cosine transform:

$$A(k) = \int_{-\infty}^{\infty} u(x)\cos(kx)\,dx$$

- The original function is represented by:

$$u(x) = \frac{1}{\pi}\int_{0}^{\infty} A(k)\cos(kx)\,dk + \frac{1}{\pi}\int_{0}^{\infty} B(k)\sin(kx)\,dk$$

- If $A(k) = A(-k)$ and $B(k) = -B(-k)$ then we can make this more symmetric:

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\cos(kx)\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} B(k)\sin(kx)\,dk$$

Wave Propagation

- To make this even more symmetric we can change slightly the definition of $A(k)$ and $B(k)$:

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \sin(kx)\, dx$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \cos(kx)\, dx$$

- Then,

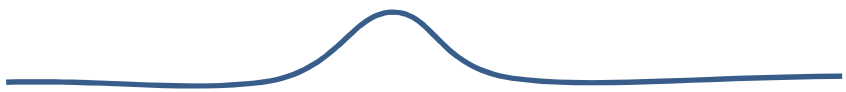
$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx)\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx)\, dk$$

Wave Propagation

- Previously, we interpreted the coefficients a_n as the amplitude of the normal mode with frequency ω_n
  - wavelength $\lambda_n = 2L/n$
  - wavenumber $k_n = 2\pi/\lambda_n = \pi n/L$
- Now, we interpret $A(k)$ and $B(k)$ as the amplitude for harmonic waves with wavenumbers between k and $k + dk$.
- It can be important to decompose a pulse into its frequency components because in real materials, the nature of wave propagation can depend on the frequency.

Example

- Consider a pulse that has a Gaussian shape:



$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- Special case:
  - Peak position is at $x = 0$
  - Width of the peak is $\sigma = 1$
- Other Gaussian functions can be transformed into this special case by linear change of variables.
- What is the continuous Fourier transform?

Example

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx)\, dx$$

- The Gaussian function $g(x)$ is an even function:

$$g(x) = g(-x)$$

- The function $\sin(kx)$ is an odd function:

$$\sin(-kx) = -\sin(kx)$$

- This integral must vanish…

$$B(k) = 0$$

Example

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\cos(kx)\,dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2}\cos(kx)\,dx$$

- From your table of integrals:

$$\int_{-\infty}^{\infty} e^{-ax^2}\cos bx\,dx = \sqrt{\frac{\pi}{a}}\,e^{-b^2/4a}$$

- In this case, $a = 1/2$ and $b = k$

$$A(k) = \frac{1}{2\pi} \times \sqrt{2\pi} e^{-k^2/2} = \frac{1}{\sqrt{2\pi}} e^{-k^2/2}$$

- This is a Gaussian distribution of wavenumbers $k = \omega/v$.

Notes about Fourier Transforms

- For the Gaussian pulse,

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

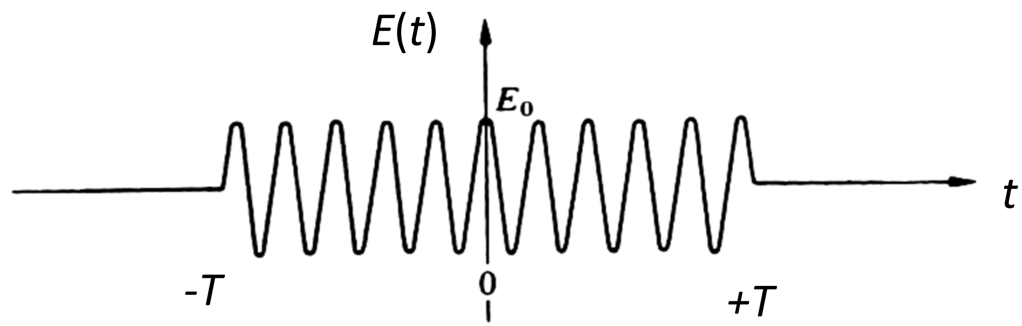
- The amplitudes of the frequency components are:

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma^2/2},\ B(k) = 0$$

- When the pulse is narrow, $\sigma \ll 1$, then the exponent in $A(k)$ is large for a large range of k
  - Since $\omega = v/k$, a narrow pulse has a wide range of frequency components.
- Conversely, a wide pulse has a narrow range of frequencies.

Another Example

- A photon can be described as a localized oscillation:



At $x = 0$, $E(t) = \begin{cases} E_0 \cos(\omega t) & \text{when } |t| < T \\ 0 & \text{otherwise} \end{cases}$

At $t = 0$,$E(x) = \begin{cases} E_0 \cos(kx) & \text{when } |x| < cT \\ 0 & \text{otherwise} \end{cases}$

$$A(k') = \frac{E_0}{\sqrt{2\pi}} \int_{-cT}^{cT} \cos(kx) \cos(k'x) \, dx$$

Another Example

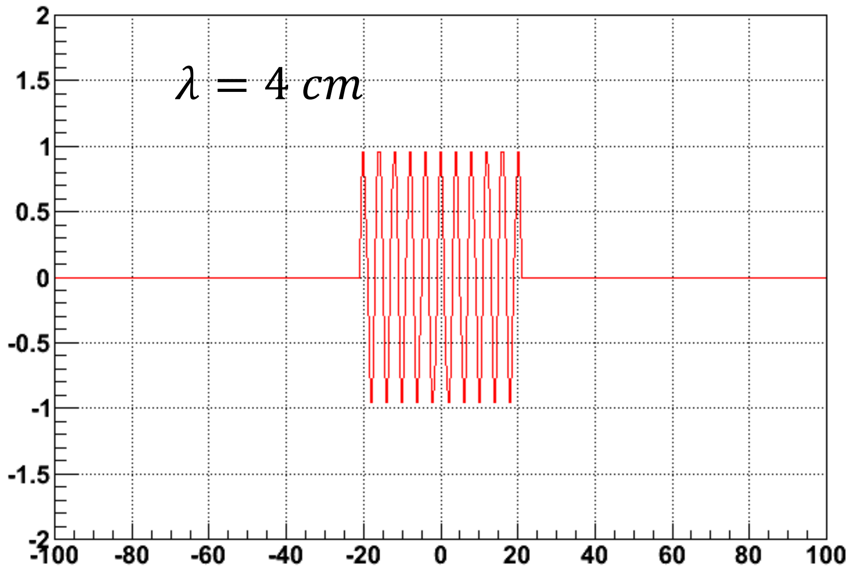
$$A(k') = \frac{E_0}{\sqrt{2\pi}} \int_{-cT}^{cT} \cos(kx)\cos(k'x)\,dx$$

- Trigonometric identity:

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$

$$A(k') = \frac{E_0}{\sqrt{2\pi}}\left[\frac{\sin\big((k-k')cT\big)}{k-k'} + \frac{\sin\big((k+k')cT\big)}{k+k'}\right]$$

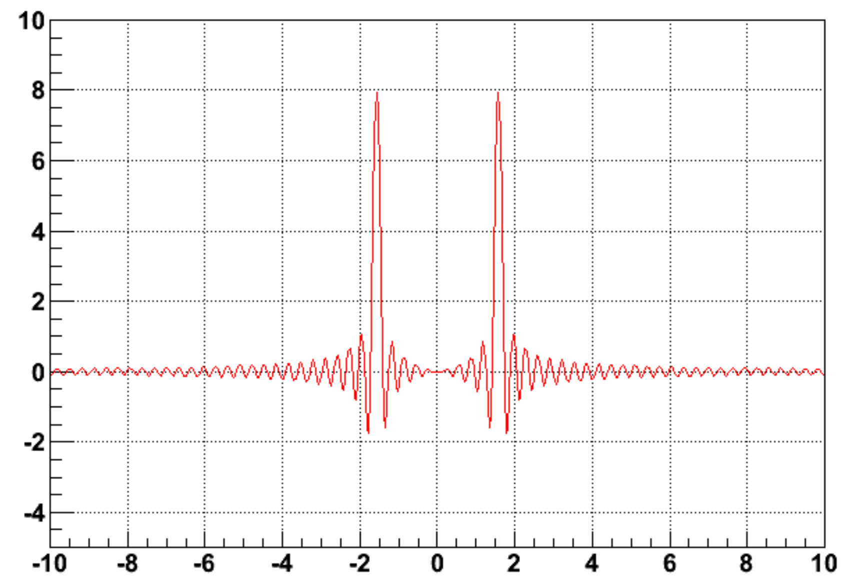
Another Example



$\lambda = 4\ cm$

$x\ \ (cm)$

$$k = \frac{2\pi}{\lambda} = 1.571\ cm^{-1}$$



$k'\ \ (cm^{-1})$

Frequency Representation

- Why would we want to represent a function in terms of its frequency components?
  - Both representations contain the same information
- Physical properties can depend on the frequency
- Examples:
  - Maximum frequency for discrete masses

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

$$\omega_{max} = 2\omega_0$$

  - Transmission lines:

$$\frac{\partial^2 V}{\partial x^2} + \frac{XY}{\omega^2}\frac{\partial^2 V}{\partial t^2} = 0 \text{ where } XY \approx -\omega^2 L' C'$$

  - Speed of light depends on wavelength: $v = c/n(\lambda)$