Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 18 – French, Chapter 6

Spring 2014 Semester

Matthew Jones

Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- Speed of propagation depends on the medium:
  - String with tension T and linear mass density μ:

    $$v = \sqrt{T/\mu}$$

  - Sound waves in air:

    $$v = \sqrt{\gamma p/\rho}$$

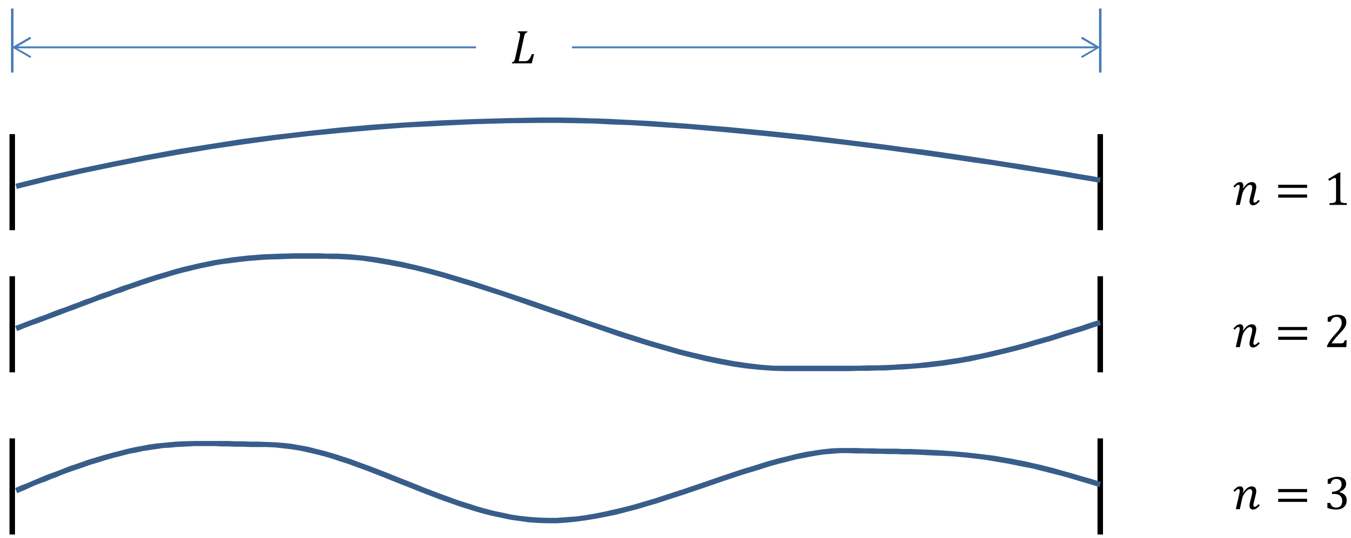
  - Electrical transmission lines:

    $$v = \sqrt{\frac{1}{L'C'}} \approx \frac{c}{\sqrt{\varepsilon_r}}$$

Boundary Conditions

- The boundary conditions at $x = 0$ and at $x = L$ determine what types of functions describe the normal modes of oscillation.
  - Both ends are fixed (no movement possible):

$$y(0,t) = y(L,t) = 0$$



$$y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Frequencies of Normal Modes

$$y_n(x,t) = y_n(x)\cos(\omega_n t + \alpha_n)$$

$$y_n(x) = \sin\left(\frac{n\pi x}{L}\right) = \sin k_n x$$

- Second derivatives:

$$\frac{\partial^2 y_n}{\partial t^2} = -\omega_n^2 y_n$$
$$\frac{\partial^2 y_n}{\partial x^2} = -k_n^2 y_n$$

- Substitute into the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \Rightarrow \omega_n = k_n v = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

Normal Modes of Oscillation

- Now we know all the normal modes of oscillation:

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t + \alpha_n)$$

- The system is linear, so any function of the form:

$$y(x,t) = \sum_n a_n\, y_n(x,t)$$

is also a solution, where a_n are real numbers.

- In particular, at $t = 0$, the initial configuration of the string is described by

$$y(x,0) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)\cos\alpha_n$$

$$\dot{y}(x,0) = -\sum_n a_n\omega_n \sin\left(\frac{n\pi x}{L}\right)\sin\alpha_n$$

Initial Conditions

$$y(x,0) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)\cos\alpha_n = \sum_n a_n' \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}(x,0) = -\sum_n a_n \omega_n \sin\left(\frac{n\pi x}{L}\right)\sin\alpha_n = \sum_n b_n' \sin\left(\frac{n\pi x}{L}\right)$$

- If we know the initial shape of the string, and its time derivative at $t = 0$, how can we determine the coefficients a_n' and b_n'?
- From these we could calculate a_n and α_n and then we would have a solution.
- Consider the special case where the velocity is initially zero…
  - Then $b_n' = -a_n\omega_n \sin\alpha_n = 0$ so we then must have $\alpha_n = 0$.

Fourier Analysis

- Simple case when initial velocity is zero:

$$y(x,0) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)$$

- If the initial shape was $y(x,0) = f(x)$, how do we calculate a_n?
- Let's first look at a similar problem…

Fourier Analysis

- Suppose we have a vector $\vec{v}$ in 3-dimensioal space.
- If we establish a set of coordinate axes then we have unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ along the x-, y-, and z-axes.

$$\hat{\imath} \cdot \hat{\imath} = 1 \qquad \hat{\imath} \cdot \hat{\jmath} = 0 \qquad \hat{\imath} \cdot \hat{k} = 0, \text{ etc...}$$

- Now we can write:

$$\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k}$$

- v_x, v_y and v_z are real numbers.
- We can calculate them using

$$v_x = \vec{v} \cdot \hat{\imath}$$
$$v_y = \vec{v} \cdot \hat{\jmath}$$
$$v_z = \vec{v} \cdot \hat{k}$$

Fourier Analysis

- Suppose we have a vector $\vec{v}$ in n-dimensional space.
- Suppose we have an orthonormal basis $\hat{u}_n$…

$$\hat{u}_n \cdot \hat{u}_m = \delta_{nm}$$

$$\delta_{nm} = \begin{cases} 1 & if\ n = m \\ 0 & if\ n \neq m \end{cases}$$

- We can write $\vec{v}$ in the form

$$\vec{v} = v_1\hat{u}_1 + v_2\hat{u}_2 + v_3\hat{u}_3 + \cdots = \sum_m v_m\hat{u}_m$$

- We can calculate v_n using

$$\vec{v} \cdot \hat{u}_n = \sum_m v_m\hat{u}_m \cdot \hat{u}_n = \sum_m v_m\,\delta_{mn} = v_n$$

Fourier Analysis

- Back to the initial value problem:
  - We have an initial function $f(x)$ that we can write
$$f(x) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)$$
  and we need to calculate the real numbers a_n…
- The functions $\sin\left(\frac{n\pi x}{L}\right)$ are sort of like a set of basis vectors… are they orthonormal?
  - How do we define the dot product in this case?

Fourier Analysis

- In this case we define the "dot product" as an integral:

$$f \cdot y_n = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Are $y_n(x)$ orthogonal?

$$y_n \cdot y_m = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) dx - \frac{1}{2}\int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right) dx$$

$= 0$ when $n \neq m$

Fourier Analysis

- But when $n = m$,

$$y_n \cdot y_m = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) dx - \cancelto{0}{\frac{1}{2}\int_0^L \cos\left(\frac{2n\pi x}{L}\right) dx}$$

$$= \frac{1}{2}\int_0^L dx = \frac{L}{2}$$

- So we can write

$$y_n \cdot y_m = \frac{L}{2}\,\delta_{nm}$$

Initial Value Problem

$$f(x) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$y_n \cdot f = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \int_0^L \left[\sum_m a_m \sin\left(\frac{m\pi x}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \sum_m a_m \; y_m \cdot y_n = \frac{L}{2} \sum_m a_m \; \delta_{mn} = \frac{L}{2} a_n$$

Initial Value Problem

$$f(x) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Now we know how to calculate a_n from the initial conditions… we have solved the initial value problem.

Example

- How to describe a square wave in terms of normal modes:

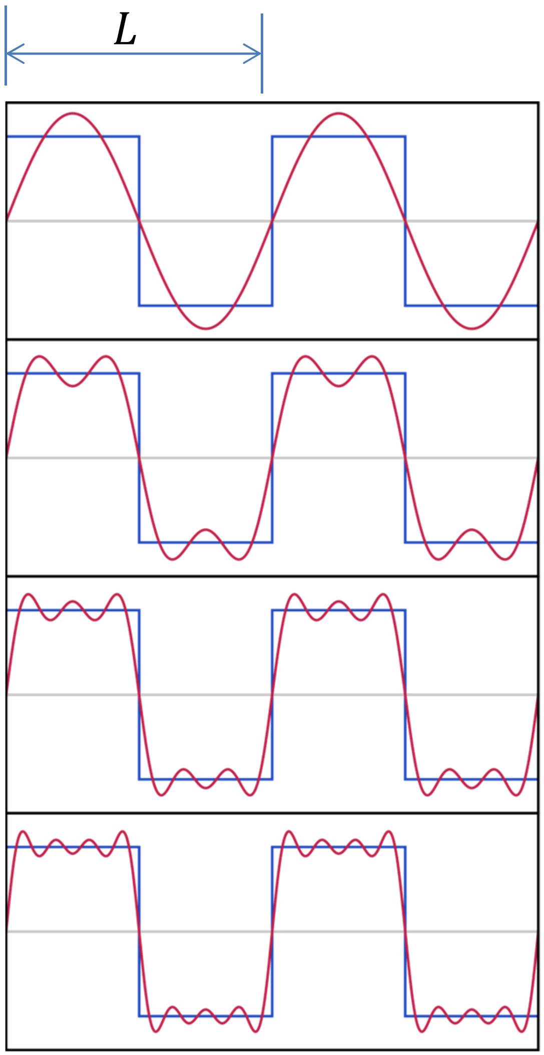
$$u(x) = \begin{cases} +1 \text{ when } 0 < x < L/2 \\ -1 \text{ when } L/2 < x < L \end{cases}$$

$$a_n = \frac{2}{L}\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right) dx - \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{n\pi}[1 - \cos(n\pi)]$$

$$a_1 = \frac{4}{\pi}, a_3 = \frac{4}{3\pi}, a_5 = \frac{4}{5\pi}, \cdots$$

Example



$$a_n = \frac{2}{n\pi}[1 - \cos(n\pi)]$$

$$a_1 = \frac{4}{\pi}, a_3 = \frac{4}{3\pi}, a_5 = \frac{4}{5\pi}, \cdots$$

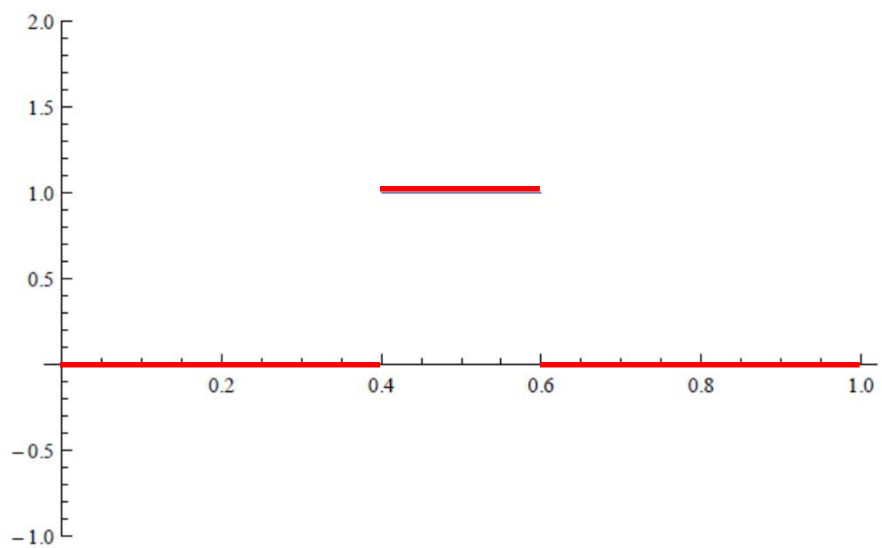
$$a_2 = 0, a_4 = 0, a_6 = 0, \cdots$$

The initial shape doesn't really satisfy the boundary conditions $y(0) = y(L) = 0$, but the approximation does.

Other Examples

- Consider an initial displacement in the middle of the string:

$$f(x) = \begin{cases} 0 \text{ when } x < 2L/5 \\ 1 \text{ when } 2L/5 < x < 3L/5 \\ 0 \text{ when } x > 3L/5 \end{cases}$$



Let's assume $L = 1$ and $v = 1$

Example

$$a_n = \frac{2}{L}\int_{2L/5}^{3L/5} \sin\left(\frac{n\pi x}{L}\right) dx$$

```
a[n_] := 2/L ∫_0^L f[x] Sin[n π x / L] ⅆx

f[x_] = Piecewise[{{0, x < 2 / 5}, {1, x > 2 / 5 && x < 3 / 5}, {0, x > 3 / 5}}]

Table[a[n], {n, M}]
```

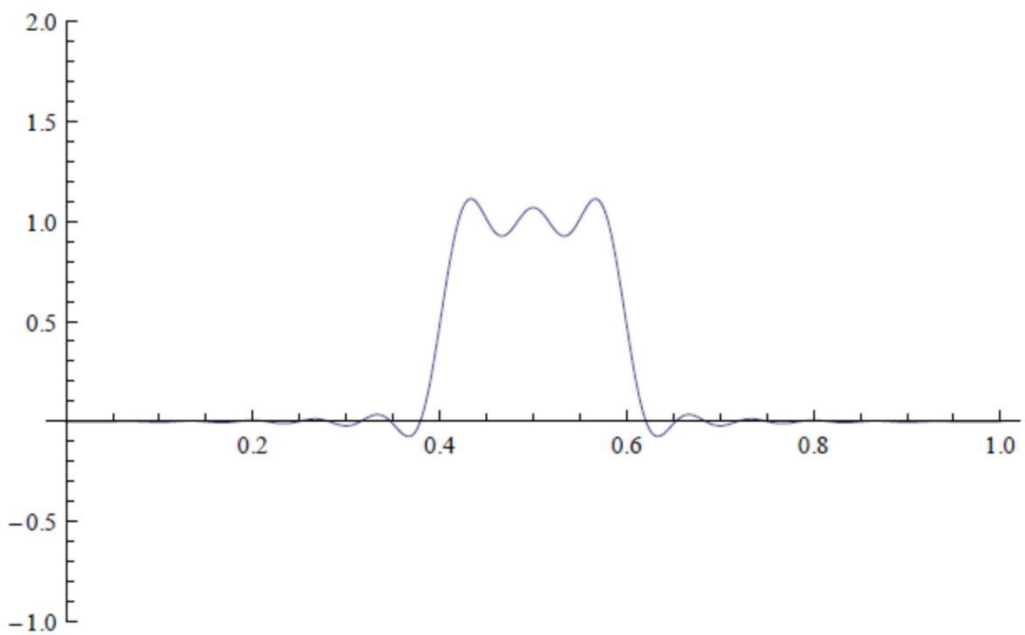
$$\left\{\frac{-1+\sqrt{5}}{\pi}, 0, \frac{-1-\sqrt{5}}{3\pi}, 0, \frac{4}{5\pi}, 0, \frac{-1-\sqrt{5}}{7\pi}, 0, \frac{-1+\sqrt{5}}{9\pi}, 0, \frac{-1+\sqrt{5}}{11\pi}, 0, \frac{-1-\sqrt{5}}{13\pi}, 0, \frac{4}{15\pi}, 0, \frac{-1-\sqrt{5}}{17\pi}, 0, \frac{-1+\sqrt{5}}{19\pi}, 0, \frac{-1+\sqrt{5}}{21\pi}, 0, \frac{-1-\sqrt{5}}{23\pi}, 0, \frac{4}{25\pi}, 0, \frac{-1-\sqrt{5}}{27\pi}, 0, \frac{-1+\sqrt{5}}{29\pi}, 0\right\}$$

- Now we know the first 30 values for a_n… we're done!

Example

- Is this a good approximation?

```
Plot[z[x, 0], {x, 0, 1}, PlotRange → {-1, 2}]
```



- A good description of sharp features require high frequencies (large n).

Example

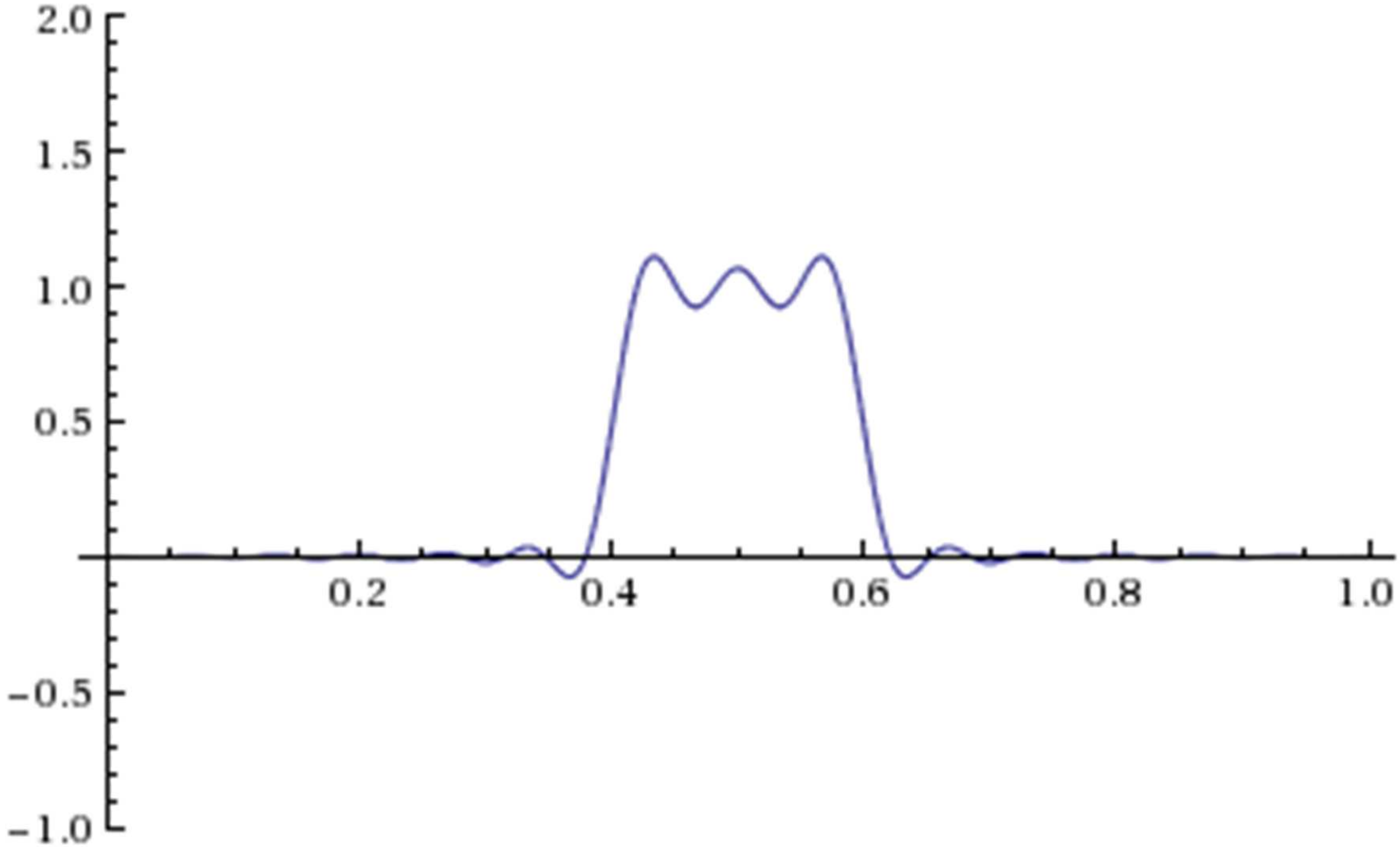
- The complete solution to the initial value problem is

$$y(x,t) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

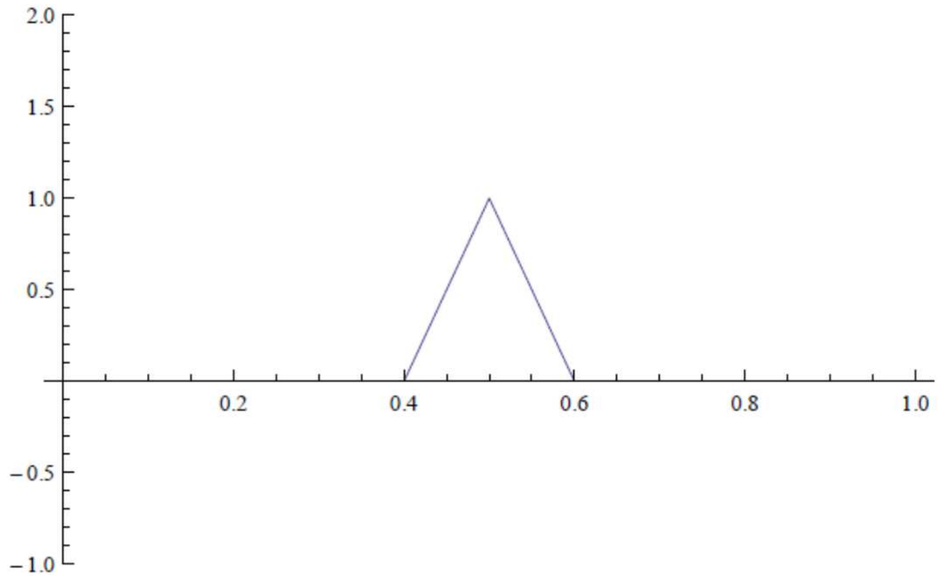
- What does this look like as a function of time?

Example



Another Example

- Consider a function that is a bit smoother:



$$f(x) = \begin{cases} 0 & x < \frac{2}{5} \\ 10\left(-\frac{2}{5} + x\right) & x > \frac{2}{5} \,\&\&\, x < \frac{1}{2} \\ 2 + 10\left(\frac{2}{5} - x\right) & x > \frac{1}{2} \,\&\&\, x < \frac{3}{5} \\ 0 & \text{True} \end{cases}$$

Example

- The integrals for the Fourier coefficients are of the form:

$$\int_a^b \sin\left(\frac{n\pi x}{L}\right) dx \text{ or } \int_a^b x \sin\left(\frac{n\pi x}{L}\right) dx$$

- These can be solved analytically, but it is a lot of work…

$$\left\{-\frac{10\left(-4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{\pi^2}, 0, \frac{10\left(-4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{9\pi^2}, 0, \frac{8}{5\pi^2}, 0, -\frac{10\left(4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{49\pi^2}, 0,\right.$$

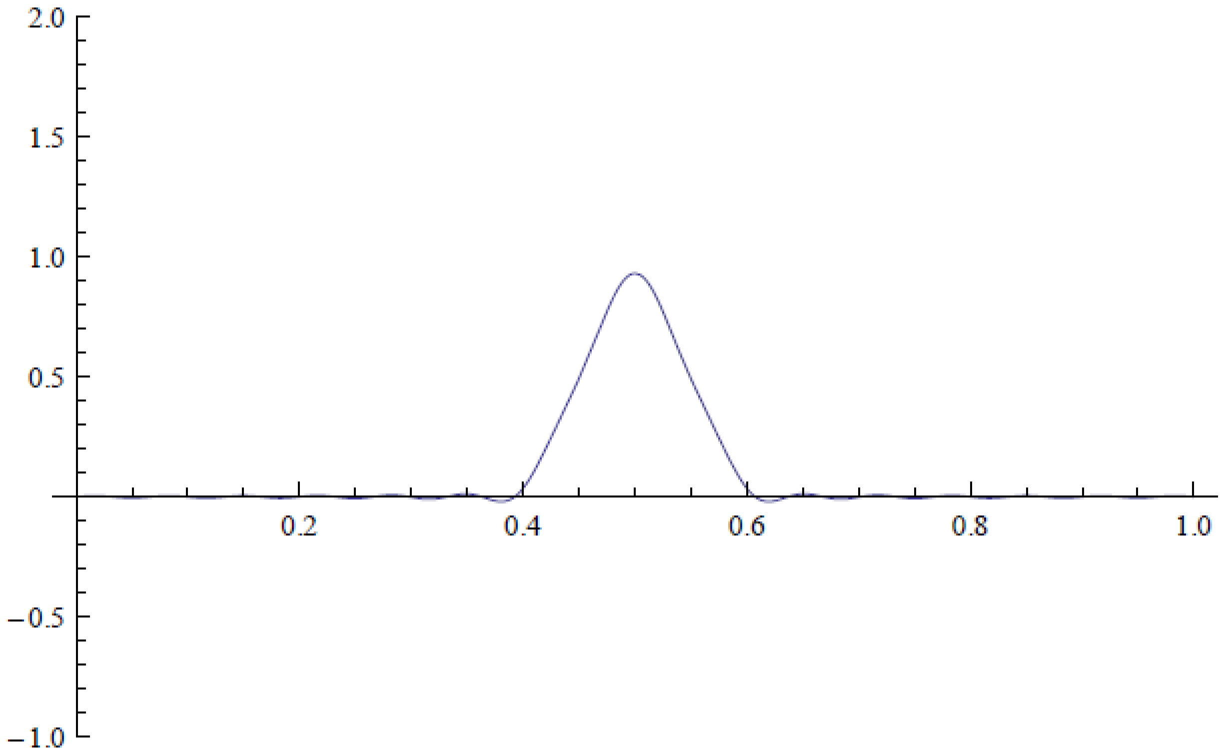
$$\frac{10\left(4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{81\pi^2}, 0, -\frac{10\left(4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{121\pi^2}, 0, \frac{10\left(4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{169\pi^2}, 0, -\frac{8}{45\pi^2},$$

$$0, -\frac{10\left(-4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{289\pi^2}, 0, \frac{10\left(-4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{361\pi^2}, 0, -\frac{10\left(-4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{441\pi^2}, 0,$$

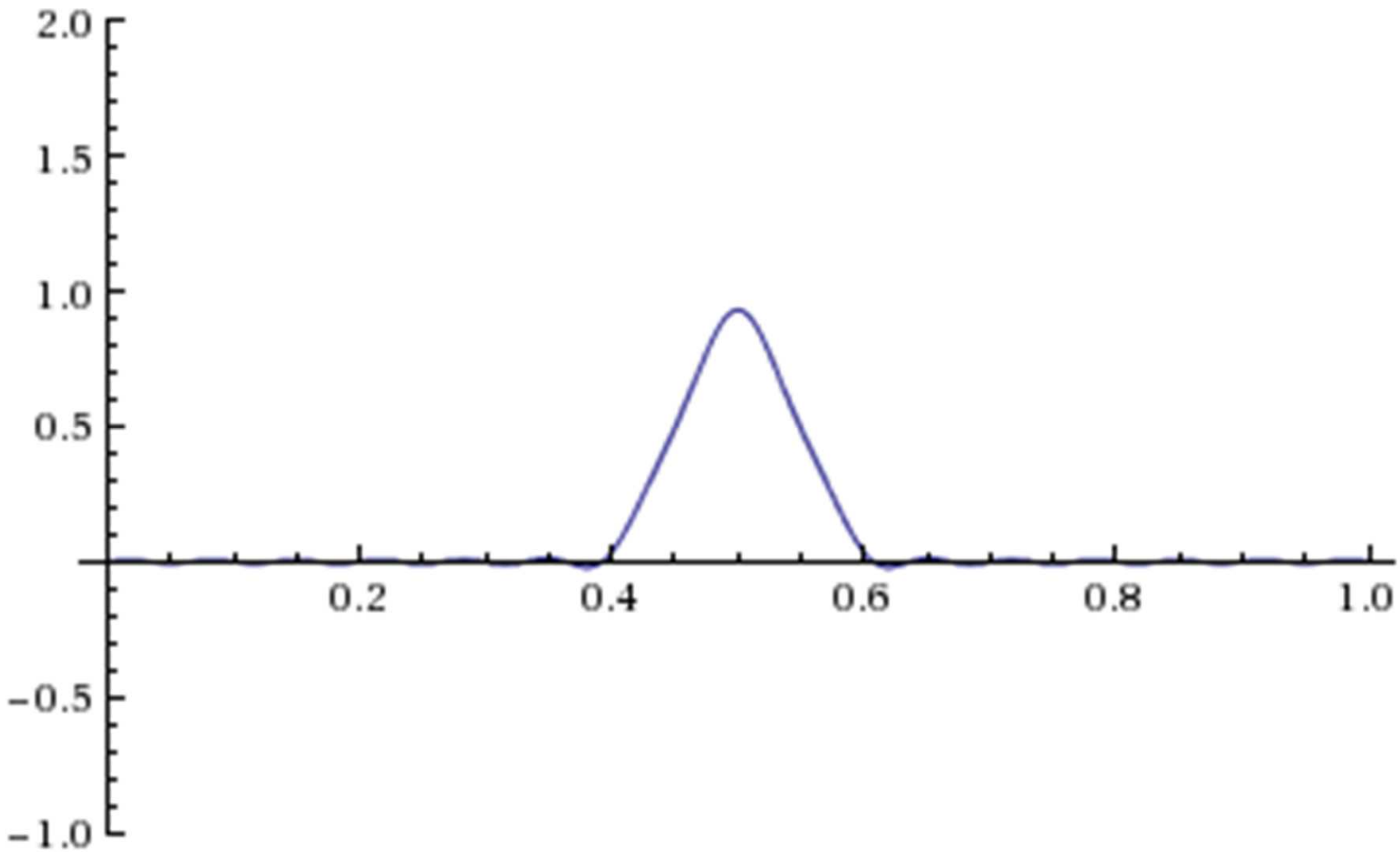
$$\left.\frac{10\left(-4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{529\pi^2}, 0, \frac{8}{125\pi^2}, 0, -\frac{10\left(4+\sqrt{2\left(5-\sqrt{5}\right)}\right)}{729\pi^2}, 0, \frac{10\left(4+\sqrt{2\left(5+\sqrt{5}\right)}\right)}{841\pi^2}, 0\right\}$$

Example

- The initial shape of the approximation with N=30 is better than for the square pulse.



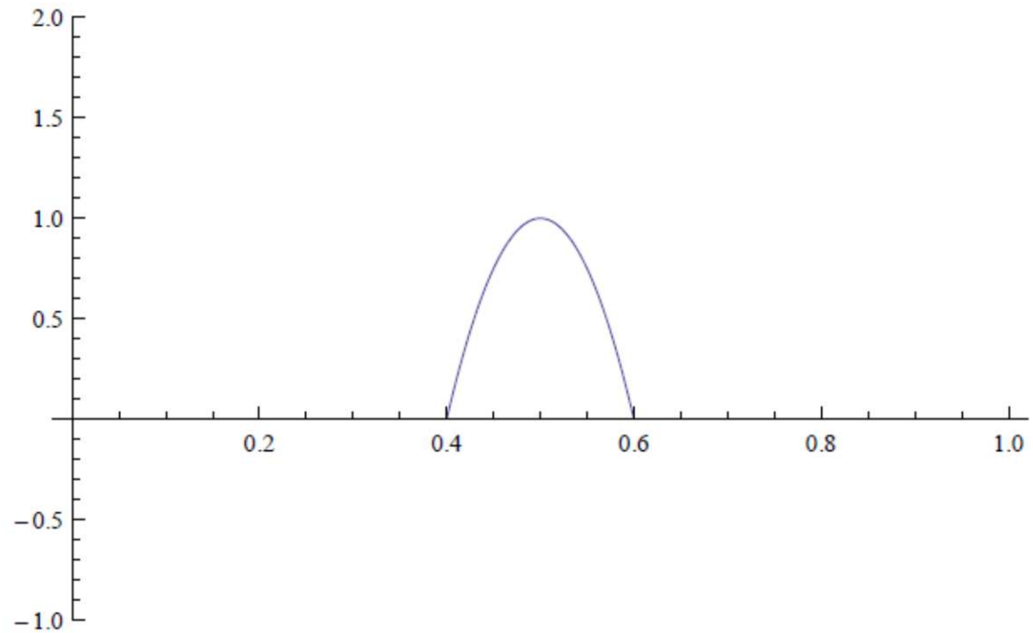
Example



Final Example

- An even smoother function:

$$f(x) = \begin{cases} 0 & x < \frac{2}{5} \\ 1 - 100\left(-\frac{1}{2} + x\right)^2 & x > \frac{2}{5}\ \&\&\ x < \frac{3}{5} \\ 0 & \text{True} \end{cases}$$



Example

- The integrals for the Fourier coefficients are of the form:

$$\int_a^b \sin\left(\frac{n\pi x}{L}\right) dx \text{ or } \int_a^b x \sin\left(\frac{n\pi x}{L}\right) dx \text{ or } \int_a^b x^2 \sin\left(\frac{n\pi x}{L}\right) dx$$

- These can be solved analytically, but it is a lot of work…

$$\Bigg\{-\frac{20\left(10-10\sqrt{5}+\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{\pi^3},\ 0,\ \frac{20\left(-10-10\sqrt{5}+3\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{27\pi^3},\ 0,\ \frac{32}{5\pi^3},$$

$$0,\ -\frac{20\left(10+10\sqrt{5}+7\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{343\pi^3},\ 0,\ \frac{20\left(-10+10\sqrt{5}+9\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{729\pi^3},\ 0,$$

$$-\frac{20\left(10-10\sqrt{5}+11\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{1331\pi^3},\ 0,\ \frac{20\left(-10-10\sqrt{5}+13\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{2197\pi^3},\ 0,\ \frac{32}{135\pi^3},$$

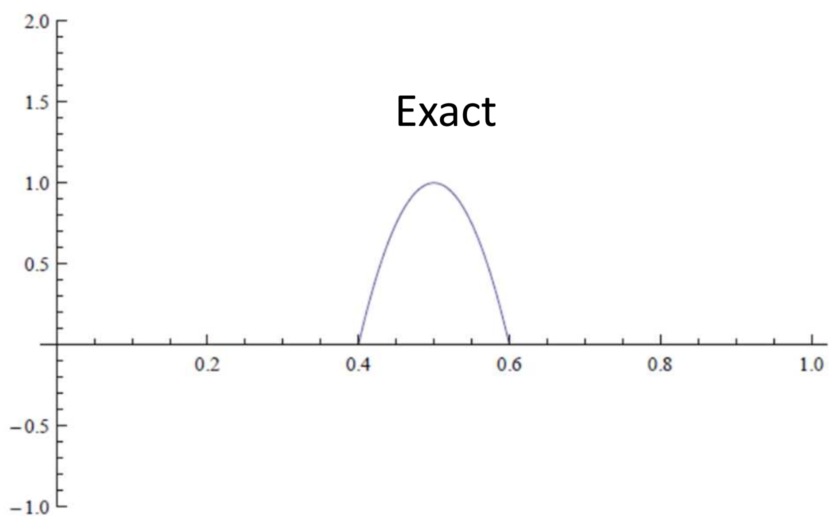
$$0,\ -\frac{20\left(10+10\sqrt{5}+17\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{4913\pi^3},\ 0,\ \frac{20\left(-10+10\sqrt{5}+19\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{6859\pi^3},\ 0,$$

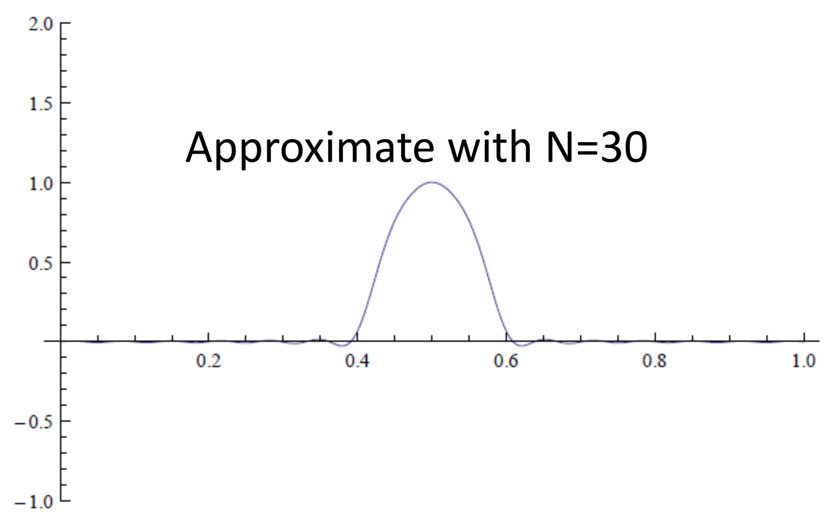
$$-\frac{20\left(10-10\sqrt{5}+21\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{9261\pi^3},\ 0,\ \frac{20\left(-10-10\sqrt{5}+23\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{12167\pi^3},\ 0,\ \frac{32}{625\pi^3},$$

$$0,\ -\frac{20\left(10+10\sqrt{5}+27\sqrt{2\left(5-\sqrt{5}\right)}\,\pi\right)}{19683\pi^3},\ 0,\ \frac{20\left(-10+10\sqrt{5}+29\sqrt{2\left(5+\sqrt{5}\right)}\,\pi\right)}{24389\pi^3},\ 0\Bigg\}$$

Example

- The initial shape of the approximation with N=30 is even better than the triangular pulse…





Example