

Physics 42200

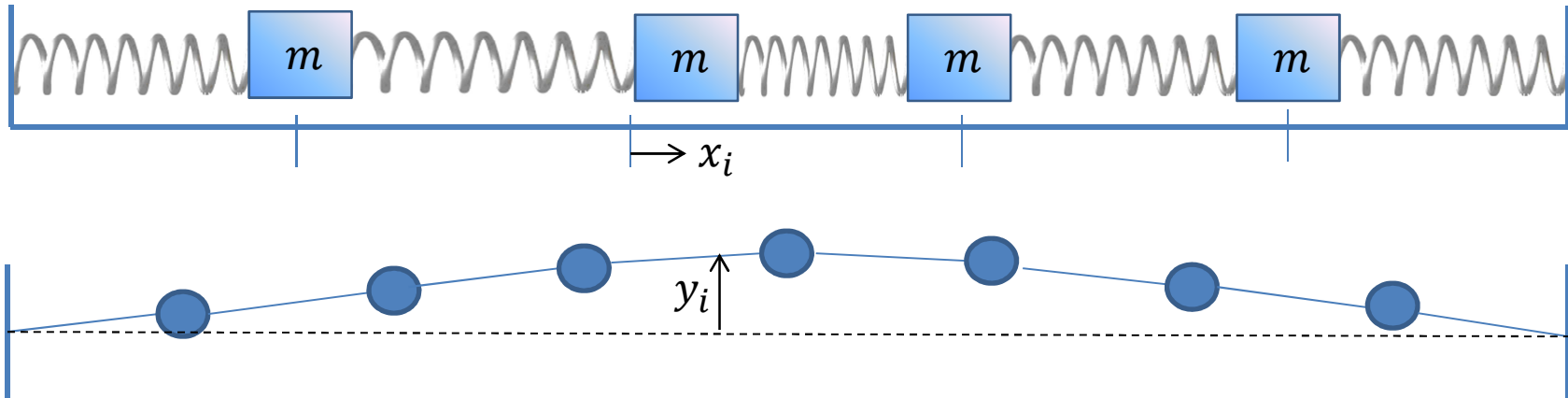
# Waves & Oscillations

Lecture 16 – French, Chapter 6

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# Continuous Systems



$$m \ddot{y}_n = F_n = \frac{T}{\ell} [(y_{n+1} - y_n) - (y_n - y_{n-1})]$$

In the limit  $m/\ell \rightarrow \mu$  this becomes the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

where  $v = \sqrt{T/\mu}$ .

# Other Continuous Systems

1. Longitudinal waves in a solid elastic rod
2. Longitudinal waves in a compressible gas
3. Transmission lines (electrical example)

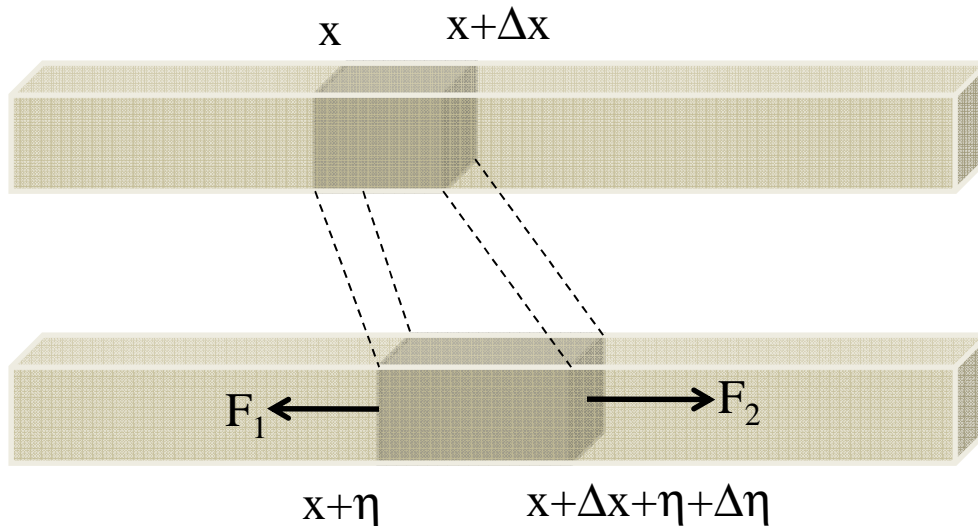
Based on our understanding of the microscopic properties of each system, we can derive the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Which physical properties determine  $v$ ?

# Other Continuous Systems

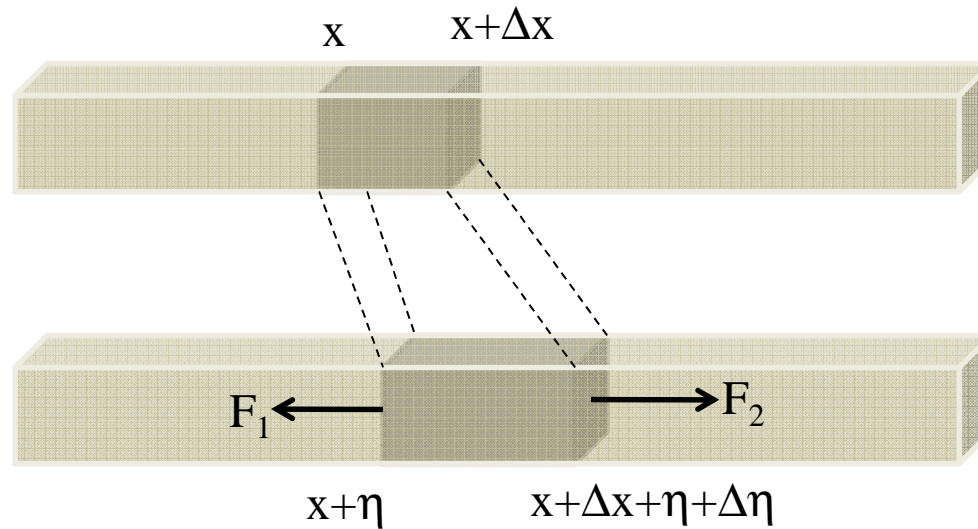
- Longitudinal waves in a solid rod:



Notation:

- $x$  labels which piece of the rod we are considering, analogous to the index  $n$  when counting discrete masses.
  - $\eta$  quantifies how much the element of mass has moved.
- Recall that strain was defined as the fractional increase in length of a small element:  $\Delta\eta/\Delta x$
  - Stress was defined as  $\Delta F/A$
  - These were related by  $\Delta F/A = Y \Delta\eta/\Delta x$

# Longitudinal Waves in a Solid Rod



$$\Delta F / A = Y \Delta \eta / \Delta x$$

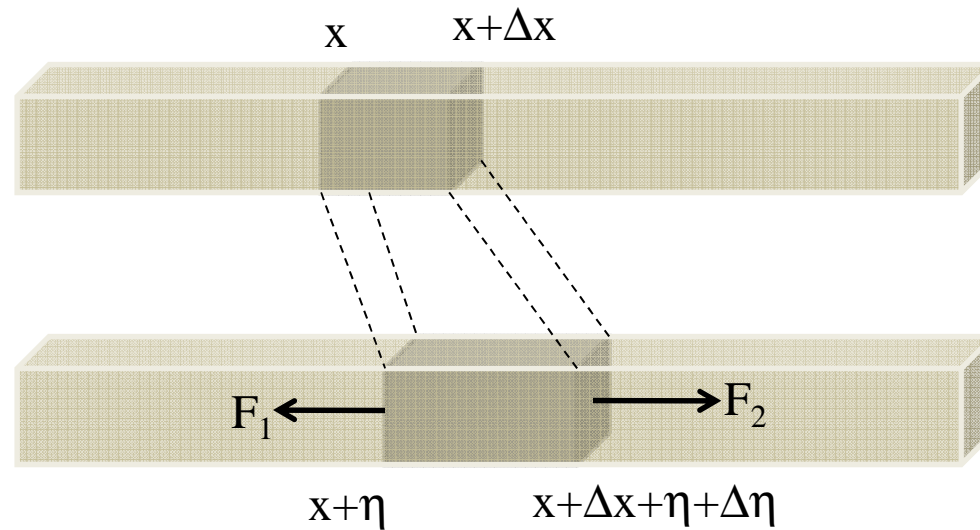
- Force on one side of the element:

$$F_1 = AY \Delta \eta / \Delta x = AY \partial \eta / \partial x$$

- Force on the other side of the element:

$$F_2 = F_1 + AY \frac{\partial^2 \eta}{\partial x^2} \Delta x$$

# Longitudinal Waves in a Solid Rod



- Newton's law:

$$m\ddot{\eta} = F_2 - F_1$$

$$F_2 - F_1 = AY \frac{\partial^2 \eta}{\partial x^2} \Delta x = \rho A \Delta x \frac{\partial^2 \eta}{\partial t^2}$$

- Wave equation:

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\rho}{Y} \frac{\partial^2 \eta}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2}$$

# Longitudinal Normal Modes

- What is the solution for a rod of length  $L$ ?

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2} \qquad v = \sqrt{Y/\rho}$$

- Boundary conditions:

- Suppose one end is fixed

$$\eta(0) = 0$$

- No force at the free end of the rod so the stress is zero there.  
Strain  $\propto$  stress, so the strain is also zero.

$$F = AY \partial \eta / \partial x$$

$$\frac{\partial \eta}{\partial x}_{x=L} = 0$$

- Look for solutions that are of the form

$$\eta(x) = f(x) \cos \omega t$$

# Longitudinal Normal Modes

$$\eta(x) = f(x) \cos \omega t$$

- Inspired by the continuous string problem, we let

$$f(x) = A \sin(kx)$$

- Derivatives:

$$\frac{\partial^2 \eta}{\partial x^2} = -k^2 \eta$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\omega^2 \eta$$

$$\frac{\partial^2 \eta}{dx^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2} \Rightarrow k = \frac{\omega}{v}$$



# Longitudinal Normal Modes

$$f(x) = A \sin\left(\frac{\omega x}{v}\right)$$

- This automatically satisfies the boundary condition at  $x = 0$ .
- At  $x = L$ ,  $\partial\eta/\partial x = 0$ :

$$\frac{\partial\eta}{\partial x}_{x=L} \propto \cos\left(\frac{\omega L}{v}\right) = 0$$

- This means that  $\frac{\omega L}{v} = (n - \frac{1}{2})\pi$
- Angular frequencies of normal modes are

$$\omega_n = \frac{\pi}{L} (n - \frac{1}{2}) \sqrt{Y/\rho}$$

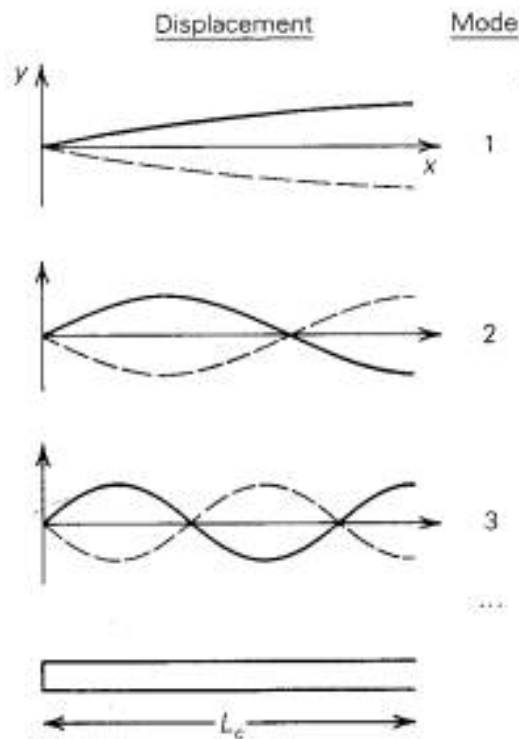
- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L} \sqrt{Y/\rho}$$

# Longitudinal Normal Modes

- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L} \sqrt{Y/\rho}$$



Lowest possible frequency:

$$\nu_1 = \frac{1}{4L} \sqrt{\frac{Y}{\rho}}$$

# Frequencies of Metal Chimes

- Suppose a set of chimes were made of copper rods, with lengths between 30 and 40 cm, rigidly fixed at one end.

- What frequencies should we expect if

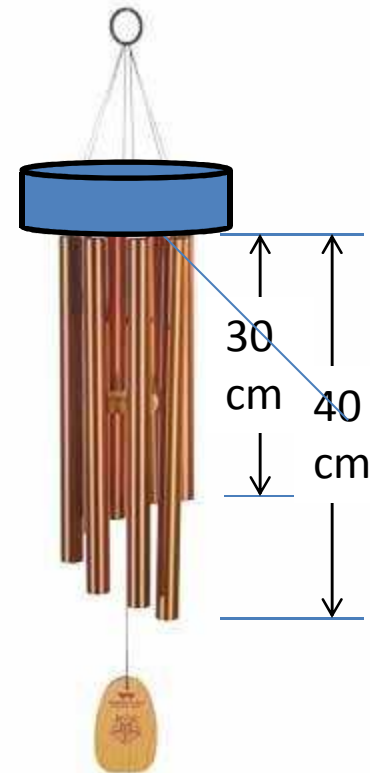
$$Y = 117 \times 10^9 \text{ N} \cdot \text{m}^{-2}$$

$$\rho = 8.96 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$$

$$\nu_1 = \frac{1}{4L} \sqrt{\frac{117 \times 10^9 \text{ N} \cdot \text{m}^{-2}}{8.96 \times 10^3 \text{ kg} \cdot \text{m}^{-3}}}$$

$$= 2260 - 3010 \text{ Hz}$$

(highest octave on a piano)



# Frequencies of Metal Chimes

- If the metal rods were not fixed at one end then the boundary conditions at both ends would be:

$$\frac{\partial \eta}{\partial x} = 0$$

- Allowed frequencies of normal modes:

$$v_n = \frac{n}{2L} \sqrt{Y/\rho}$$

Open at Both Ends



Harmonic

Wavelength  $\lambda$

Frequency  $f$

1<sup>st</sup>

$2L$

$f_1$

2<sup>nd</sup>

$L$

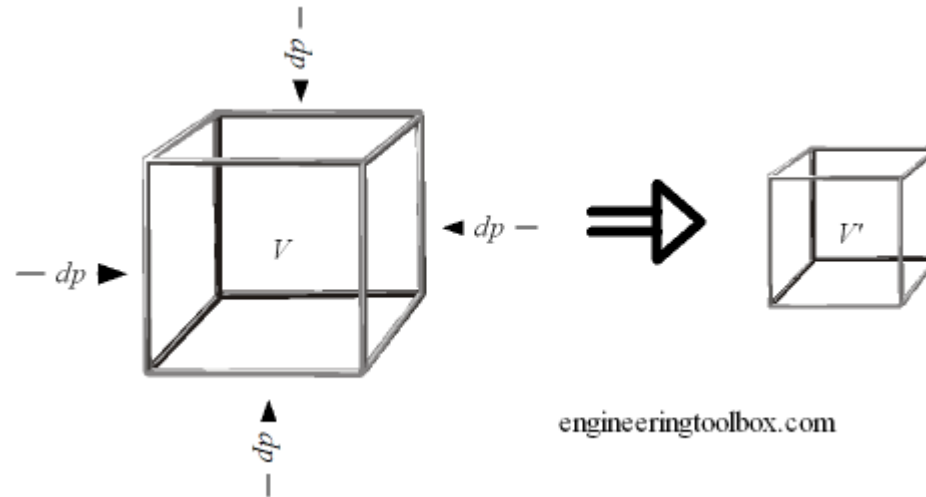
$2f_1$

3<sup>rd</sup>

$2L/3$

$3f_1$

# Longitudinal Waves in a Gas



- Increased pressure on a volume of gas decreases its volume
- Bulk modulus of elasticity is defined

$$K = -V \frac{dp}{dV}$$

# Longitudinal Waves in a Gas

- Equations of state for a gas:
  - Ideal gas law:  $pV = NkT$
  - Adiabatic gas law:  $pV^\gamma = \text{constant}$
- In an adiabatic process, no heat is absorbed
  - Absorbing heat would remove mechanical energy from a system
  - Propagation of sound waves through a gas is an example of an adiabatic process
- Bulk modulus calculated from equation of state:

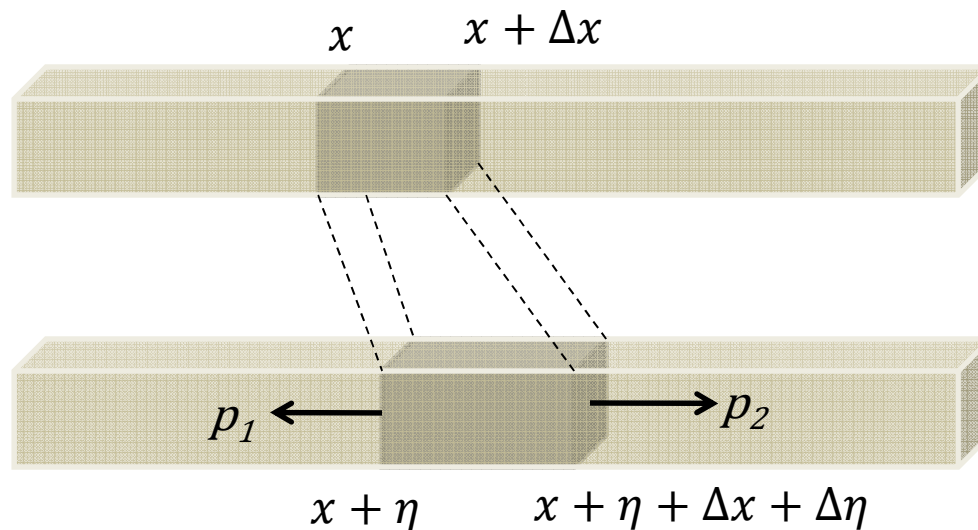
$$V^\gamma dp + \gamma p V^{\gamma-1} dV = 0$$

$$\frac{dp}{dV} = -\gamma p/V$$

$$K = -V \frac{dp}{dV} = \gamma p$$

# Longitudinal Waves in a Gas

- By analogy with the solid rod, we consider an element of gas at position  $x$  of thickness  $\Delta x$  that is displaced by a distance  $\eta(x)$ :



# Longitudinal Waves in a Gas

- Wave equation:

$$\frac{\partial^2 \eta}{dx^2} = \frac{1}{v^2} \frac{\partial^2 \eta}{\partial t^2}$$

- For a solid rod,  $v = \sqrt{Y/\rho}$
- For a gas,  $v = \sqrt{K/\rho} = \sqrt{\gamma p/\rho}$
- Changes in pressure and density are very small compared with the average pressure and density.
- At standard temperature and pressure, air has

$$\gamma = 1.40$$

$$p = 101.3 \text{ kPa}$$

$$\rho = 1.2 \text{ kg/m}^3$$

$$v = \sqrt{\frac{(1.40)(101.3 \times 10^3 \text{ N/m}^2)}{(1.2 \text{ kg/m}^3)}} = 343 \text{ m/s}$$

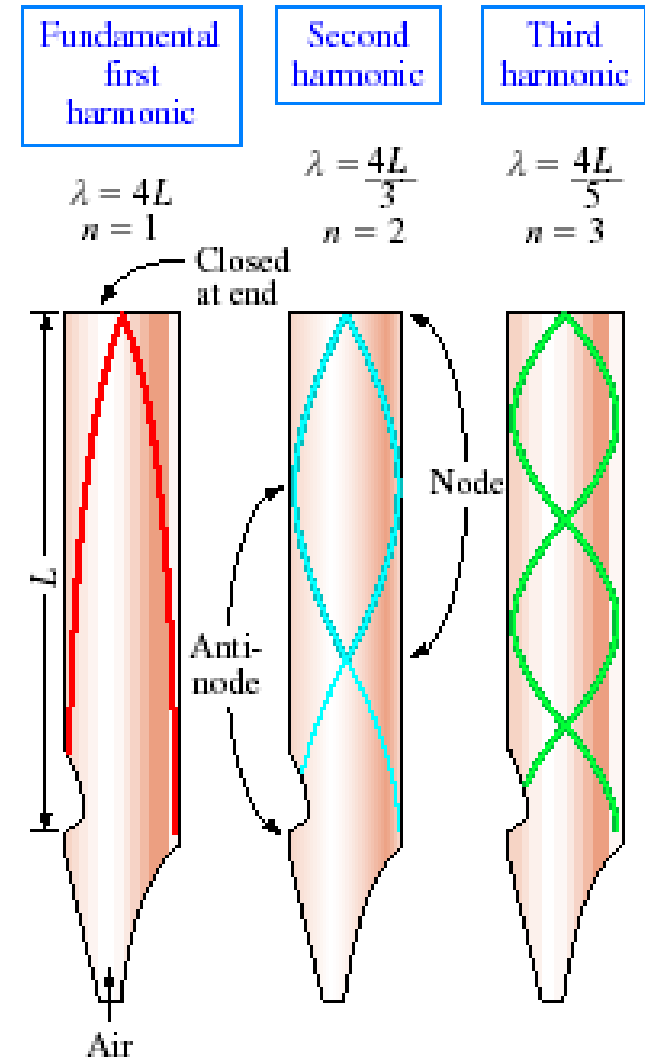


# The Physics of Organ Pipes



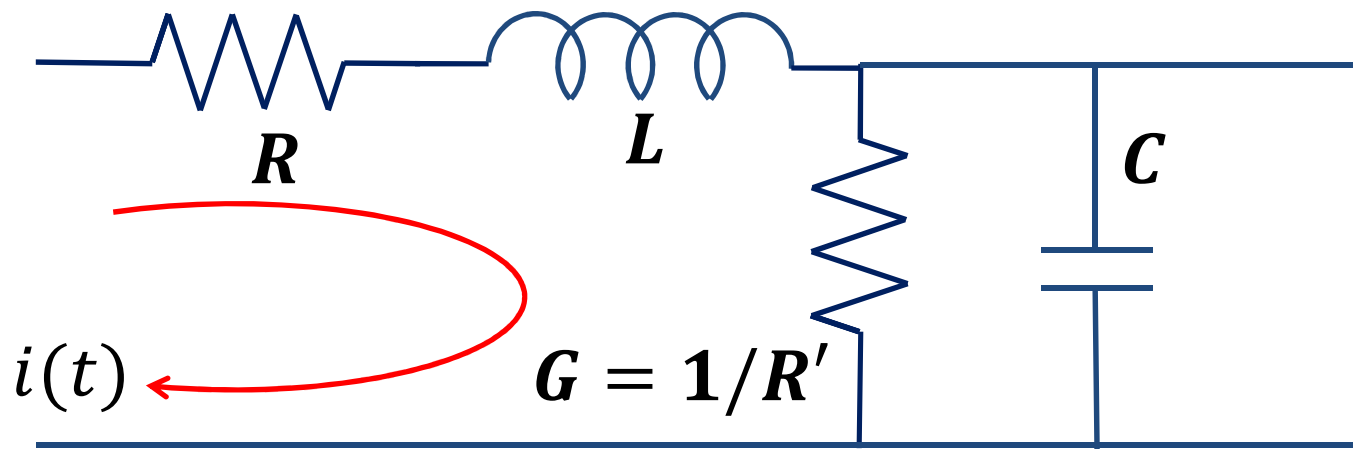
# Resonant Cavities

- Air under pressure enters at the bottom
  - Entering air rapidly oscillates between the pipe and the lip
  - The lower end is a pressure anti-node
- Top end can be open or closed
  - Open end is a pressure node/displacement anti-node
  - Closed end a displacement node/pressure anti-node



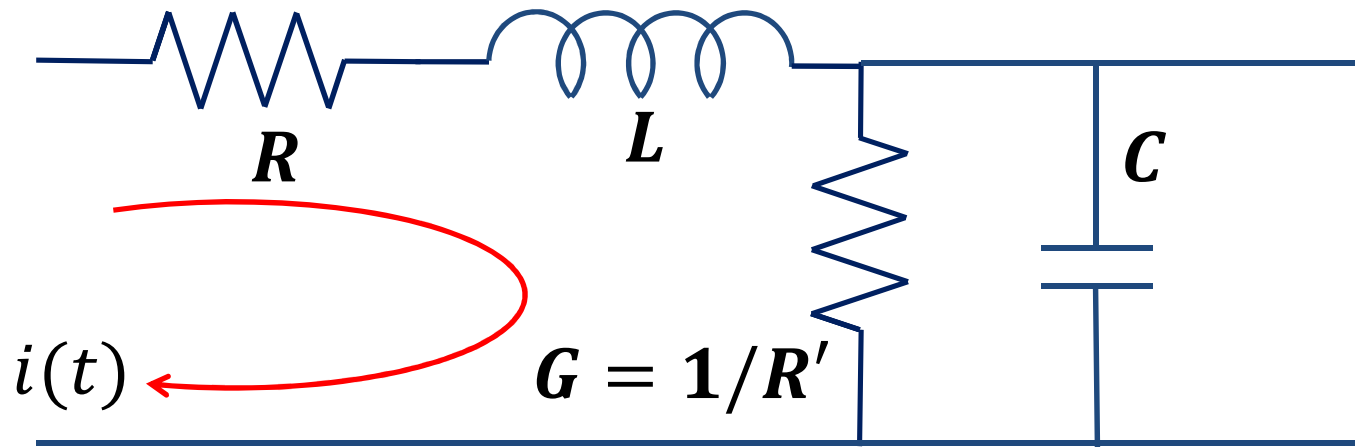
# Electrical Circuits

- First, consider one “lump” of a circuit:



- It is convenient to describe the resistor that is in parallel with the capacitor in terms of its conductance,  $G = 1/R'$ .

# Electrical Circuits



- Calculate the total impedance of the lump:

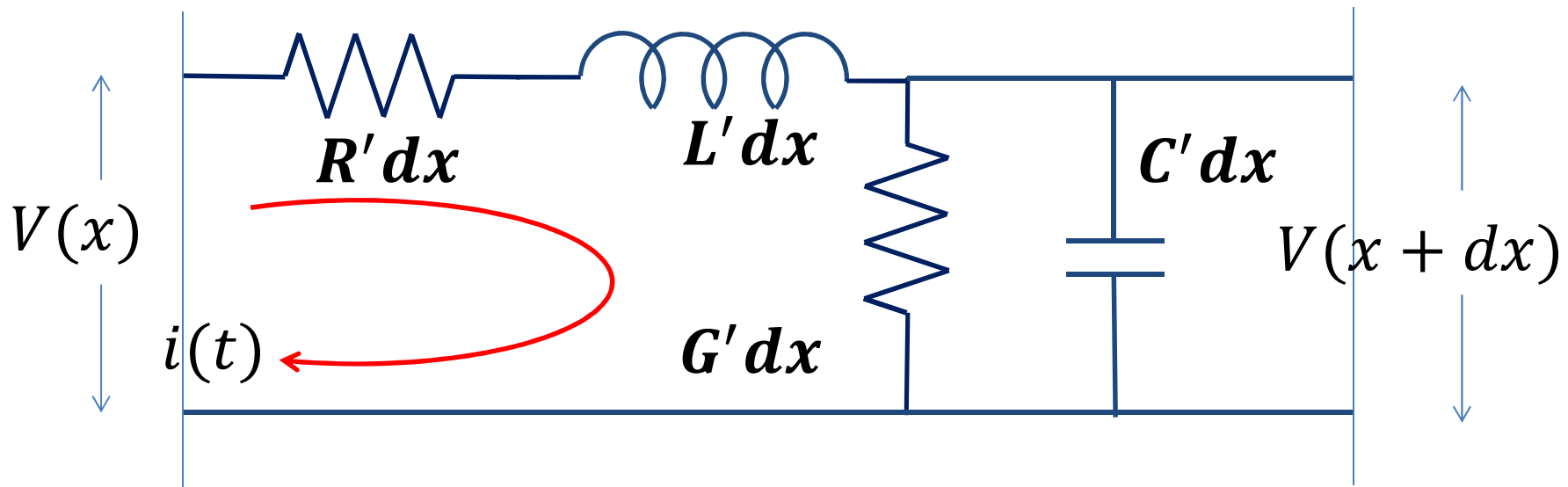
$$\left. \begin{aligned} Z_R &= R \\ Z_L &= i\omega L \\ Z_C &= \frac{1}{i\omega C} \\ Z_G &= 1/G \end{aligned} \right\}$$

$$X = R + i\omega L$$

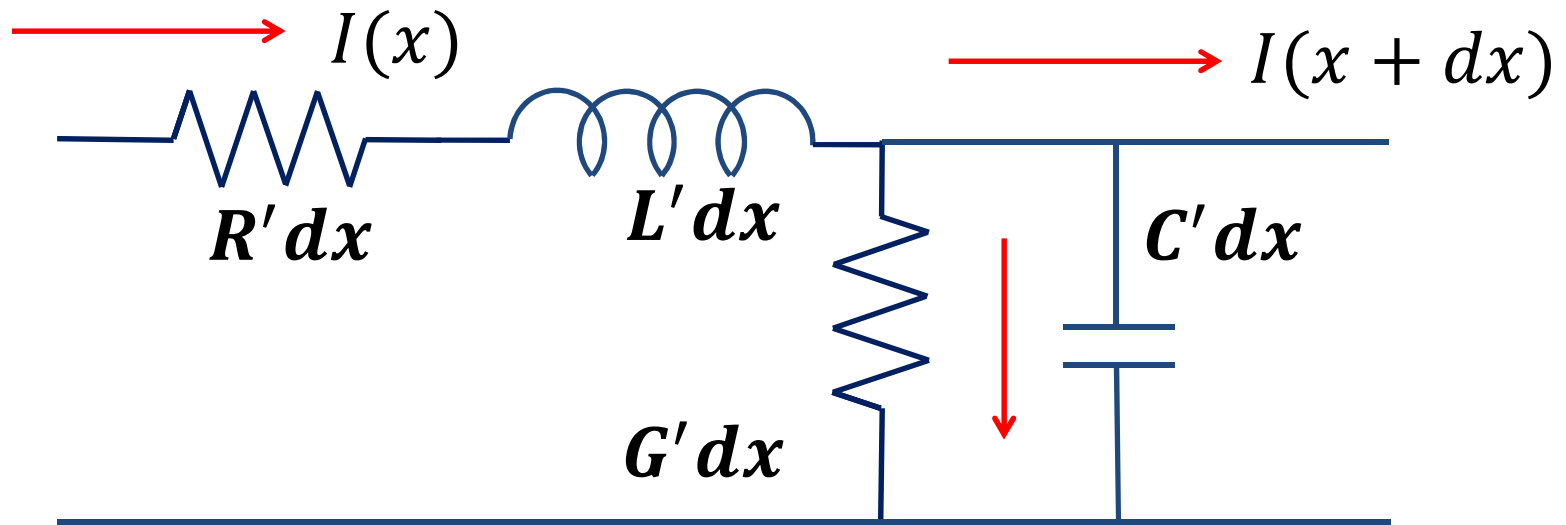
$$Y = G + i\omega C$$

# Electrical Circuits

- Suppose the resistance, inductance, capacitance and conductance were distributed uniformly with length:
  - Let  $R'$  be the resistance per unit length,  $L'$  be the inductance per unit length, etc...
- Consider the voltage on either side of the lump:



# Electric Circuits



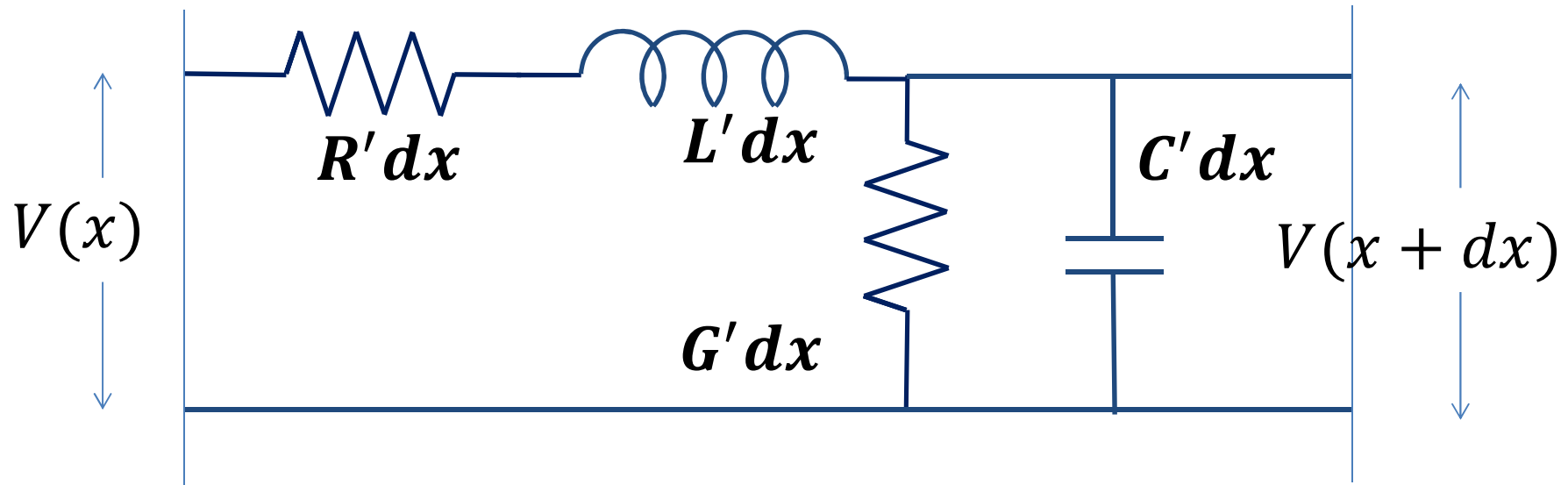
- Current flowing through  $G'$  and  $C'$  is

$$\Delta I = \frac{V(x)}{Z_{G'+C'}} = V(x)Y$$

$$I(x+dx) = I(x) - V(x)Y$$

$$\frac{\partial I}{\partial x} = \frac{I(x+dx) - I(x)}{dx} = -V(x)Y$$

# Electrical Circuits



- Voltage drop across the lump:

$$V(x + dx) = V(x) - I(x)X$$

$$\frac{\partial V}{\partial x} = \frac{V(x + dx) - V(x)}{dx} = -I(x)X$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\partial I}{\partial x}X = XY V(x)$$

# Electrical Circuits

- When we assume that the voltage is of the form

$$V(x, t) = V(x)e^{i\omega t}$$
$$\frac{\partial^2 V}{\partial t^2} = -\omega^2 V(x)$$

- Using the previous result,  $\frac{\partial^2 V}{\partial x^2} = XY V(x)$  we get:

$$\frac{\partial^2 V}{\partial x^2} + \frac{XY}{\omega^2} \frac{\partial^2 V}{\partial t^2} = 0$$

- Does this resemble the wave equation?
  - Expand out  $XY = (R' + i\omega L')(G' + i\omega C')$
  - When  $R'$  and  $G'$  are small, which is frequently the case then  $XY \approx -\omega^2 L' C'$



# Electrical Circuits

- Wave equation:

$$\frac{\partial^2 V}{\partial x^2} = L' C' \frac{\partial^2 V}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2}$$

- Speed of wave propagation is

$$v = \frac{1}{\sqrt{L' C'}}$$