

Physics 42200

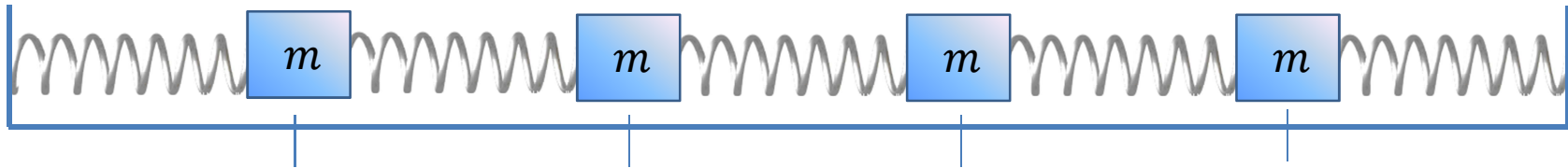
# **Waves & Oscillations**

Lecture 15 – French, Chapter 6

Spring 2014 Semester

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# Vibrations of Continuous Systems



- Equations of motion for masses in the middle:

$$m \ddot{x}_i + 2kx_i - k(x_{i-1} + x_{i+1}) = 0$$
$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0$$

- Proposed solution:

$$x_i(t) = A_i \cos \omega t$$
$$\frac{A_{i-1} + A_{i+1}}{A_i} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- We solved this to determine  $A_i$  and  $\omega_j$ ...

# Vibrations of Continuous Systems

- Amplitude of mass  $n$  for normal mode  $k$ :

$$A_{n,k} = C \sin \left( \frac{nk\pi}{N+1} \right)$$

- Frequency of normal mode  $k$ :

$$\omega_k = 2\omega_0 \sin \left( \frac{k\pi}{2(N+1)} \right)$$

- Solution for normal modes:

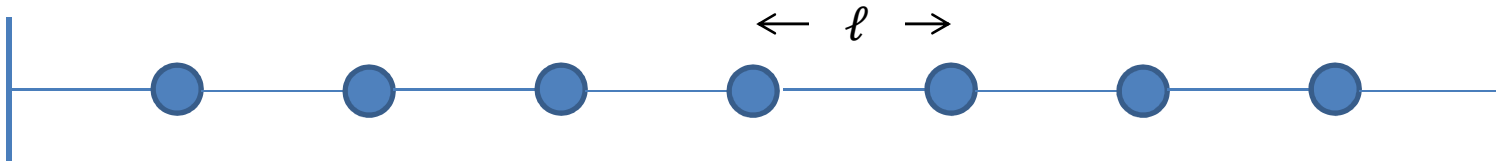
$$q_{n,k}(t) = A_{n,k} \cos \omega_k t$$

- General solution:

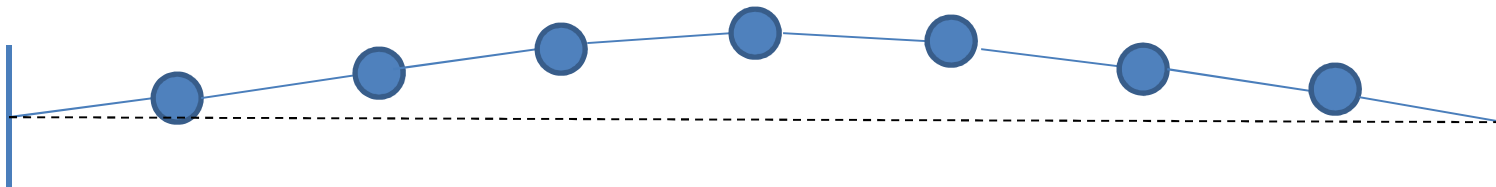
$$x_n(t) = \sum_{k=1}^N a_k \sin \left( \frac{nk\pi}{N+1} \right) \cos(\omega_k t - \theta_k)$$

# Another Example

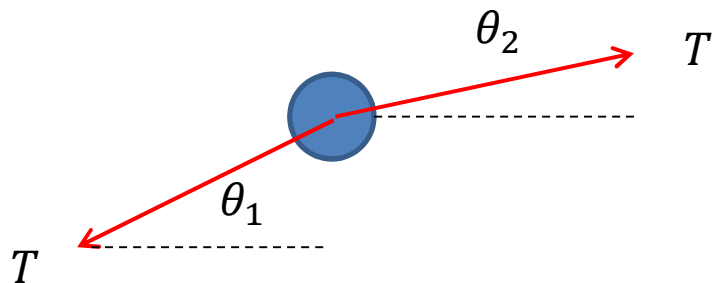
- Discrete masses on an elastic string with tension  $T$ :



- Consider transverse displacements:

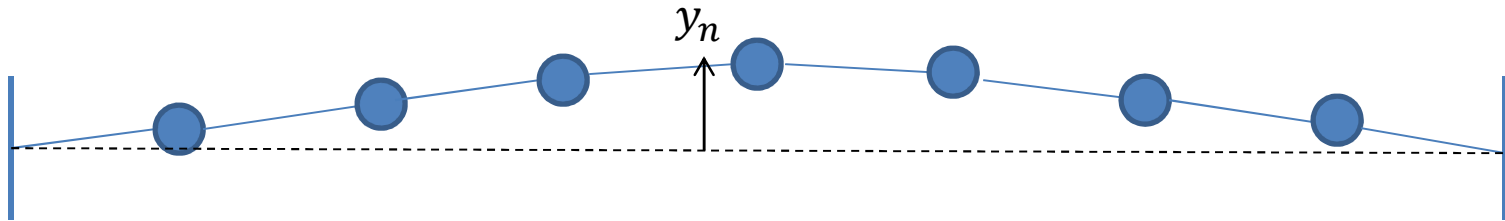


- Vertical force on one mass:



$$\begin{aligned}
 F_n &= T \sin \theta_2 - T \sin \theta_1 \\
 &= T(\theta_2 - \theta_1) \\
 &= \frac{T}{\ell} [(y_{n+1} - y_n) - (y_n - y_{n-1})]
 \end{aligned}$$

# Another Example



- Equation of motion for mass  $n$ :

$$m \ddot{y}_n = F_n = \frac{T}{\ell} [(y_{n+1} - y_n) - (y_n - y_{n-1})]$$

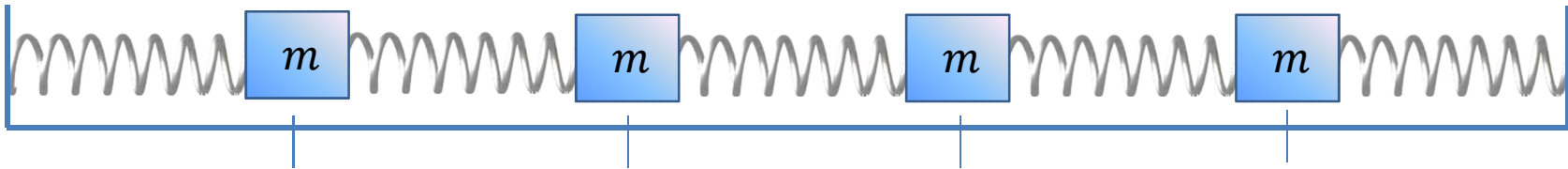
$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = \frac{T}{m\ell}$$

- Normal modes:

$$y_{n,k}(t) = A_{n,k} \cos(\omega_k t - \theta_k)$$

# Example

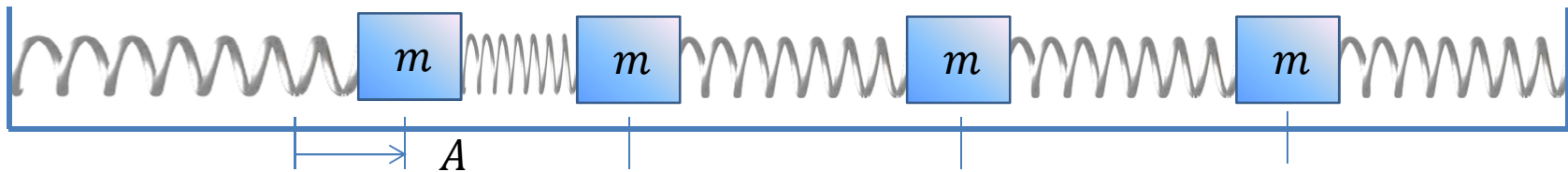


- Solutions are of the form

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \theta_k)$$

- The constants  $a_k$  and  $\theta_k$  must be chosen to satisfy the initial conditions.
- Consider, for example, an initial state where all masses are in their equilibrium position except for the mass at  $x_1$  which is initially displaced by a distance  $A$  ...

# Example



- Consider, for example, an initial state where all masses are in their equilibrium position except for the mass at  $x_1$  which is initially displaced by a distance  $A$  ...

$$\begin{aligned}x_1(0) &= \sum_{k=1}^N a_k \sin\left(\frac{k\pi}{N+1}\right) \cos \theta_k = A \\x_2(0) &= \sum_{k=1}^N a_k \sin\left(\frac{2k\pi}{N+1}\right) \cos \theta_k = 0 \\&\vdots \\x_N(0) &= \sum_{k=1}^N a_k \sin\left(\frac{Nk\pi}{N+1}\right) \cos \theta_k = 0 \\&\dot{x}_n(0) = 0\end{aligned}$$

# Example

- We have  $2N$  equations
  - initial positions of  $N$  masses
  - initial velocities of  $N$  masses
- We have  $2N$  unknowns:  $a_k$  and  $\theta_k$
- How do we solve this linear system of equations?
- Properties of the normal modes:
  - Eigenvalues:  $\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$
  - Eigenvectors:  $A_{n,k} = \sin\left(\frac{nk\pi}{N+1}\right)$
- Eigenvectors are orthogonal:

$$\sum_{k=1}^N A_{n,k} A_{m,k} = 0 \text{ when } n \neq m$$



# Discrete Sine Transform

- The eigenvectors are orthogonal so it must be true that

$$\sum_{k=1}^N \sin\left(\frac{nk\pi}{N+1}\right) \sin\left(\frac{mk\pi}{N+1}\right) = 0$$

when  $n \neq m$ .

This term sums  
to zero...

- When  $n = m$  we just have

$$\sum_{k=1}^N \sin^2\left(\frac{nk\pi}{N+1}\right) = \sum_{k=1}^N \frac{1}{2} \left( 1 + \cos\left(\frac{2nk\pi}{N+1}\right) \right) = \frac{N}{2}$$

# Discrete Sine Transform

- We can summarize this in a useful form:

$$\sum_{k=1}^N \sin\left(\frac{nk\pi}{N+1}\right) \sin\left(\frac{mk\pi}{N+1}\right) = \frac{N}{2} \delta_{nm}$$

- The symbol  $\delta_{nm}$  is called the Kronecker Delta:

$$\delta_{nm} = \begin{cases} 0 & \text{when } n \neq m \\ 1 & \text{when } n = m \end{cases}$$

- How will this help us solve for the constants of integration, given the initial conditions?

# Example

- General solution:

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \theta_k)$$

- At time  $t = 0$ ,

$$x_n(0) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\theta_k)$$

- Consider the expression:

$$\begin{aligned} \sum_{n=1}^N x_n(0) \sin\left(\frac{nk'\pi}{N+1}\right) &= \sum_{n,k=1}^N a_k \cos \theta_k \sin\left(\frac{nk'\pi}{N+1}\right) \sin\left(\frac{nk\pi}{N+1}\right) \\ &= \frac{N}{2} \sum_{k=1}^N a_k \cos \theta_k \delta_{k'k} = \frac{N}{2} a_{k'} \cos \theta_{k'} \end{aligned}$$

# Example

- Likewise, consider the time derivatives:

$$\dot{x}_n(t) = - \sum_{k=1}^N a_k \omega_k \sin\left(\frac{nk\pi}{N+1}\right) \sin(\omega_k t - \theta_k)$$

$$\dot{x}_n(0) = \sum_{k=1}^N \underbrace{a_k \omega_k \sin \theta_k}_{\text{constants}} \sin\left(\frac{nk\pi}{N+1}\right)$$

$$\sum_{n=1}^N \dot{x}_n(0) \sin\left(\frac{nk'\pi}{N+1}\right) = \frac{N}{2} a_{k'} \omega_{k'} \sin \theta_{k'}$$

- If the initial velocities were all zero, then  
 $\theta_k = 0$  for  $k = 1, \dots, N$

# Example

- Now we know that  $\theta_k$  are all zero...

$$\sum_{n=1}^N x_n(0) \sin\left(\frac{nk'\pi}{N+1}\right) = \frac{N}{2} a_{k'}$$

$$a_k = \frac{2}{N} \sum_{n=1}^N x_n(0) \sin\left(\frac{nk\pi}{N+1}\right)$$

- In this example,  $x_1(0) = A$ ,  $x_{n \neq 1}(0) = 0$
- Therefore,

$$a_k = \frac{2A}{N} \sin\left(\frac{k\pi}{N+1}\right)$$

And we're done!

## A slightly different example...

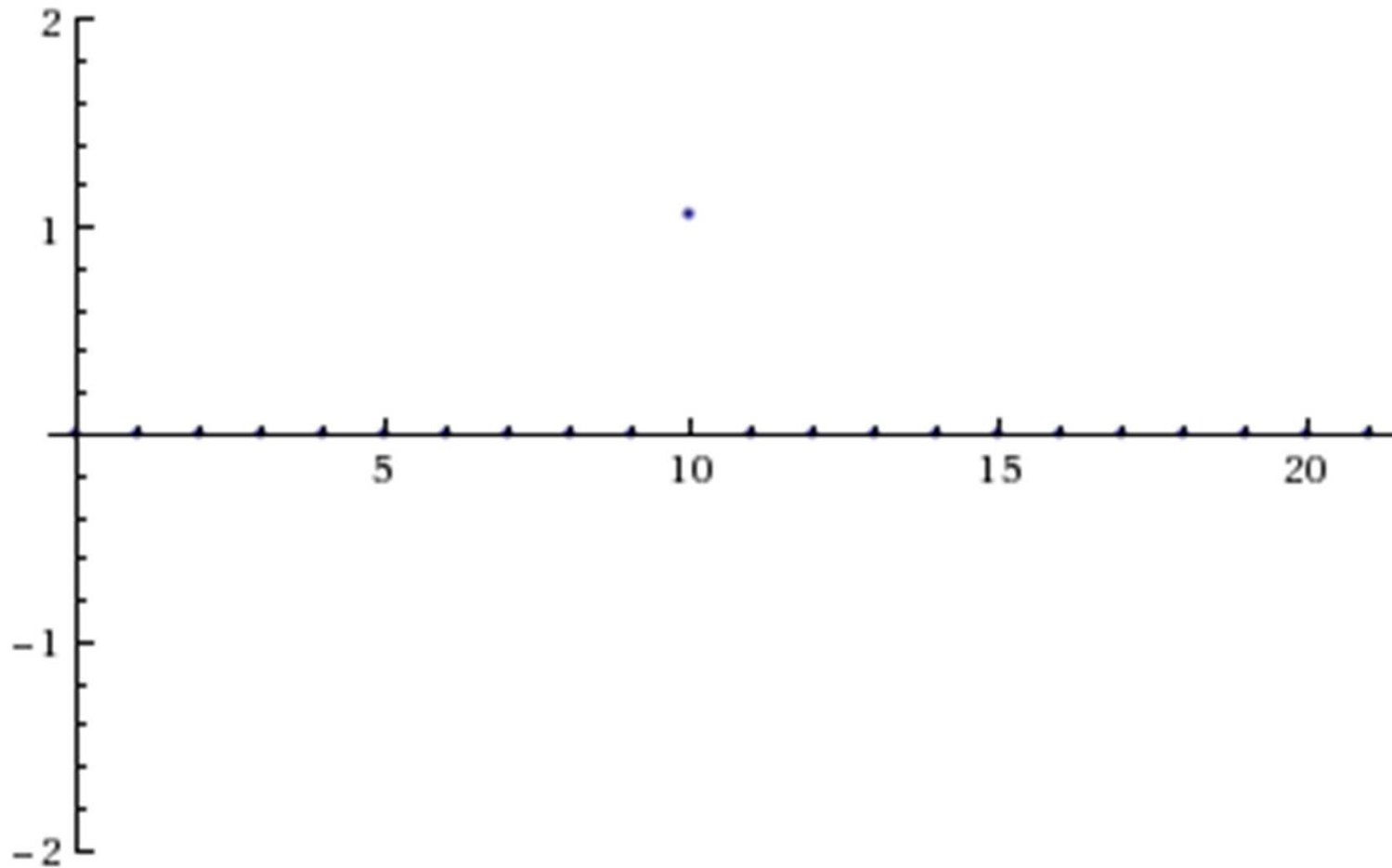
- Instead of the mass at one end being initially displaced, suppose it was the mass in the middle. In this case,

$$a_k = \frac{2A}{N} \sin \left( \frac{(N/2)k\pi}{N+1} \right)$$

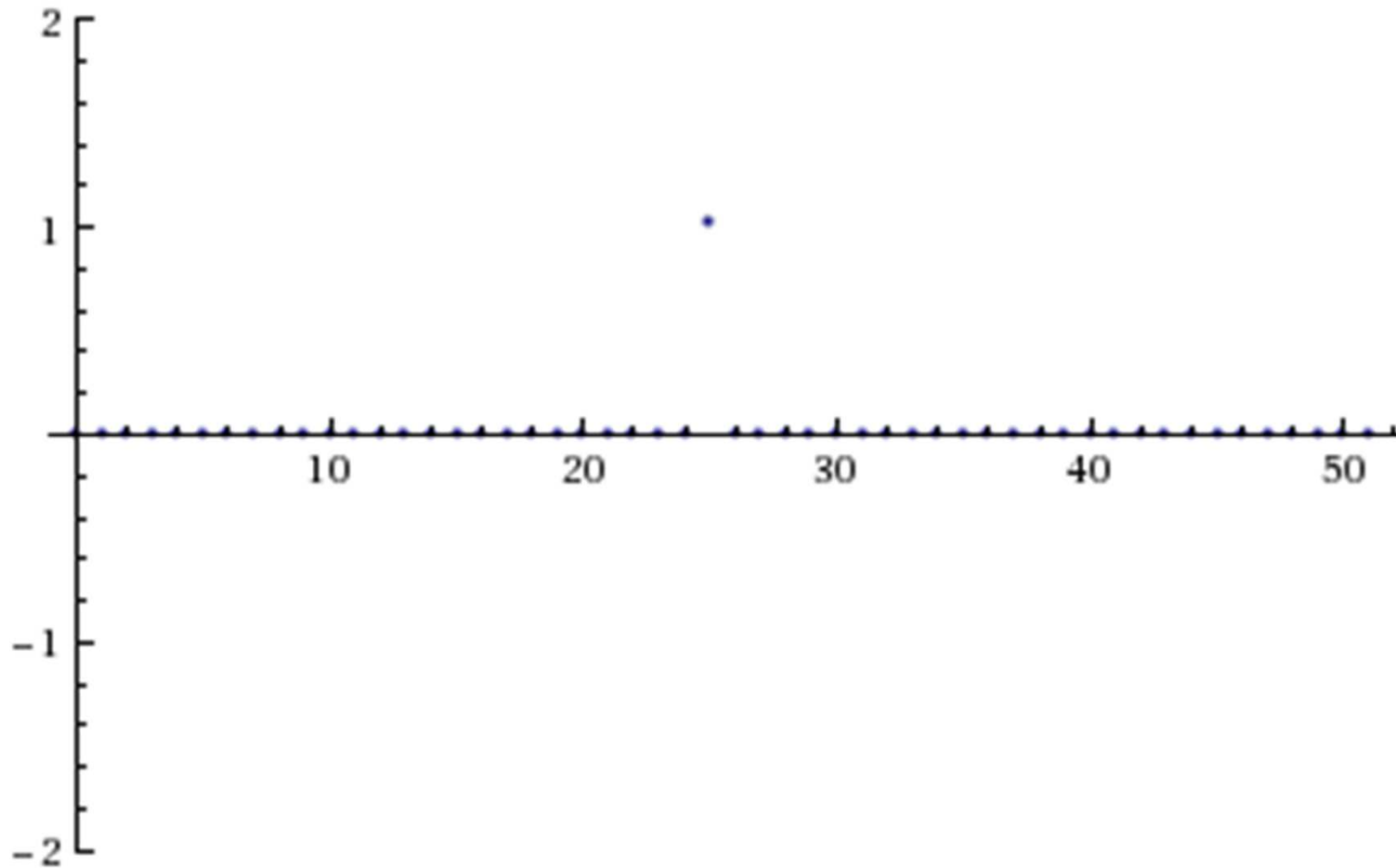
$$\omega_k = 2\omega_0 \sin \left( \frac{k\pi}{2(N+1)} \right)$$

$$x_n(t) = \sum_{k=1}^N a_k \sin \left( \frac{nk\pi}{N+1} \right) \cos(\omega_k t)$$

# Example with $N=20$



# Example with $N=50$





# Review

- We calculated the eigenvalues for a system with  $N$  identical masses

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- We found the normal modes of vibration (eigenvectors):

$$A_{n,k} = \sin\left(\frac{nk\pi}{N+1}\right)$$

- The general form of the solution is

$$x_n(t) = \sum_{k=1}^N a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \theta_k)$$

# Review

- We determined the constants of integration from the initial conditions:

$$a_k \cos \theta_k = \frac{2}{N} \sum_{n=1}^N x_n(0) \sin \left( \frac{nk\pi}{N+1} \right)$$

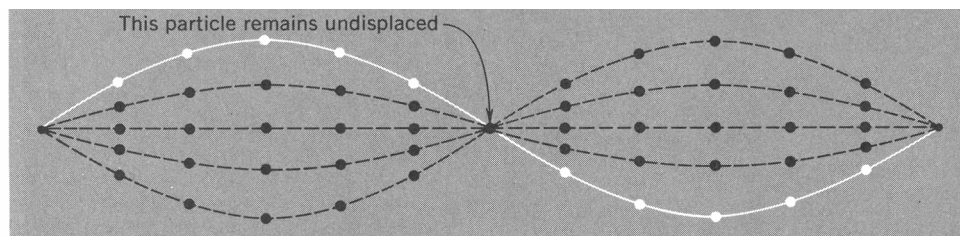
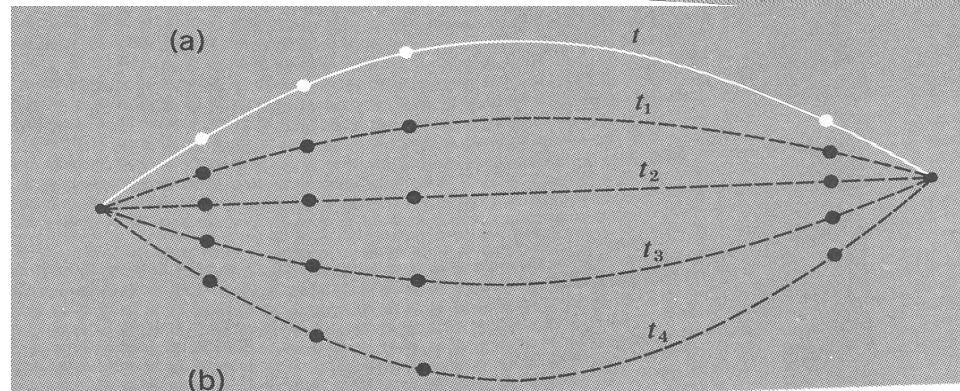
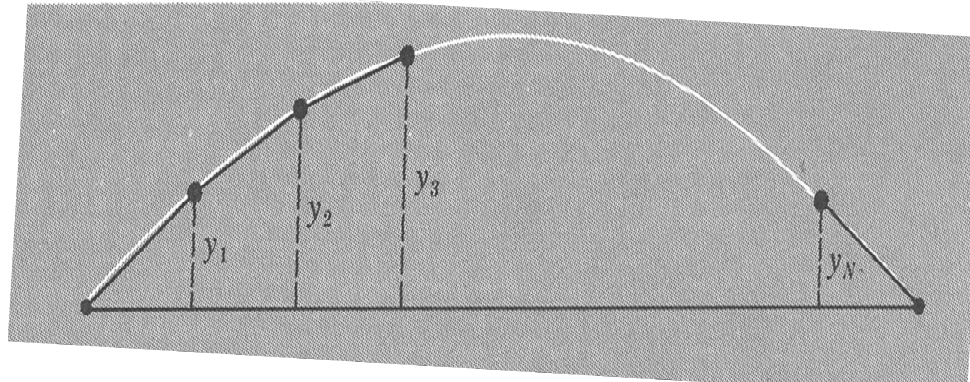
$$a_k \sin \theta_k = \frac{2}{N\omega_k} \sum_{n=1}^N \dot{x}_n(0) \sin \left( \frac{nk\pi}{N+1} \right)$$

- Put these back into the general form of the solution:

$$x_n(t) = \sum_{k=1}^N a_k \sin \left( \frac{nk\pi}{N+1} \right) \cos(\omega_k t - \theta_k)$$

And we're done...

# Masses on a String



First normal mode

Second normal mode

# Continuous Systems

- What happens when the number of masses goes to infinity, while the linear mass density remains constant?

$$m \ddot{y}_n = \frac{T}{\ell} [(y_{n+1} - y_n) - (y_n - y_{n-1})]$$

$$\frac{m}{\ell} \rightarrow \mu$$

$$\frac{y_{n+1} - y_n}{\ell} \rightarrow \left( \frac{\partial y}{\partial x} \right)_{x+\Delta x} \quad \frac{(y_n - y_{n-1})}{\ell} \rightarrow \left( \frac{\partial y}{\partial x} \right)_x$$

$$\mu \ell \frac{\partial^2 y}{\partial t^2} = T \left[ \left( \frac{\partial y}{\partial x} \right)_{x+\Delta x} - \left( \frac{\partial y}{\partial x} \right)_x \right]$$

# Continuous Systems

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\ell}$$

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}$$

The Wave Equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \qquad v = \sqrt{T/\mu}$$

# Solutions

- When we had  $N$  masses, the solutions were

$$y_{n,k}(t) = A_{n,k} \cos(\omega_k t - \delta_k)$$

- $n$  labels the mass along the string
- With a continuous system,  $n$  is replaced by  $x$ .

- Proposed solution to the wave equation for the continuous string:

$$y(x, t) = f(x) \cos \omega t$$

- Derivatives:

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x) \cos \omega t$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} \cos \omega t$$

# Solutions

- Substitute into the wave equation:

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \\ \frac{\partial^2 f}{\partial x^2} &= -\frac{\omega^2}{v^2} f(x) \\ \frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{v^2} f(x) &= 0\end{aligned}$$

- This is the same differential equation as for the harmonic oscillator.
- Solutions are  $f(x) = A \sin(\omega x/v) + B \cos(\omega x/v)$

# Solutions

$$f(x) = A \sin(\omega x/v) + B \cos(\omega x/v)$$

- Boundary conditions at the ends of the string:

$$f(0) = f(L) = 0$$

$$f(x) = A \sin(\omega x/v) \text{ where } \omega L/v = n\pi$$

- Solutions can be written:

$$f_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right)$$


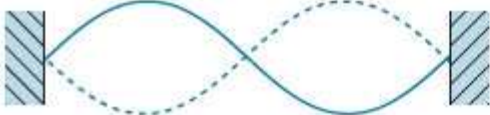


- Complete solution describing the motion of the whole string:

$$y_n(x, t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$



# Properties of the Solutions

$$y_n(x, t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

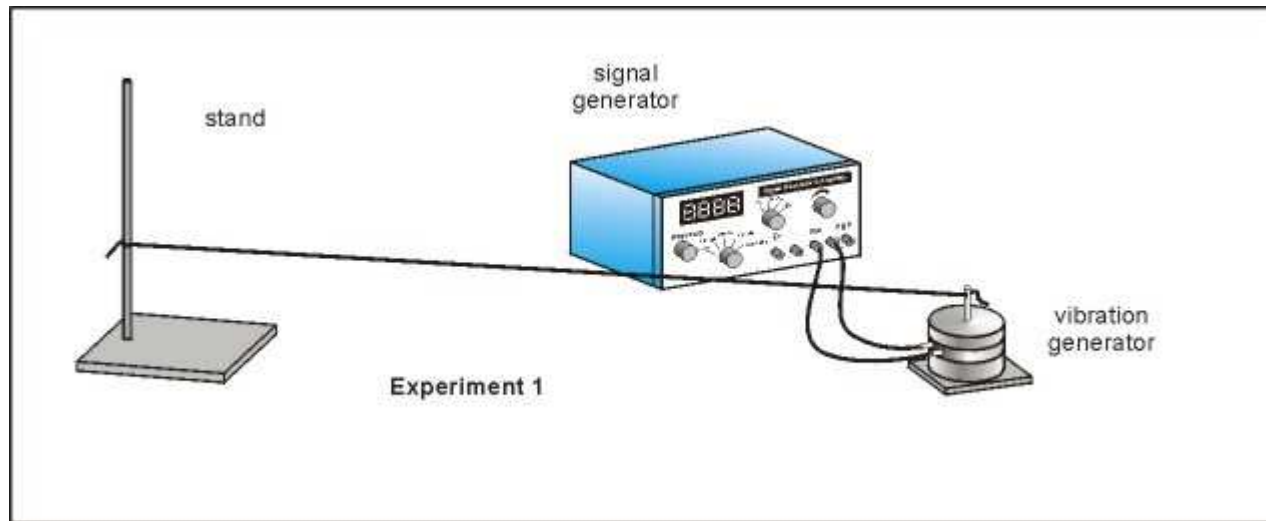
	mode	wavelength	frequency
	first	$2L$	$\frac{v}{2L}$
	second	$L$	$\frac{v}{L}$
	third	$\frac{2L}{3}$	$\frac{3v}{2L}$
	fourth	$\frac{L}{2}$	$\frac{2v}{L}$

$$\lambda_n = \frac{2L}{n}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

# Forced Oscillations



- One end of the string is fixed, the other end is forced with the function  $Y(t) = B \cos \omega t$ .

$$y(0, t) = B \cos \omega t$$

$$y(L, t) = 0$$

- The wave equation still holds so we expect solutions to be of the form

$$y(x, t) = f(x) \cos \omega t$$

# Forced Oscillations

- This time we can't constrain  $f(x)$  to be zero at both ends.
- Now, let  $f(x) = A \sin(kx + \alpha)$ 
  - The constant  $k$  is just  $\omega/v$ .
  - We need to solve for  $A$  and  $\alpha$
- Boundary condition at  $x = L$ :

$$\sin\left(\frac{\omega L}{v} + \alpha\right) = 0 \Rightarrow \frac{\omega L}{v} + \alpha = p\pi$$
$$\alpha_p = p\pi - \frac{\omega L}{v}$$

- Condition at  $x = 0$ :

$$B = A_p \sin \alpha_p$$

# Forced Oscillations

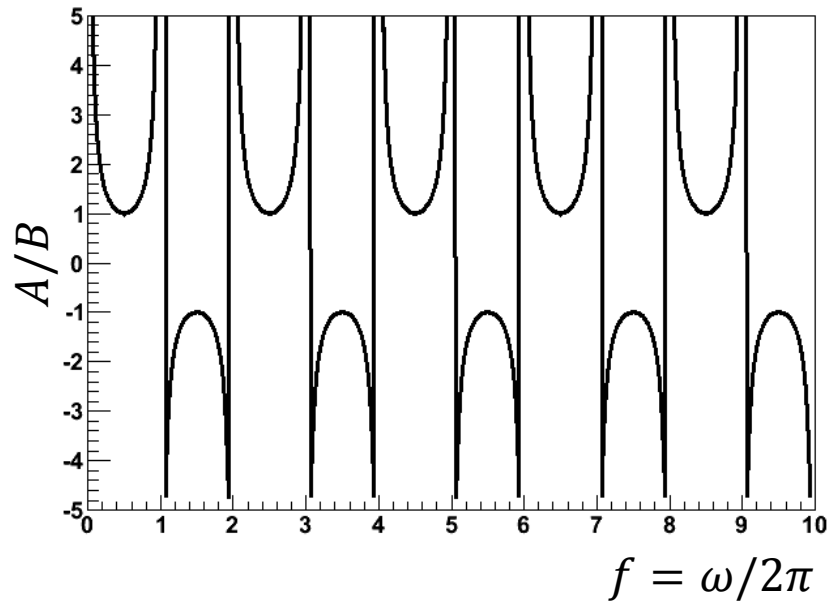
- Amplitude of oscillations:

$$A_p = \frac{B}{\sin(p\pi - \omega L/v)}$$

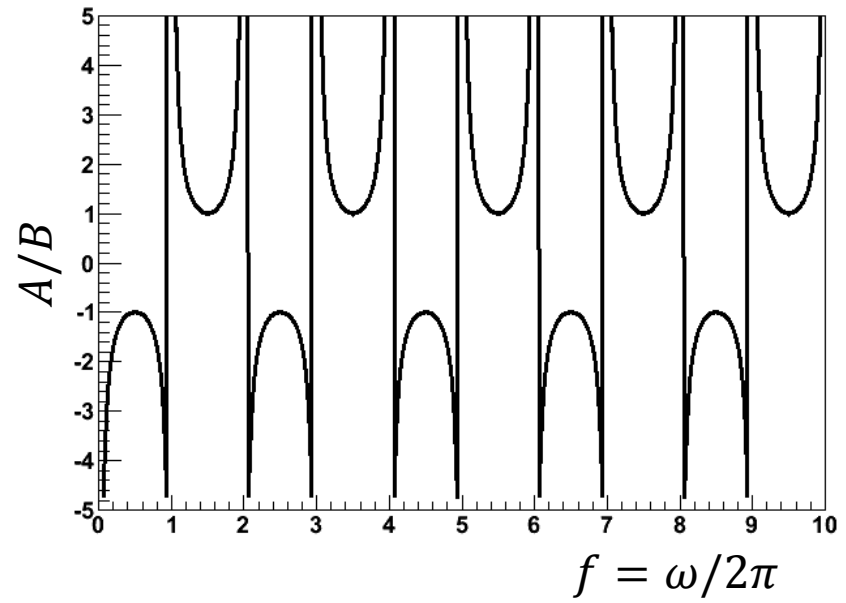
- What does this mean?
  - The driving force can excite many normal modes of oscillation
  - When  $\omega = p\pi v/L$ , the amplitude gets very large

# Forced Oscillations

$$p = 1$$



$$p = 2$$



$$L = 5 \text{ m}$$
$$v = 10 \text{ m/s}$$