1. Show that \( z(t) = ae^{i\omega t} + be^{i\beta e^{i\omega t}} \) can be written as
\( z(t) = re^{i(\omega t + \varphi)} \)
where \( a, b, \alpha, \beta, r, \varphi \) and \( \omega \) are real numbers.

This is easier to think about using phasors or by drawing vectors in the complex plane:

\[
|z(t)| = \sqrt{z^* z} = \sqrt{(ae^{i(\omega t + \alpha)} + be^{i(\omega t + \beta)})^*(ae^{i(\omega t + \alpha)} + be^{i(\omega t + \beta)})}
= \sqrt{(ae^{-i(\omega t + \alpha)} + be^{-i(\omega t + \beta)})^*(ae^{i(\omega t + \alpha)} + be^{i(\omega t + \beta)})}
= \sqrt{(a^2 + ab(e^{i(\alpha - \beta)} + e^{-i(\alpha - \beta)}) + b^2)^{1/2}}
= \sqrt{(a^2 + 2ab \cos(\alpha - \beta) + b^2)^{1/2}}
\]

Recall that
\[
\cos \Theta = \frac{e^{i\Theta} + e^{-i\Theta}}{2}
\]
The phase, $\varphi$, can be calculated by considering the case when $t=0$.

In this case,

$$\tan \varphi = \frac{\text{Im} \, z}{\text{Re} \, z}$$

when $t=0$, $z = ae^{i\alpha} + be^{i\beta} = a(cos \alpha + i sin \alpha) + b(cos \beta + i sin \beta) = acos \alpha + bcos \beta + i(asin \alpha + bsin \beta)$

So, $\text{Im}(z) = a \sin \alpha + b \sin \beta$

and $\text{Re}(z) = a \cos \alpha + b \cos \beta$.

Thus, $\tan \varphi = \frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta}$

If you want to write a formal expression for $\varphi$ itself, then

$$\varphi = \tan^{-1} \left( \frac{a \sin \alpha + b \sin \beta}{a \cos \alpha + b \cos \beta} \right)$$
2. Although there is no unique way to define the position, $x(t)$, some choices are more convenient than others. In this case, it is convenient to measure $x(t)$ with respect to the equilibrium position of the spring so that the force exerted by the spring vanishes when $x = 0$.

(a)

\[ \begin{array}{c}
\text{x} \\
\downarrow
\end{array} \]

\[ \begin{array}{c}
L \\
\downarrow
\end{array} \]

$x$ is positive when the spring is compressed and negative when the spring is stretched.

(b) The force exerted by the spring is

\[ F(x) = -kx \]

With the mass stuck to the spring,

\[ F(x) = m\ddot{x} = -kx \]

So \[ m\ddot{x} + kx = 0 \]

or \[ \ddot{x} + \omega^2x = 0 \] where \[ \omega = \sqrt{\frac{k}{m}} \] .

Solutions are of the form

\[ x(t) = A \sin \omega t + B \cos \omega t \]

\[ \text{3} \]
At time $t=0$, $x=0$ so $B=0$.

The velocity is $\dot{x}(t) = A \omega \cos \omega t$ and at $t=0$, $\dot{x}(t) = v_0$.

Thus, $A \omega = v_0$ so $A = \frac{v_0}{\omega}$.

The solution is $x(t) = \frac{v_0}{\omega} \sin \omega t$ for $t>0$. 
3. When in equilibrium, the net force is zero. Thus
\[ F_s - mg = 0 \]
where the sign convention is such that positive forces point up.
The amount the spring has been compressed is \( L-z \) so \( F_s = k(L-z) \), which points up when \( z < L \).
Hence,
\[ k(L-z_0) - mg = 0 \]
\[ kL - mg = kz_0 \]
So the equilibrium position is
\[ z_0 = L - \frac{mg}{k} \]
(b) The free-body diagram looks like this:
\[ \begin{array}{c}
F_s = k(L-z) \\
\uparrow \\
\downarrow mg
\end{array} \]
(c) The net force on the mass is
\[ F = -mg + k(L-z) = -kz + kL - mg \]
(d) From Newton's second law, \( F = ma \).
But then \( \ddot{z} + \omega^2 z = \frac{kL}{m} - g \neq 0 \), so it is not of the form \( \ddot{z} + \omega^2 z = 0 \).
(e) Suppose that we measure distances with respect to the equilibrium position, \( z_0 \).

Then \( z = z_0 + z' \).

The acceleration is \( \ddot{z} = \ddot{z}' \).

But \( \omega^2 z = \frac{k}{m} z = \frac{k}{m} (z_0 + z') \)

\[
= \frac{k}{m} \left( L - \frac{mg}{k} \right) + \frac{k}{m} z'
\]

So \( \ddot{z} + \omega^2 z = \frac{kL}{m} - g \)

\[
\ddot{z}' + \frac{k}{m} z' + \left( \frac{kL}{m} - g \right) = \frac{kL}{m} - g
\]

Therefore, \( \ddot{z}' + \omega^2 z' = 0 \).

(f) When the spring is uncompressed, \( z = L = z_0 + z' \)

So \( z' = L - z_0 = L - \left( L - \frac{mg}{k} \right) = \frac{mg}{k} \).

Solutions to \( \ddot{z}' + \omega^2 z' = 0 \) are of the form

\[ z'(t) = A \sin \omega t + B \cos \omega t \]

When \( t = 0 \), \( \dot{z}' = \frac{mg}{k} \) so \( B = \frac{mg}{\omega k} \).
When \( t = 0 \), \( \ddot{z} = Aw \) but the mass was released from rest, so \( \dot{z}'(t) = 0 \) at \( t = 0 \).

Thus, \( A = 0 \) so the solution is

\[
\dot{z}'(t) = \frac{mg \cos \omega t}{k},
\]

where \( \omega = \frac{k}{m} \).

Expressed in terms of \( z \),

\[
z(t) = L - \frac{mg}{k} + \dot{z}'(t)
\]

\[
= L - \frac{mg}{k} \left( 1 - \cos \omega t \right)
\]