1. Consider the following reference frame in which the log is displaced by a distance $z$ beyond its equilibrium position, measured from the surface of the water.

The buoyant force acting on the log is

$$F_b = -\rho g z \tilde{a}$$

but this is equal to $M$ times the acceleration in an inertial reference frame.

If $x$ is the position of the log in an inertial reference frame, then

$$F = M \ddot{x} = M(\dddot{h} + \ddot{z})$$

Hence,

$$M \ddot{z} + \rho g z \tilde{a} = -M \dddot{h}.$$  

If the waves have amplitude $A$ and angular frequency $\omega$, then we can write

$$h(t) = A \cos \omega t$$

and hence,

$$\dddot{h} = -A\omega^2 \cos \omega t.$$  

The equation of motion is then

$$M \ddot{z} + \rho g z \tilde{a} = M A\omega^2 \cos \omega t.$$
(a) The time dependent driving force is then
\[ F(t) = MAw^2 \cos \omega t \]

(b) If we also consider the effects of damping, the equation of motion can be written
\[ \ddot{z} + \gamma \dot{z} + \omega_0^2 z = A \omega^2 \cos \omega t \]

Where \( \omega_0^2 = \frac{pga}{M} \)

To determine whether there would be large amplitude oscillations, we can evaluate the expression \( \frac{\omega}{\omega_0} \), which would approach 1 at resonance.

\[ \frac{\omega}{\omega_0} = \frac{2\pi}{T} = \frac{2\pi}{(6s)\sqrt{\frac{10^3 \text{kg} \cdot \text{m}^3}{\frac{10^3 \text{kg} \cdot \text{m}^3}{\text{m}^2}} (9.81 \text{m/s}^2)(0.05 \text{m}^2)}} \]

\[ = 2.3 \]

The Q-value for this oscillator is
\[ Q = \frac{\omega_0}{\gamma} = \left( 10^2 \text{s} \right) \sqrt{\frac{(100 \text{ kg})}{\sqrt{(10^3 \text{kg} \cdot \text{m}^3)(9.81 \text{m/s}^2)(0.05 \text{m}^2)}}} \]

\[ = 221 \]

Recall that the FWHM of the power as a function of \( \frac{\omega}{\omega_0} \) is just \( \gamma \). Therefore, since \( \frac{\omega}{\omega_0} = 2.3 \gg 1 \), the driving frequency is far from resonance so there will be no large amplitude oscillations.
2. The position of the pin on the shaft is

\[ z(t) = r \cos \omega t \]

and if the length of the rubber band (minus the equilibrium length) is \[ x(t) \] then the position of the mass in an inertial reference frame is \[ z(t) + x(t) \]. The equation of motion is then

\[ M(\ddot{x} + \dot{z}) + b\dot{x} + kx = 0 \]

or \[ M\ddot{x} + b\dot{x} + kx = -M\ddot{z} = Mr\omega^2 \cos \omega t \]

which can be written in the standard way:

\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = r \omega^2 \cos \omega t \]

(a) If the motor stopped, the mass would oscillate with \[ x(t) \] given by

\[ x(t) = A e^{-\frac{\gamma}{2}t} \cos (\omega_{\text{free}} t) \]

where \[ \omega_{\text{free}} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \]

If the amplitude of oscillations decreased by \( \frac{1}{e} \) in time \( T \) then

\[ e^{-\frac{\gamma}{2}T} = e^{-1} \quad \Rightarrow \quad \gamma = 2/T \]

The \( Q \) value of the system can be expressed as

\[ Q = \frac{\omega_0}{\gamma} = \frac{2}{T} \sqrt{\frac{k}{M}} \]
(b) We already worked out the vertical component of the position of the pin. It was
\[ z(t) = r \cos \omega t \]

(c) We already worked out the differential equation. It was
\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = r \omega^2 \cos \omega t \]

(d) The maximal amplitude occurs when \( \omega \approx \omega_0 \). When \( \gamma \ll \omega_0 \) this will occur when \( \omega \approx \sqrt{k/m} \).

To calculate the exact frequency for which the amplitude is maximal, we consider the expression for the amplitude of an oscillator subject to a force \( F(t) = F_0 \cos \omega t \). This was
\[ A(\omega) = \frac{F_0 / m}{(\omega_0^2 - \omega^2)^2 + (\omega \omega_0 / Q)^2)^{1/2}} \]

In the current problem, the force term is of the form \( F(t) = m r \omega^2 \cos \omega t \) so the amplitude is
\[ A(\omega) = \frac{r \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega \omega_0 / Q)^2)}^{1/2} \]
\[ = \frac{r \omega^2 / \omega_0^2}{((1 - \omega^2 / \omega_0^2)^2 + (\omega / \omega_0 Q)^2)^{1/2}} \]

Let \( u = \omega^2 / \omega_0^2 \), \( A(u) = \frac{ru}{((1-u)^2 + u/Q^2)^{1/2}} \)

\( A(u) \) is maximal when \( \frac{dA}{du} = 0 \).
\[
\frac{dA}{du} \frac{r}{((1-u)^2 + u/Q^2)^{1/2}} + \frac{ru(1-u) - ru/2Q^2}{((1-u)^2 + u/Q^2)^{3/2}} = 0
\]

The denominator is never zero, so

\[
(1-u)^2 + u/Q^2 + u(1-u) - u/2Q^2 = 0
\]

\[
1-2u + u/Q^2 + u - u^2 - u/2Q^2 = 0
\]

\[
1 - u + u/2Q^2 = 0
\]

\[
1 - u(1 - 1/2Q^2) = 0
\]

\[u = \frac{1}{1 - 1/2Q^2}\]

Therefore, maximal amplitude occurs when

\[\omega = \omega_0 \sqrt{\frac{1}{1 - 1/2Q^2}}\]

This is a higher frequency than \(\omega_0\).

In the case where the forcing term was

\[F(t) = F_0 \cos \omega t\]

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the maximal amplitude occurs at

\[\omega = \omega_0 \sqrt{1 - 1/2Q^2}\]
In the current problem, when \( \omega = \omega_0 \sqrt{\frac{1}{1 - \frac{1}{2}a^2}} \),
the maximal amplitude is

\[
A_{\text{max}} = \frac{r}{(1 - \frac{1}{2}a^2) \left( ((1 - \frac{1}{2}a^2) + \frac{\sqrt{Q^2}}{1 - \frac{1}{2}a^2}) \right)^{\frac{1}{2}}}
\]

\[
= \frac{r}{\left( ((1 - \frac{1}{2}a^2 - 1)^2 + \frac{1}{Q^2} \left( 1 - \frac{1}{2}a^2 \right) \right)^{\frac{1}{2}}}
\]

\[
= \frac{r}{\left( \frac{1}{4Q^4} + \frac{1}{Q^2} - \frac{1}{2Q^4} \right)^{\frac{1}{2}}}
\]

\[
= \frac{rQ}{\left( 1 - \frac{1}{2}a^2 \right)^{\frac{1}{2}}}
\]

When \( Q \) is large, \( A_{\text{max}} \approx rQ \).
The deviation of each spring from their equilibrium length is \( x_1(t) \) and \( x_2(t) \), where

\[
x_1(t) = z(t) - \frac{1}{2} x(t)
\]

\[
x_2(t) = z(t) + \frac{1}{2} x(t)
\]

The net force acting on the beam is

\[
F = -kx_1 - kx_2 = M \ddot{z}
\]

The net torque on the beam is

\[
N = -\frac{kx_1L}{2} + \frac{kx_2L}{2} = I \ddot{x}
\]

Thus,

(a) \( M \ddot{z} + 2kz = 0 \)

\( I \ddot{x} - \frac{kL^2x}{2} = 0 \)

(b) The frequencies of the normal modes are

\[
\omega_1 = \sqrt{\frac{2k}{M}}
\]

and

\[
\omega_2 = \sqrt{\frac{kL^2}{2I}}
\]
(c) When one spring is compressed by a distance $d$, while the other is in its equilibrium position, we have

$$x_1(0) = d$$
$$x_2(0) = 0$$

Hence, $z(0) = \frac{1}{2} (x_1(0) + x_2(0)) = \frac{1}{2} d$

and $\alpha(0) = \frac{1}{L} (x_2(0) - x_1(0)) = -\frac{d}{L}$

Solutions to the equations of motion are

$$z(t) = A \cos (\omega_z t + \theta)$$
$$\alpha(t) = B \cos (\omega_x t + \phi)$$

At $t=0$, the velocity is

$$\dot{z}(0) = -A \omega_z \sin \theta$$
$$\dot{\alpha}(0) = -B \omega_x \sin \phi$$

If these are both zero at $t=0$, then we must have $\theta = \phi = 0$.

Therefore, $A = \frac{1}{2} d$ and $B = -\frac{d}{L}$.

The solution is

$$z(t) = \frac{d}{2} \cos \omega_z t$$
$$\alpha(t) = -\frac{d}{L} \cos \omega_x t$$

where

$$\omega_z = \sqrt{\frac{2k}{m}}$$
$$\omega_x = \sqrt{\frac{kL^2}{2I}}$$
$\mathbf{x}(t)$ is defined in an inertial reference frame. In this frame we can define the coordinates of the masses:

\begin{align*}
y_1(t) &= \mathbf{x}(t) + x_1(t) \\
y_2(t) &= \mathbf{x}(t) + x_1(t) + x_2(t)
\end{align*}

The tension in each string supports the masses. In the top string, for $x_1 \ll L$, $T_1 = 2mg$
and in the bottom string, for $x_2 \ll L$, $T_2 = mg$

The $x$-component of the force acting on the top mass is

\[ F_1 = -\frac{T_1}{L} x_1 + \frac{T_2}{L} x_2 = -\frac{2mg}{L} x_1 + \frac{mg}{L} x_2 \]

and the force acting on the bottom mass is

\[ F_2 = -\frac{T_2}{L} x_2 = -\frac{mg}{L} x_2 \]
These forces give rise to accelerations in the inertial reference frame, and the equations of motion are

\[ m \ddot{y}_1 = m \ddot{x} + m \dot{x}_1 = -\frac{2mg}{L} x_1 + \frac{mg}{L} x_2 \]

\[ m \ddot{y}_2 = m \ddot{x} + m \dot{x}_1 + m \dot{x}_2 = -\frac{mg}{L} x_2 \]

\[ \Rightarrow \ddot{x}_1 + \frac{2g}{L} x_1 - \frac{g}{L} x_2 = -\ddot{x}_2 \]

\[ \ddot{x}_1 + \ddot{x}_2 + \frac{g}{L} x_2 = -\ddot{x}_2 \]

Resonance will occur when \( \omega^2 \) is an eigenvalue. In this case, suppose

\[ x_1(t) = A \cos \omega t \]
\[ x_2(t) = B \cos \omega t \]

Then \( \ddot{x}_1(t) = -\omega^2 x_1 \)
\( \ddot{x}_2(t) = -\omega^2 x_2 \) and the homogeneous system of equations is

\[ (-\omega^2 + \frac{2g}{L}) x_1 - \frac{g}{L} x_2 = 0 \]
\[ (-\omega^2) x_1 + (-\omega^2 + \frac{g}{L}) x_2 = 0 \]

or

\[ \begin{pmatrix} -\omega^2 + \frac{2g}{L} & -\frac{g}{L} \\ -\omega^2 & -\omega^2 + \frac{g}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \]

Thus,

\[ \begin{vmatrix} -\omega^2 + \frac{2g}{L} & -\frac{g}{L} \\ -\omega^2 & -\omega^2 + \frac{g}{L} \end{vmatrix} = 0 \]
The determinant is
\[
\begin{vmatrix}
2g/L - \omega^2 & -g/L \\
-\omega^2 & g/L - \omega^2
\end{vmatrix} = (2g/L - \omega^2)(g/L - \omega^2) - \omega^2 g/L = 0
\]

Let \( \lambda = \omega^2 \). Then,
\[
(2g/L - \lambda)(g/L - \lambda) - \lambda g/L = 0
\]
\[
\lambda^2 - \frac{4g}{L} \lambda + \frac{2g^2}{L^2} = 0
\]
\[
\lambda = \frac{2g}{L} \pm \sqrt{\frac{4g^2}{L^2} - \frac{2g^2}{L^2}}
\]
\[
= \frac{2g}{L} \pm \frac{g\sqrt{2}}{L} = \frac{g}{L} (2 \pm \sqrt{2})
\]

Resonance will occur when
\[
\omega_1 = \sqrt{\frac{g}{L}} \sqrt{2 - \sqrt{2}}
\]
or
\[
\omega_2 = \sqrt{\frac{g}{L}} \sqrt{2 + \sqrt{2}}
\]
The eigenvectors can be determined by substituting the eigenvalues into the equation:

\[
\begin{pmatrix}
2g/L - 9g/L(2-\sqrt{2}) & -9g/L \\
-9g/L(2-\sqrt{2}) & 9g/L - 9g/L(2-\sqrt{2})
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1
\end{pmatrix} = 0
\]

\[
\begin{pmatrix}
9g/L \sqrt{2} & -9g/L \\
-9g/L(2-\sqrt{2}) & -9g/L(1-\sqrt{2})
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1
\end{pmatrix} = 0
\]

\[
\Rightarrow \sqrt{2}A_1 - B_1 = 0
\]

\[
C_0 A_1 = B_1 / \sqrt{2}
\]

Likewise

\[
\begin{pmatrix}
2g/L - 9g/L(2+\sqrt{2}) & -9g/L \\
-9g/L(2+\sqrt{2}) & 9g/L - 9g/L(2+\sqrt{2})
\end{pmatrix}
\begin{pmatrix}
A_2 \\
B_2
\end{pmatrix} = 0
\]

\[
\begin{pmatrix}
-9g/L \sqrt{2} & -9g/L \\
-9g/L(2+\sqrt{2}) & -9g/L(1+\sqrt{2})
\end{pmatrix}
\begin{pmatrix}
A_2 \\
B_2
\end{pmatrix} = 0
\]

\[
\Rightarrow A_2 \sqrt{2} + B_2 = 0
\]

\[
A_2 = -B_2 / \sqrt{2}
\]

In the first normal mode (lower frequency), both masses move in the same direction, and the top mass has a smaller amplitude.

In the second normal mode of oscillation, the masses move in opposite directions, but the top mass still has a smaller amplitude.

1: \( \omega^2 = 9g/L(2-\sqrt{2}) \)

2: \( \omega^2 = 9g/L(2+\sqrt{2}) \)