PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 9 – French, Chapter 4

Spring 2013 Semester

Matthew Jones

Forced Harmonic Motion

$$\boldsymbol{m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t}$$

- Frequency of free oscillations:

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad\qquad \omega_{free} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$$

- Driving frequency: ω
- Steady state solution ($t \gg 1/\gamma$):

$$\boldsymbol{x(t) = A\, cos(\omega t - \delta)}$$

- Amplitude of steady-state oscillations:

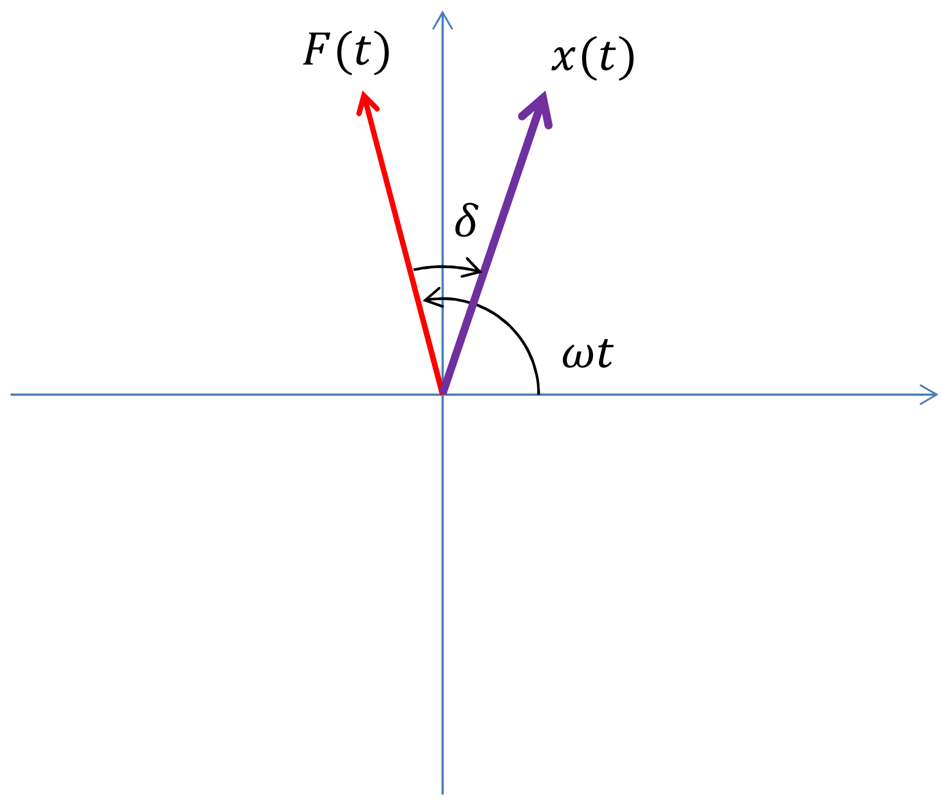
$$A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\gamma)^2}}$$

- Phase difference:

$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{(\omega_0)^2 - \omega^2}\right)$$

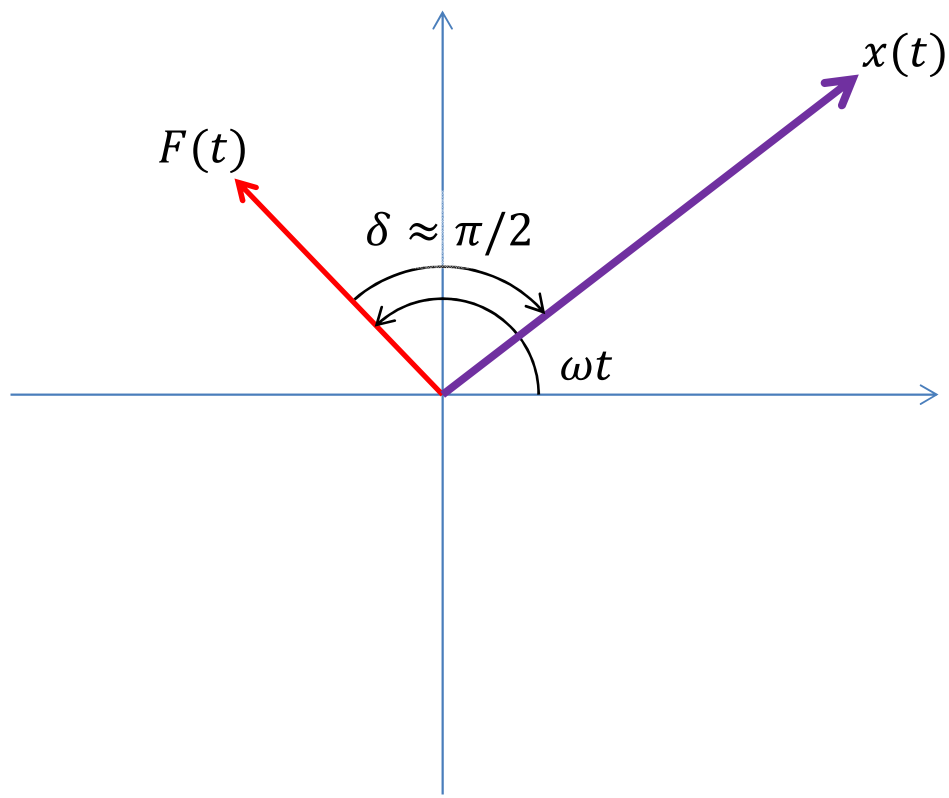
Forced Harmonic Motion

- Phasor diagram: $\omega < \omega_0$



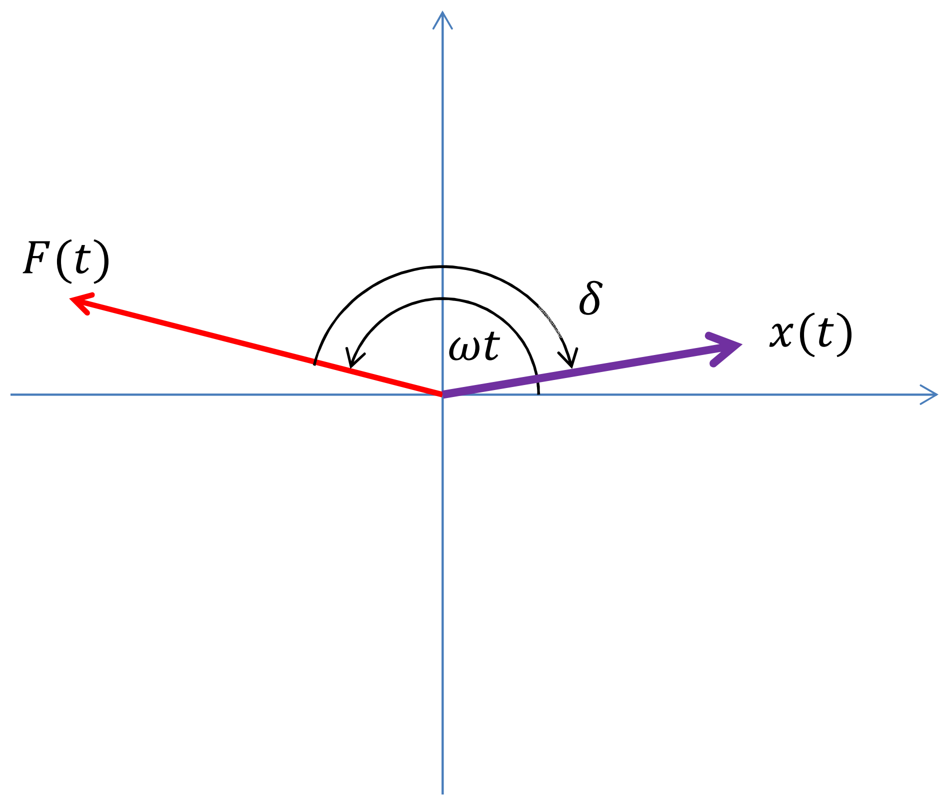
Forced Harmonic Motion

- Phasor diagram: $\omega \approx \omega_0$



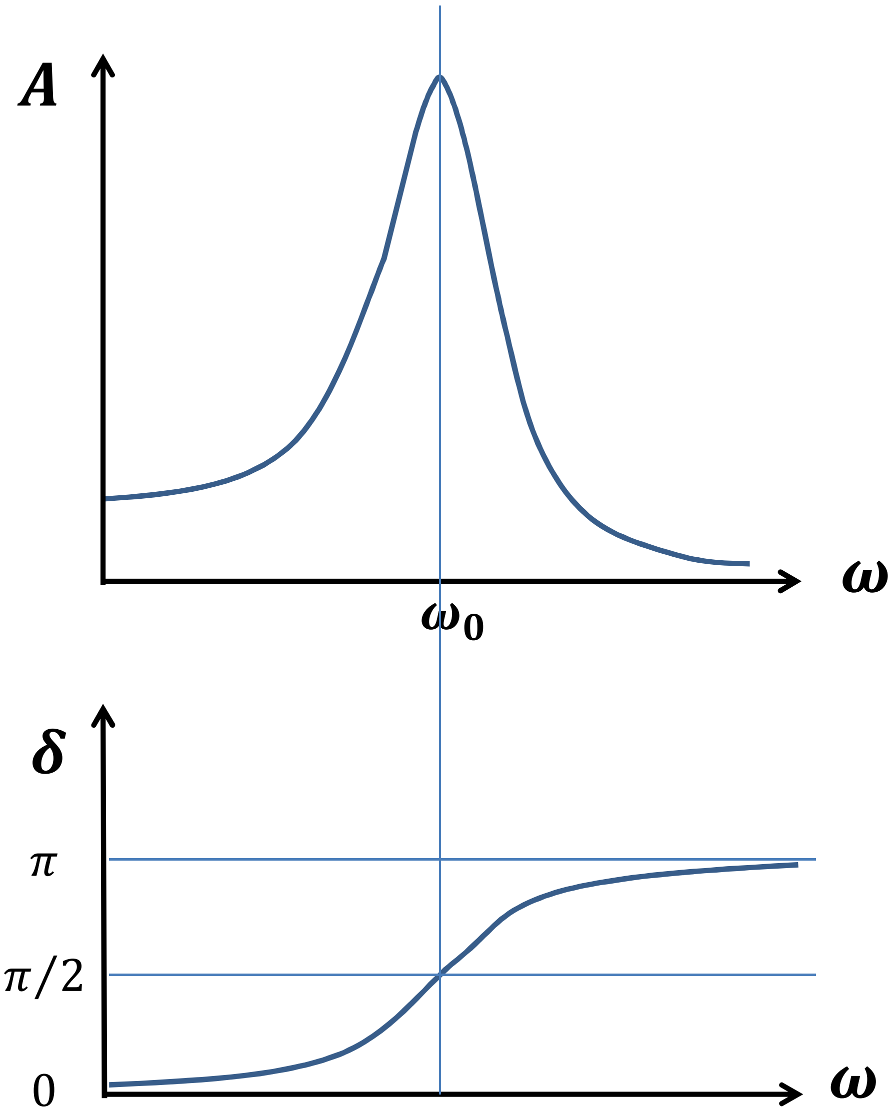
Forced Harmonic Motion

- Phasor diagram: $\omega > \omega_0$



Resonance

- The peak occurs at a frequency that is close to, but not exactly equal to ω_0.
- At resonance, the phase shift is $\delta = \pi/2$.
- The force pushes the mass in the direction it is already moving adding energy to the system.



“Quality Factor”

- Instead of using $\gamma = b/m$ and $\omega_0 = \sqrt{k/m}$, it is convenient to describe the shape of the resonance curve using the variables ω_0 and $Q = \omega_0/\gamma$.
- $Q = \omega_0/\gamma$ is called the “quality factor”.
- Written in terms of ω_0 and Q, the amplitude is

$$A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\omega_0)^2/Q^2}}$$

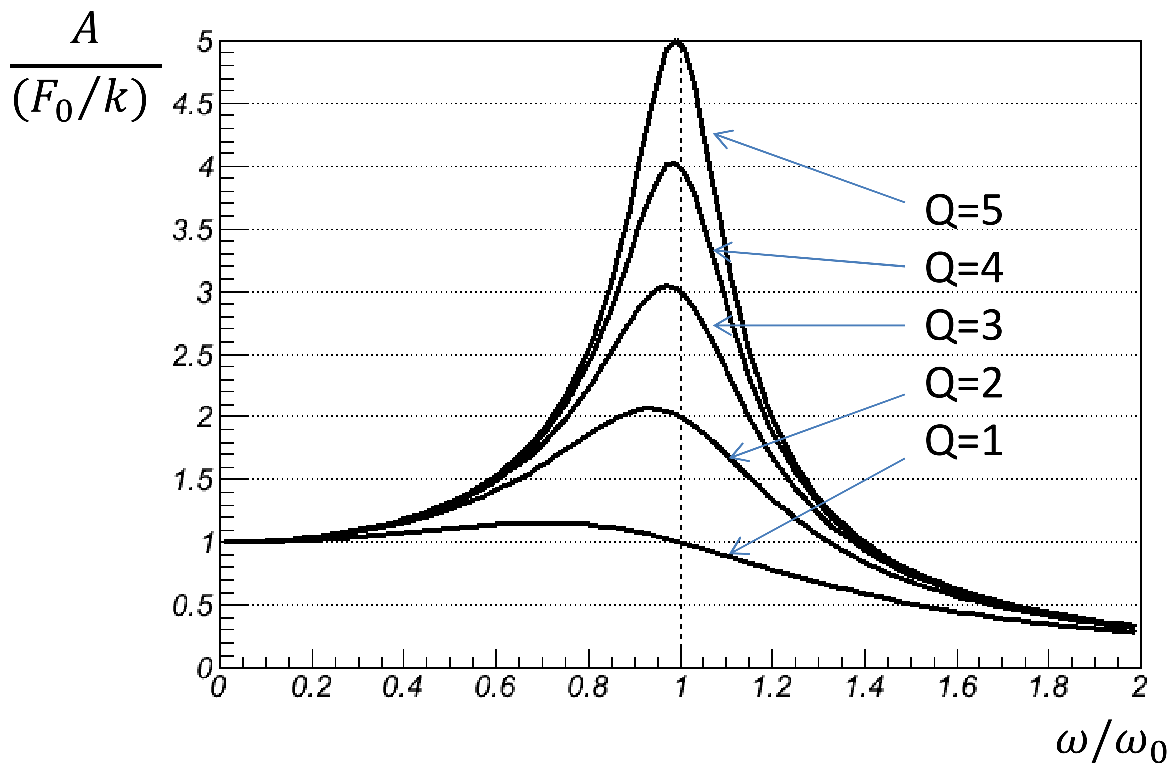
$$= \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

Quality Factor

$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

- Why is this a convenient form?
  - Dimensionless quantities are easier to analyze
  - The scale of the amplitude is determined by F_0/k
  - The shape of the curve is determined by the dimensionless quantities ω/ω_0 and Q

Quality Factor



The normalized height is approximately Q

The maximum occurs when $\omega/\omega_0 \approx 1$

At resonance, the motion is amplified by the factor Q.

Energy

- An oscillator stores energy
- The driving force adds energy to the system
- The damping force dissipates energy
- Instantaneous rate at which energy is added:

$$P = \frac{dW}{dt} = F\frac{dx}{dt} = F\dot{x}$$

$$F(t) = F_0 \cos \omega t$$

$$x(t) = A\, cos(\omega t - \delta)$$

$$\dot{x}(t) = -A\omega \sin(\omega t - \delta)$$

$$P = -F_0 A\omega \cos(\omega t) \sin(\omega t - \delta)$$

- Average rate at which energy is added:

$$\bar{P}(\omega) = \frac{1}{2} F_0 A\omega \sin \delta$$

- Maximal when $\delta = \pi/2$

Energy

$$\bar{P}(\omega) = \frac{1}{2} F_0 A \omega \sin\delta$$

- Some algebra:

$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

$$\sin\delta = \frac{1/Q}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

- Average power:

$$\bar{P}(\omega) = \frac{(F_0)^2 \omega_0}{2kQ} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$

Energy

- When $\omega \approx \omega_0$ we can simplify further:

$$\omega = \omega_0 + \Delta\omega$$

$$\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \approx \frac{2}{\omega_0}\Delta\omega$$

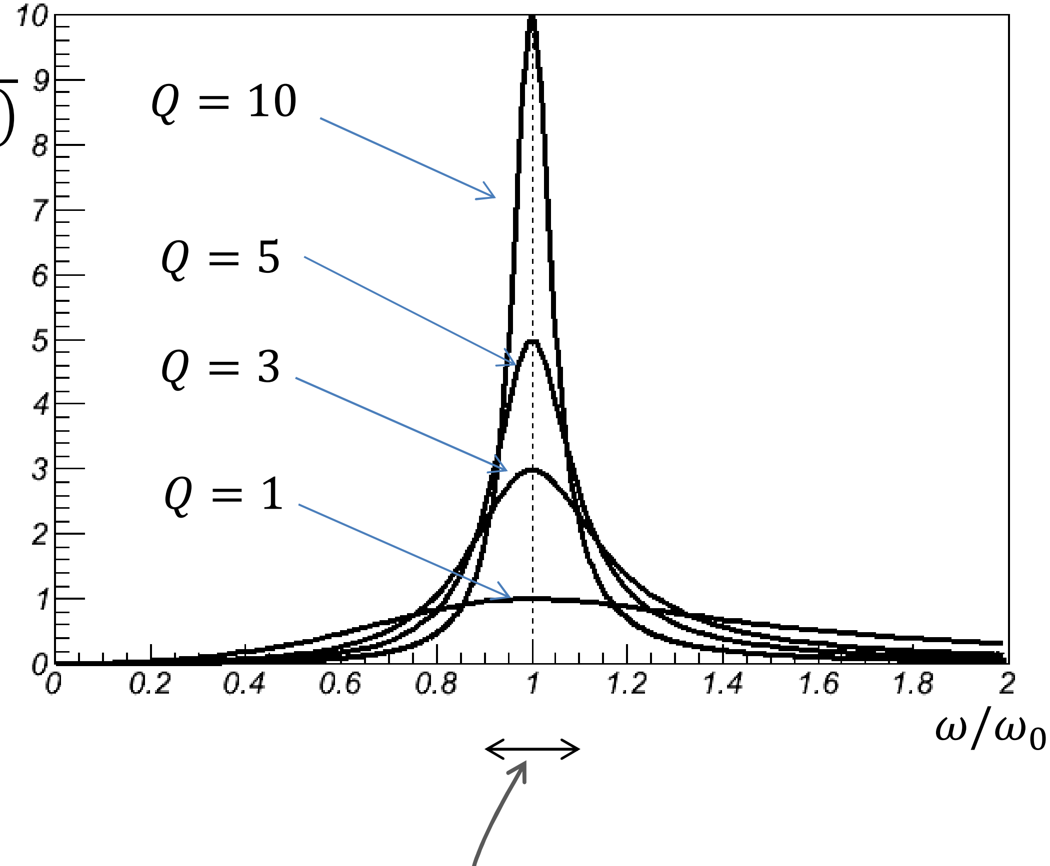
$$\bar{P}(\omega) = \frac{(F_0)^2(\omega_0/Q)}{2k/(\omega_0)^2}\frac{1}{4(\Delta\omega)^2 + (\omega_0/Q)^2}$$

- What value of $\Delta\omega$ will reduce the peak power by a factor of ½?

$$\frac{1}{4(\Delta\omega)^2+(\omega_0/Q)^2} = \frac{1}{2}\frac{1}{(\omega_0/Q)^2} \Longrightarrow 2\Delta\omega = \omega_0/Q$$

Power Resonance Shape

$$\frac{\bar{P}(\omega)}{\left((F_0)^2\omega_0/2k\right)}$$



$$\frac{1}{Q}\frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$

FWHM = $2\Delta\omega = \frac{\omega_0}{Q} = \gamma = \frac{b}{m}$

Resonance Curves

- General properties:
  - Amplitude at resonance: Static displacement x Q
  - FWHM power bandwidth: $\gamma = \omega_0/Q$
  - When Q is large, a small force at the resonant frequency produces large oscillations
  - Large amplitudes persist only when the driving force is near the oscillation frequency

Transient Phenomena

- So far, we only considered the form of solutions when t was very large:

$$x_1(t) = A\, cos(\omega t - \delta)$$

- What is the form of the solution when t is small?
- The solution with no forcing term was

$$x_2(t) = \boldsymbol{B} e^{-\gamma t/2} \cos \omega_0 t + \boldsymbol{C}\, e^{-\gamma t/2} \sin \omega_0 t$$

- Complete solution:

$$x(t) = x_1(t) + x_2(t)$$

- Initial conditions determine $\boldsymbol{B}$ and $\boldsymbol{C}$ in the complete solution.

Transient Phenomena

- Suppose a mass is already in motion at $t = 0$:

$$x(0) = A_0$$
$$\dot{x}(0) = 0$$

- Suppose that γ is small, so that this motion persists for a long time: ignore the $e^{-\gamma t/2}$ terms

$$x_2(t) = B\cos\omega_0 t$$

- Steady state solution:

$$x_1(t) = A\,cos(\omega t - \delta), \qquad A = \frac{F_0/m}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + (\omega\omega_0)^2/Q^2}}$$

- Complete solution:

$$x(t) = B\cos\omega_0 t + A\,cos(\omega t - \delta)$$

Transient Phenomena

- At $t = 0$, $\dot{x}(t) = 0$:

$$\dot{x}(t) = -B\omega_0 \, sin \, \omega_0 t - A\omega \, sin(\omega t - \delta)$$

- The phase of the driving force must be 0 or π.
- Amplitude at $t = 0$:

$$A_0 = B + A$$

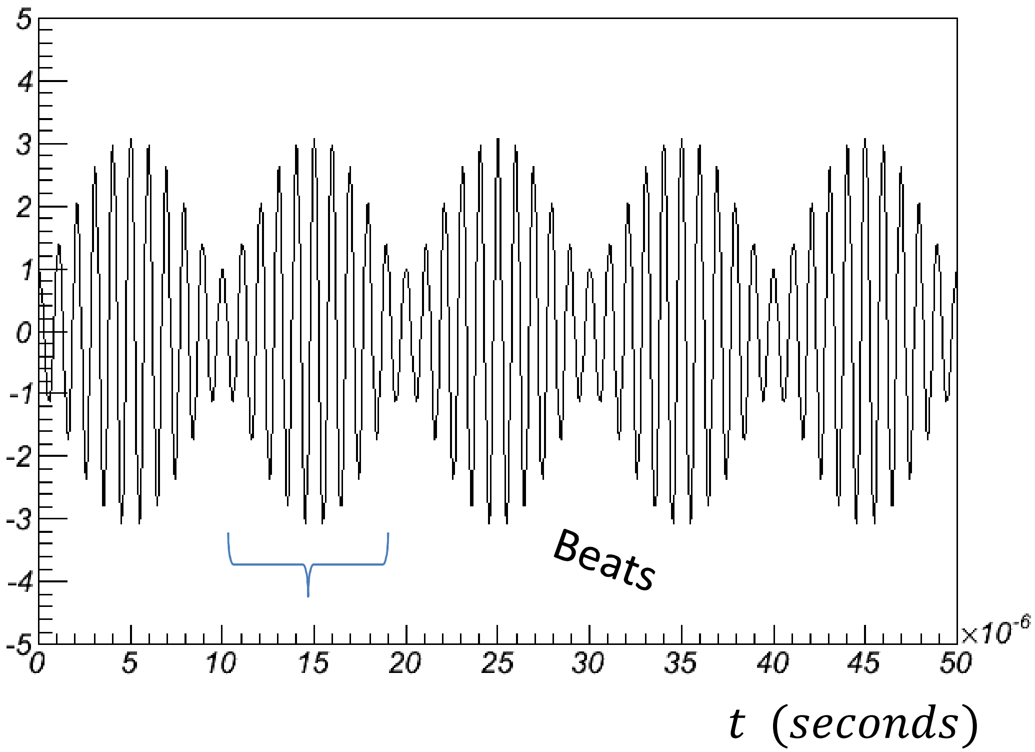
- A and A_0 are given, so $B = A_0 - A$

$$x(t) = (A_0 - A\,) \cos \omega_0 t + A \, cos \, \omega t$$

$$= A_0 \cos \omega_0 t + A(\cos \omega t - \cos \omega_0 t)$$

$$= A_0 \cos \omega_0 t + 2\mathrm{A} \sin(\overline{\omega} t) \sin\left(\frac{1}{2} \Delta\omega t\right)$$

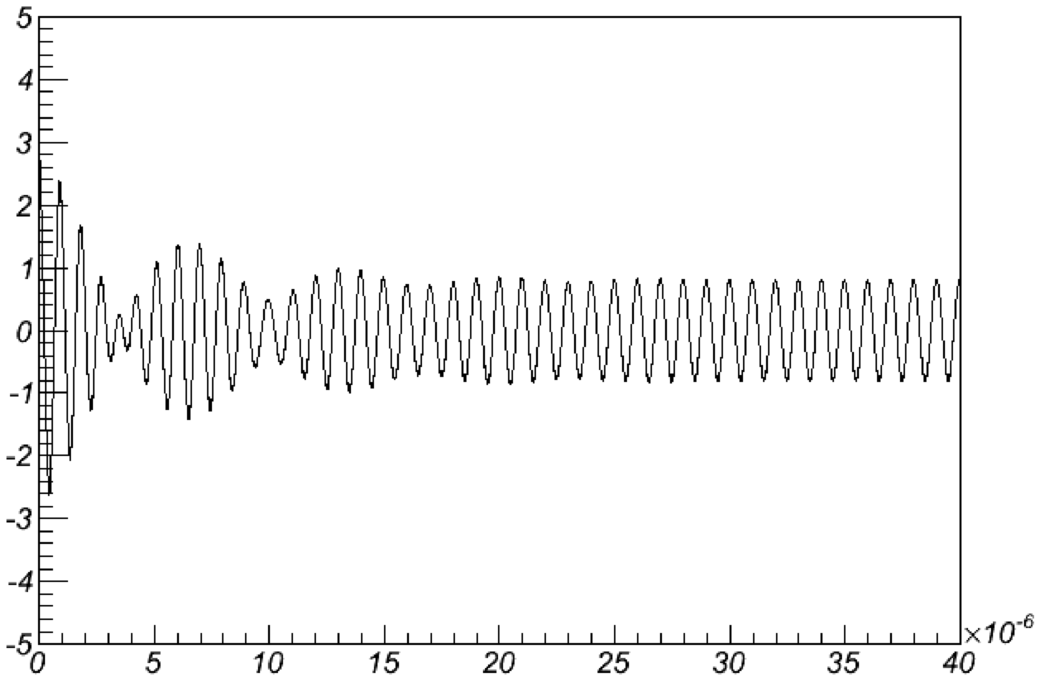
Transient Phenomena



$$\omega_0 = 2\pi \times (1.0 \text{ Mhz})$$
$$\omega = 2\pi \times (1.1 \text{ Mhz})$$
$$Q = 2$$

Transient Phenomena

- Accounting for energy dissipation looks like this:



$t \ \ (seconds)$

$$\omega_0 = 2\pi \times (1.0 \text{ Mhz})$$
$$\omega = 2\pi \times \ (1.15 \text{ Mhz})$$
$$Q = 20$$
$$\frac{1}{\gamma} = \frac{Q}{\omega_0} = 3.2\ \mu s$$