

Physics 42200

Waves & Oscillations

Lecture 7 – French, Chapter 3

Spring 2013 Semester

Matthew Jones

Announcement

- Assignment #2 is on the web page:

http://www.physics.purdue.edu/~mjones/phys42200_Spring2013/Assignment_2_Spring2013.pdf

- Just two questions
- Due Friday, January 25th

- The teaching assistants are:

- Qiang Zhang (zhan1217@purdue.edu)
- Jie Hui (hui4@purdue.edu)
- They should be available Wednesday mornings in room Phys 6A or Phys 4.

Damped Harmonic Motion

- Viscous damping: $F = -b\dot{x}$ (force proportional to velocity)
- Differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(t) = Ae^{\alpha t}$$

$$(\alpha^2 m + \alpha b + k)Ae^{\alpha t} = 0$$

$$\alpha^2 m + \alpha b + k = 0$$

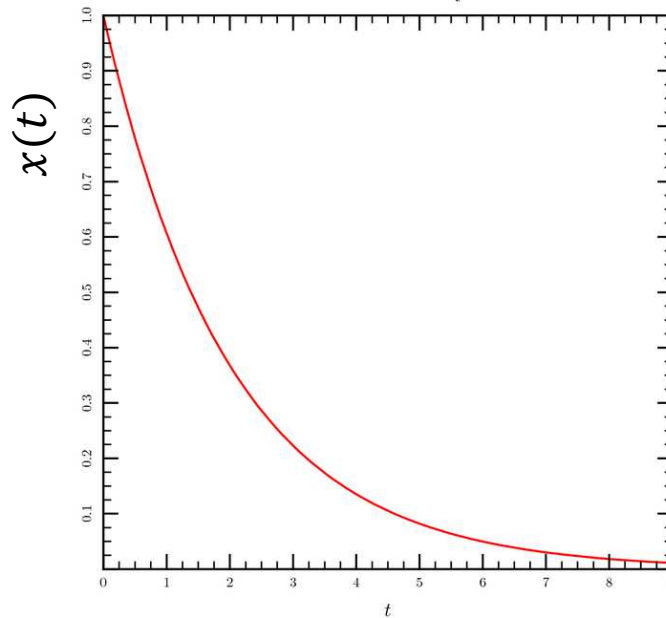
$$\alpha = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

- Three possible forms of the solution depend on the sign of the discriminant.

Overdamped Case

$$\frac{b^2}{4m^2} - \frac{k}{m} = \frac{\gamma^2}{4} - (\omega_0)^2 > 0$$

$$x(t) = \mathbf{A} e^{-\frac{\gamma}{2}t} e^{t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}} + \mathbf{B} e^{-\frac{\gamma}{2}t} e^{-t\sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}}$$

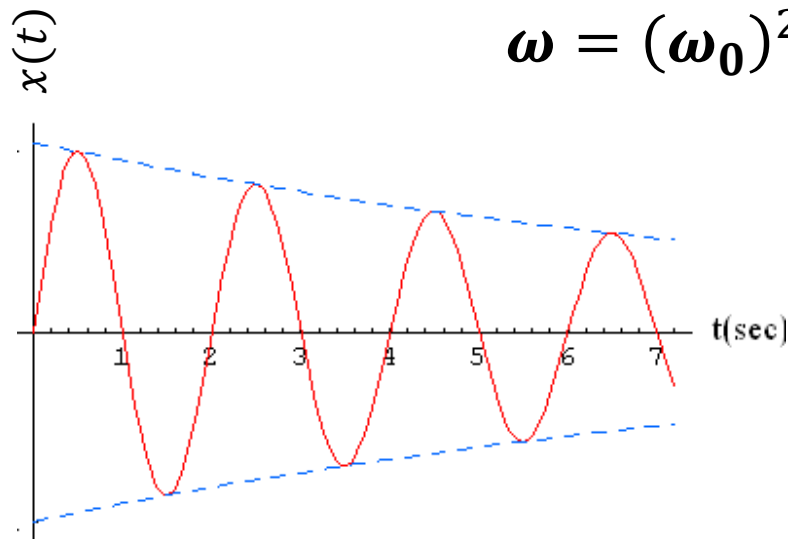


Underdamped Case

$$\frac{b^2}{4m^2} - \frac{k}{m} = \frac{\gamma^2}{4} - (\omega_0)^2 < 0$$

$$x(t) = \mathbf{A}e^{-\frac{\gamma}{2}t} \sin \omega t + \mathbf{B}e^{-\frac{\gamma}{2}t} \cos \omega t$$

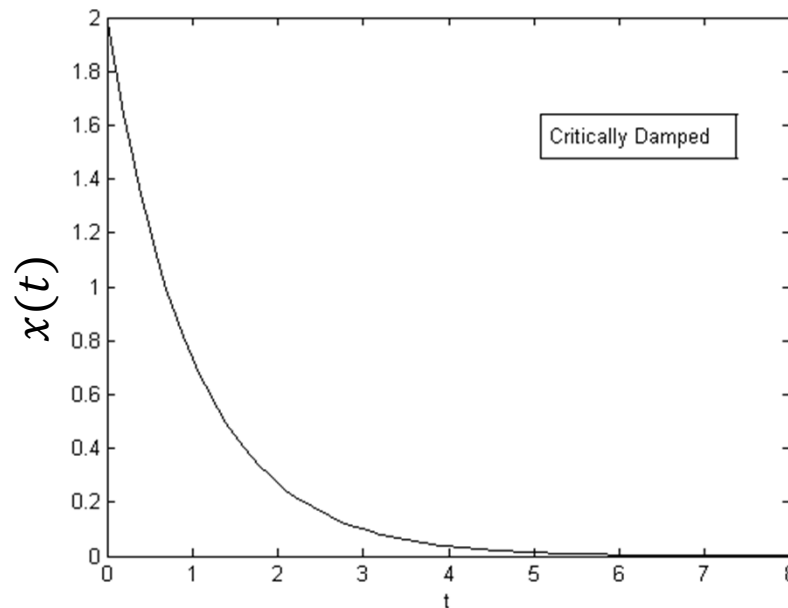
$$\omega = \sqrt{(\omega_0)^2 - \frac{\gamma^2}{4}}$$



Critically Damped Case

$$\frac{b^2}{4m^2} - \frac{k}{m} = 0$$

$$x(t) = (\textcolor{red}{A} + \textcolor{red}{B}t)e^{-\frac{\gamma}{2}t}$$

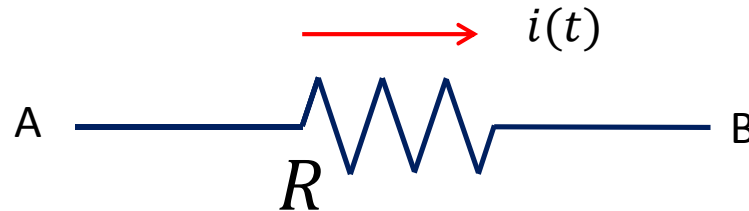


RLC Circuits

Review of electricity and magnetism:

- Capacitors store energy in an electric field
- Resistors dissipate energy by heating
- Inductors store energy in a magnetic field
- Voltage
 - Energy per unit charge
 - SI units: *Volt = J/C*
- Current
 - Charge passing a point in a circuit per unit time
 - SI units: *Ampere = C/s*

Resistors

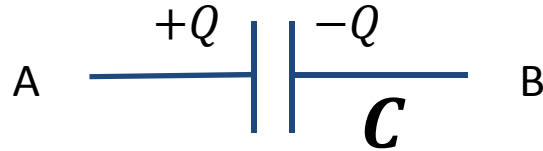


- Current flows from A to B in the direction indicated by the arrow.
- Charges at point B have less energy than at point A because some of their energy was dissipated as heat.
- Potential difference:

$$\Delta V = V_A - V_B = i(t) R$$

- Resistance, R , is measured in *ohms* in SI units

Capacitors



- Potential difference:

$$\Delta V = V_A - V_B = \frac{Q}{C}$$

- No current can flow across the capacitor, so any charge that flows onto the plates accumulates there:

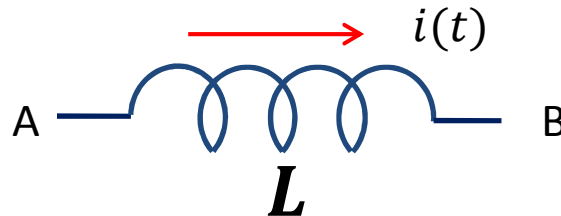
$$Q(t) = Q_0 + \int_0^t i(t) dt$$

- Potential difference:

$$\Delta V = \frac{1}{C} \int_0^t i(t) dt$$

- SI units for capacitance: *farad*

Inductors



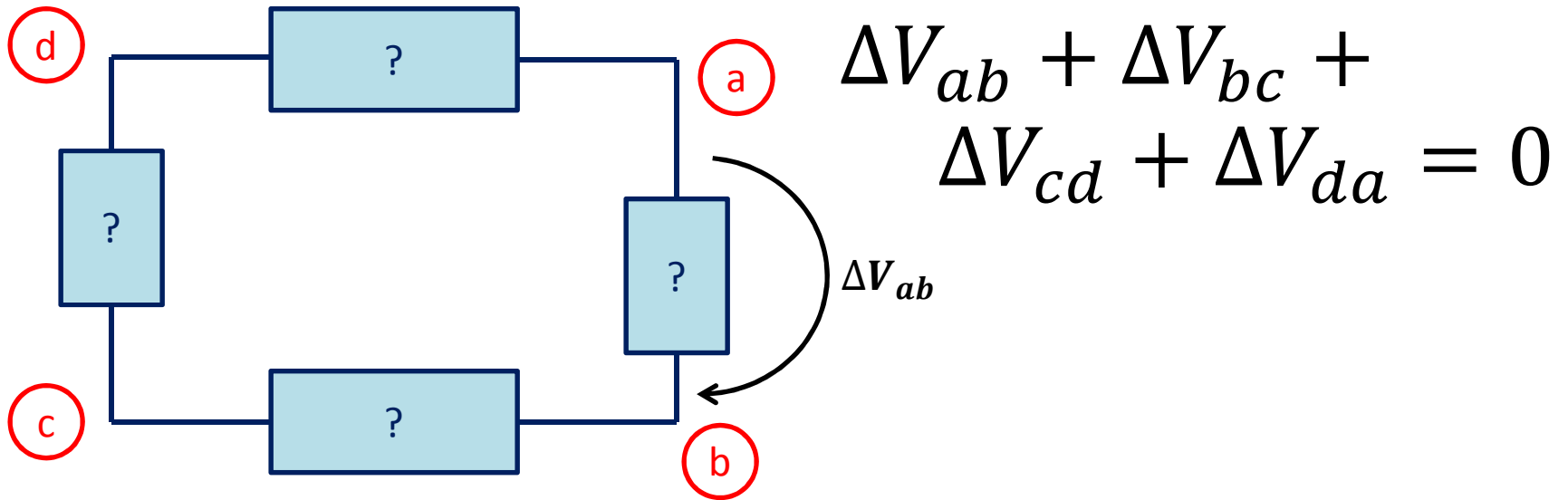
- The inductor will establish a potential difference that opposes any change in current.
 - If $di/dt < 0$ then $V_B > V_A$
 - magnetic field is being converted to energy
 - If $di/dt > 0$ then $V_B < V_A$
 - energy is being stored in the magnetic field

$$\Delta V = V_A - V_B = L \frac{di}{dt}$$

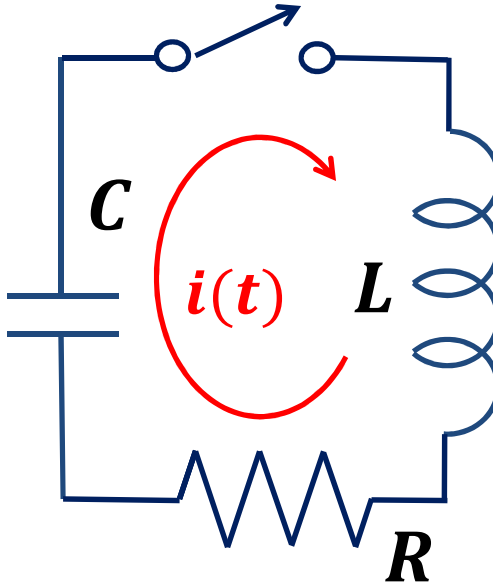
- SI units for inductance: *henry*

Kirchhoff's Loop Rule

- The sum of the potential differences around a loop in a circuit must equal zero:



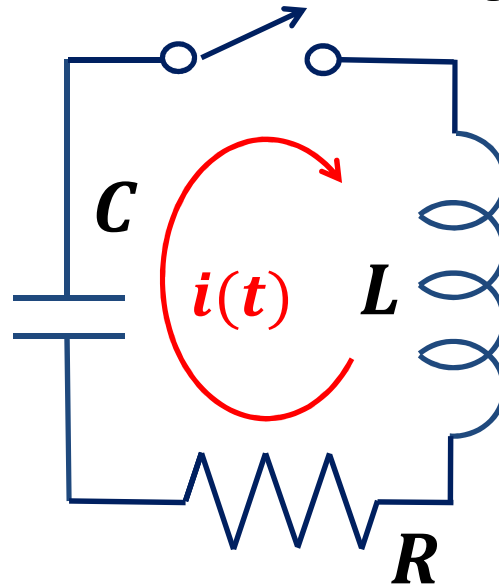
Kirchhoff's Loop Rule



Sum of potential differences:

$$-L \frac{di}{dt}$$

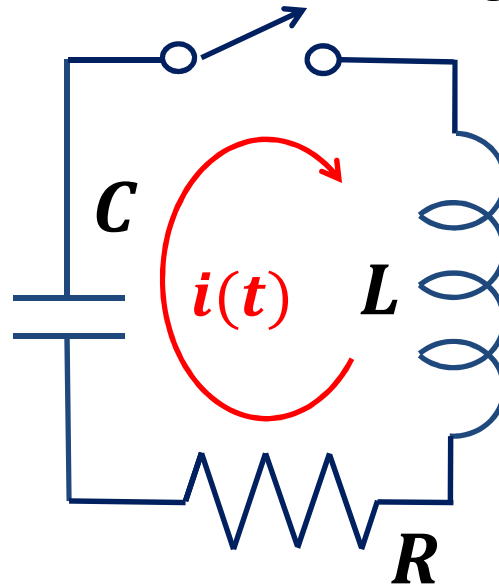
Kirchhoff's Loop Rule



Sum of potential differences:

$$-L \frac{di}{dt} - i(t)R$$

Kirchhoff's Loop Rule



Sum of potential differences:

$$-L \frac{di}{dt} - i(t)R - \frac{1}{C} \left(Q_0 + \int_0^t i(t) dt \right) = 0$$

Initial charge, Q_0 , defines the initial conditions.

Kirchhoff's Loop Rule

$$L \frac{di}{dt} + i(t)R + \frac{1}{C} \left(Q_0 + \int_0^t i(t) dt \right) = 0$$

Differentiate once with respect to time:

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0$$

This is of the same form as the equation for a damped harmonic oscillator:

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx(t) = 0$$

Solutions

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0$$

Suppose $i(t) = Ae^{\alpha t}$

Then $\frac{di}{dt} = \alpha Ae^{\alpha t}$ and $\frac{d^2 i}{dt^2} = \alpha^2 Ae^{\alpha t}$

Substitute into the differential equation:

$$\left(\alpha^2 L + \alpha R + \frac{1}{C} \right) Ae^{\alpha t} = 0$$

True for any t only if $\alpha^2 L + \alpha R + \frac{1}{C} = 0$.

Solutions

$$\alpha^2 L + \alpha R + \frac{1}{C} = 0$$

Roots of the polynomial:

$$\alpha = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Define some new symbols:

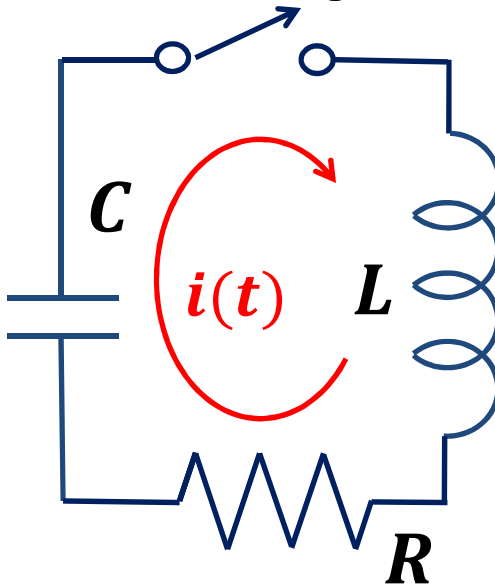
$$\omega_0 = \sqrt{1/LC}$$

$$\gamma = R/L$$

Then the roots can be written

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

Example



- If $L = 2.2 \mu H$ and $C = 10 nF$ what is the frequency of oscillations when $R = 0$?
- What is the largest value of R that will still allow the circuit to oscillate?

Example

$$\omega_0 = \sqrt{1/LC} = \sqrt{\frac{1}{(2.2 \mu H)(0.01 \mu F)}} = 6.74 \times 10^6 \text{ s}^{-1} = 2.15 \text{ MHz}$$

$$\gamma = R/L$$

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

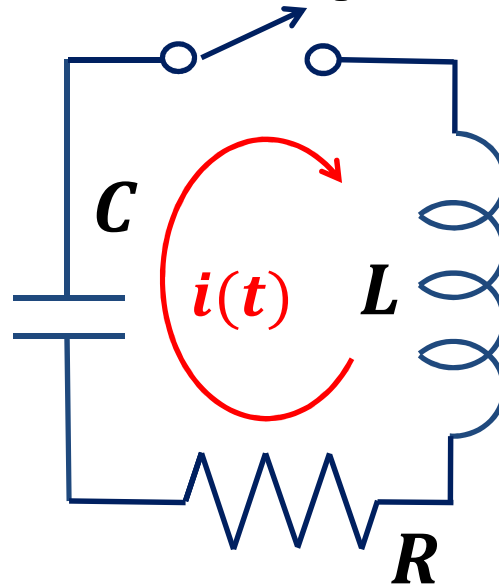
- Critical damping:

$$\frac{\gamma^2}{4} - (\omega_0)^2 = 0$$

$$\gamma = \frac{R}{L} = 2 \sqrt{\frac{1}{LC}}$$

$$R = 2 \sqrt{\frac{L}{C}} = 2 \sqrt{\frac{2.2 \mu H}{0.01 \mu F}} = 29.7 \Omega$$

Example



$$L = 2.2 \mu H$$

$$C = 10 nF$$

$$R = 2 \Omega$$

- Suppose the initial charge on the capacitor was 10 nC... What voltage is measured across R as a function of time?

Example

- Calculate the discriminant:

$$\frac{R^2}{4L^2} - \frac{1}{LC} = \frac{(2\ \Omega)^2}{4(2.2\ \mu H)^2} - \frac{1}{(2.2\ \mu H)(0.01\ \mu F)} < 0$$

- The circuit will oscillate with frequency:

$$f = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 1.07\ MHz$$

- Time constant:

$$\frac{\gamma}{2} = \frac{R}{2L} = \frac{(2\ \Omega)}{2(2.2\ \mu H)} = 4.55 \times 10^5\ s^{-1}$$

- Current will be:

$$i(t) = i_0 e^{-\gamma t/2} \cos(\omega t + \varphi)$$

Example

$$i(t) = i_0 e^{-\gamma t/2} \cos(\omega t + \varphi)$$

- Initial conditions:

- $i(0) = 0$ because the inductor produces a potential difference that opposes the change in current.

- Therefore, $\varphi = \pi/2$

- Initial potential across capacitor:

$$\Delta V = \frac{Q}{C} = \frac{10 \text{ nC}}{10 \text{ nF}} = 1 \text{ Volt}$$

- Initial voltage across inductor:

$$\Delta V = L \frac{di}{dt} = 1 \text{ Volt}$$

$$\frac{di}{dt} = \frac{1 \text{ V}}{2.2 \text{ } \mu\text{H}} = 4.55 \times 10^5 \text{ A/s} = i_0 \omega$$

- $i_0 = \frac{4.55 \times 10^5 \text{ A/s}}{6.73 \times 10^6 /s} = 68 \text{ mA}$

Example

- Current in circuit:

$$i(t) = i_0 e^{-\gamma t/2} \sin \omega t$$

where $\gamma/2 = 4.55 \times 10^5 \text{ s}^{-1}$ and $i_0 = 68 \text{ mA}$

- Potential difference across the resistor:

$$\Delta V = i(t)R$$

$$v(t) = v_0 e^{-\gamma t/2} \sin \omega t$$

where $v_0 = 136 \text{ mV}$.

