

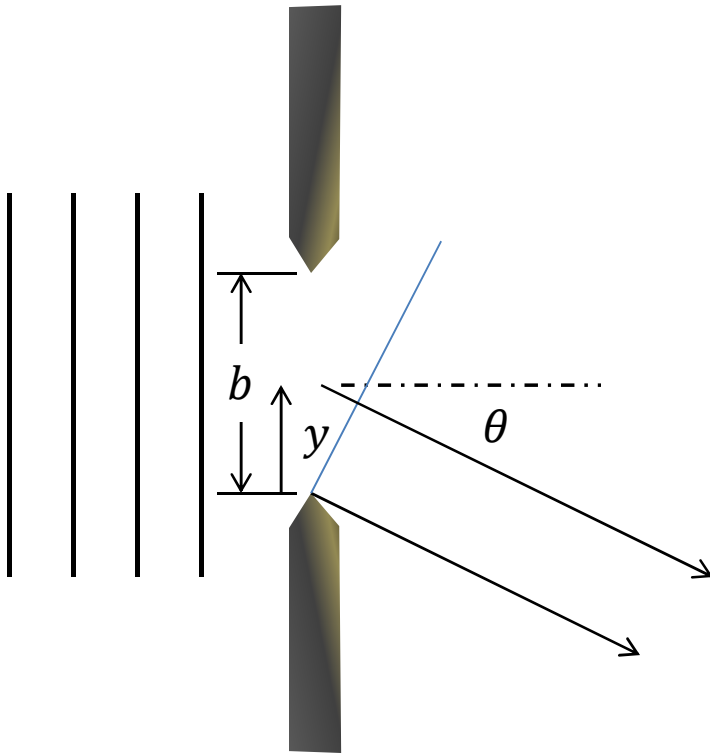
Physics 42200
Waves & Oscillations

Lecture 40 – Diffraction

Spring 2013 Semester

Matthew Jones

Fraunhofer Diffraction



Fraunhofer diffraction:

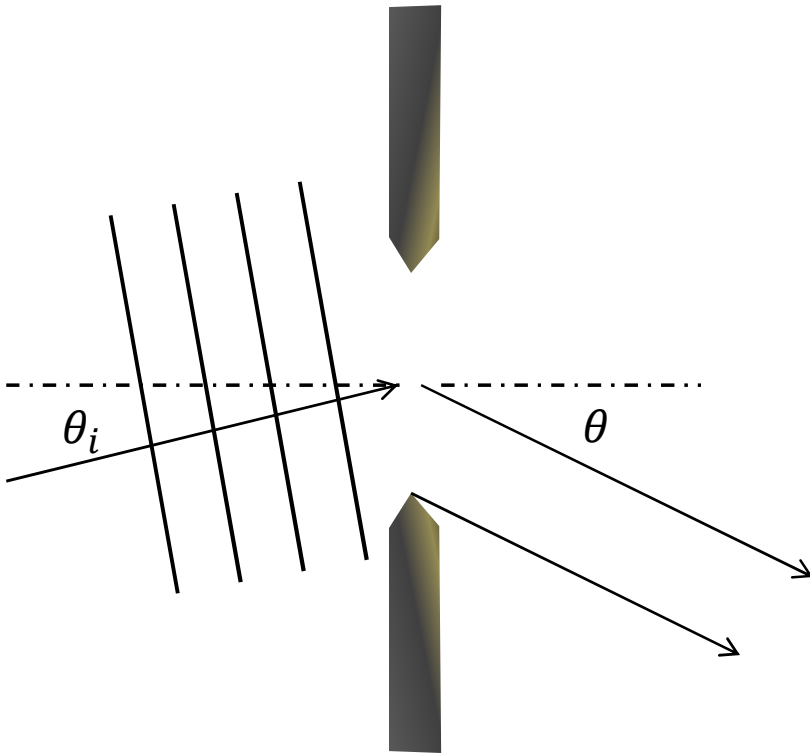
- The phase varies linearly across the aperture
- The intensity of light arriving from each part of the aperture is equal

$$dE = \frac{\epsilon_L e^{ik\mathbf{y} \sin \theta} dy}{R}$$

Fraunhofer Diffraction

Fraunhofer diffraction:

- The phase varies linearly across the aperture
- The intensity of light arriving from each part of the aperture is equal

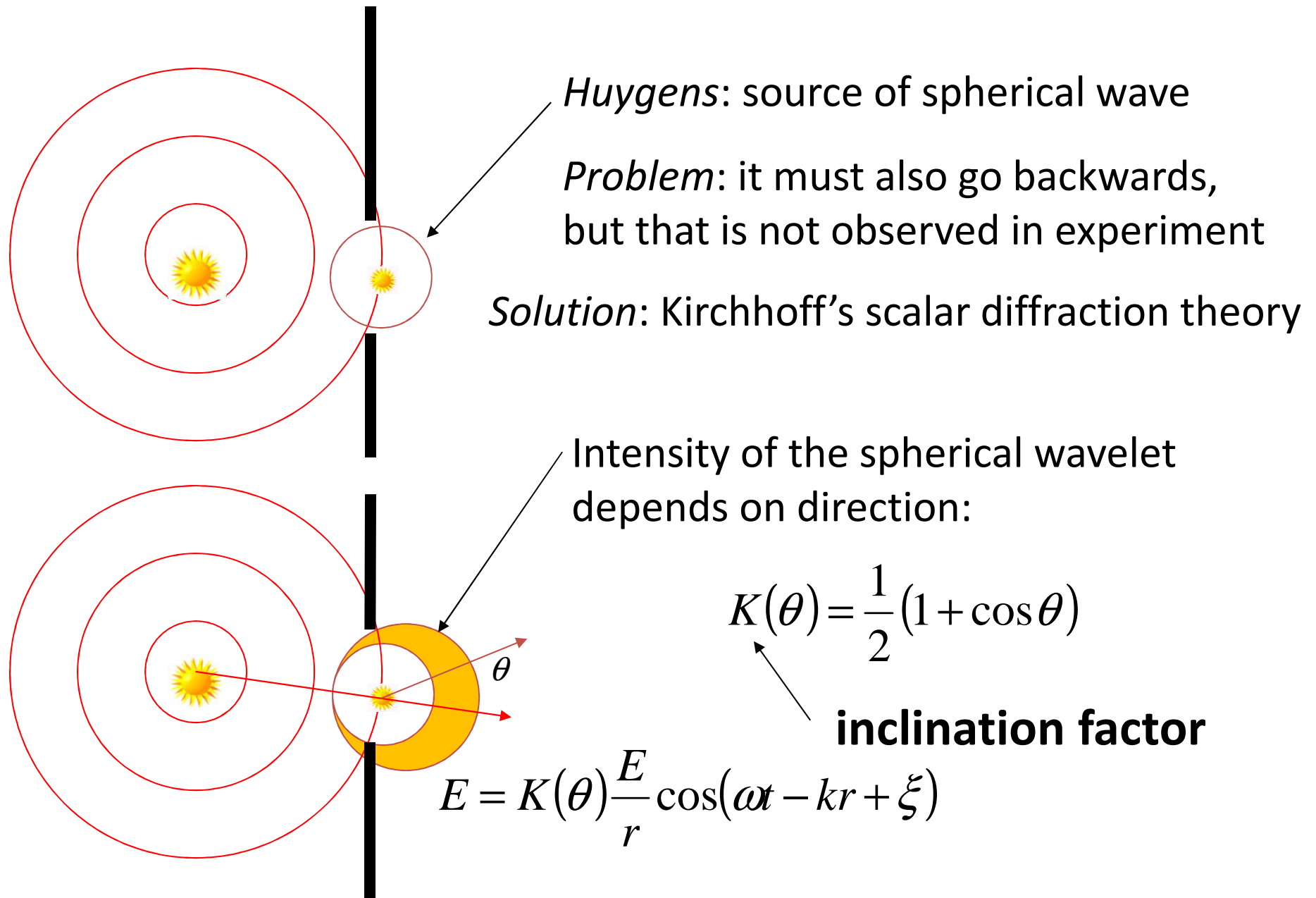


$$dE = \frac{\epsilon_L e^{ik\mathbf{y}} (\sin \theta - \sin \theta_i) dy}{R}$$

Huygens-Fresnel Principle

- Each point on a wavefront is a source of spherical waves that are in phase with the incident wave.
- The light at any point in the direction of propagation is the sum of all such spherical waves, taking into account their relative phases and path lengths.
- *The secondary spherical waves are preferentially emitted in the forward direction.*

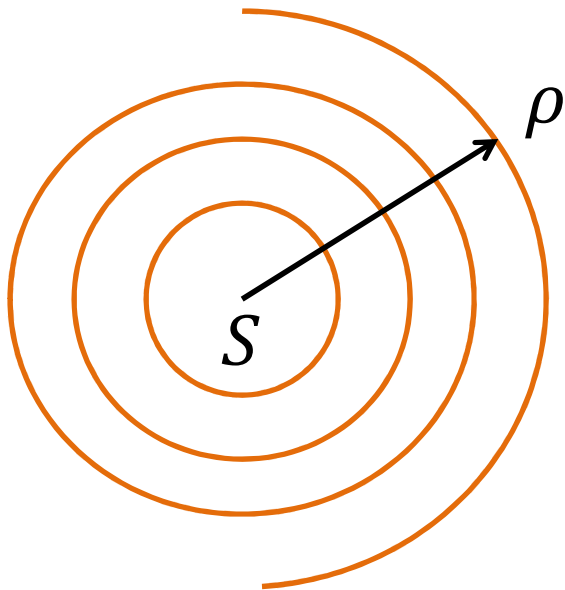
Huygens-Fresnel Principle



Propagation of Spherical Waves

- Consider a spherical wave emitted from a source S at time $t = 0$.

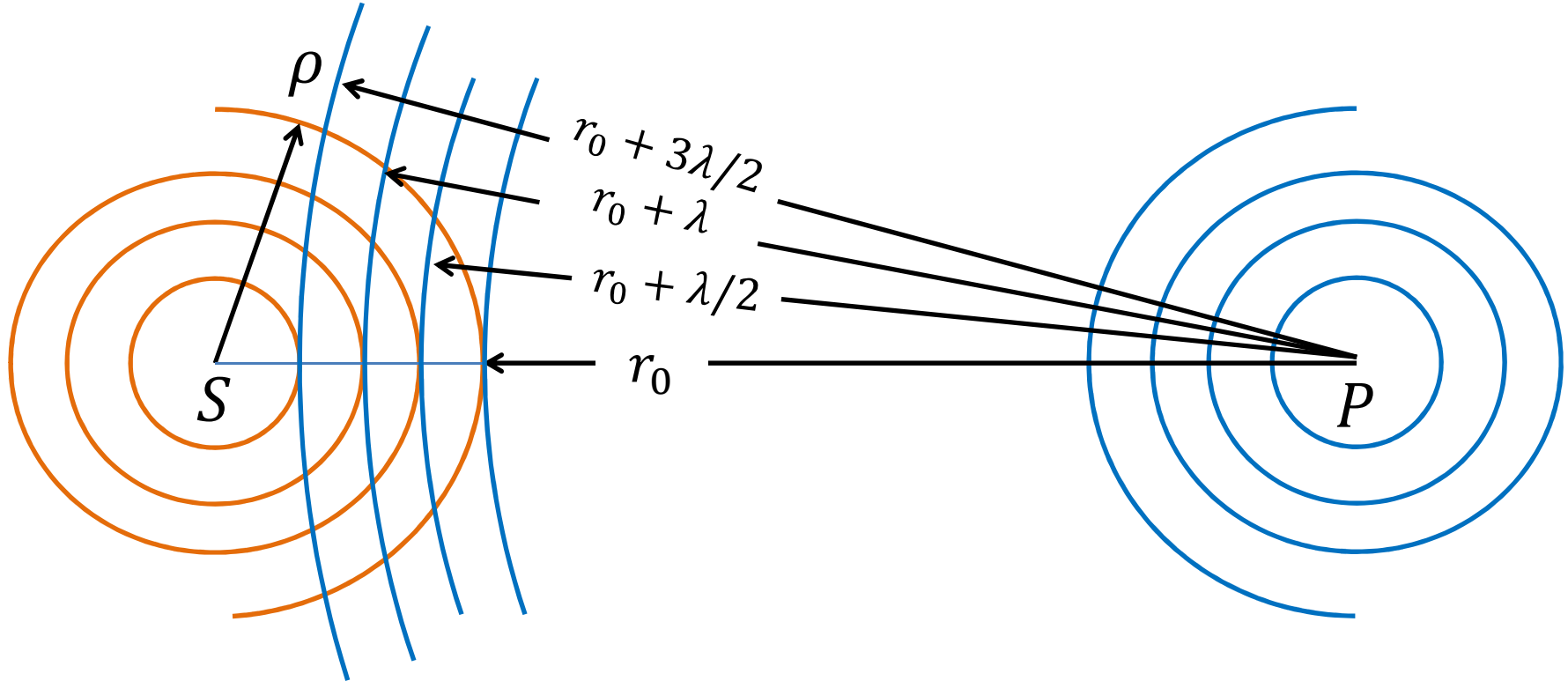
$$E(\rho, t') = \frac{\varepsilon_0}{\rho} \cos(\omega t' - k\rho)$$



- These spherical waves expand outward from S

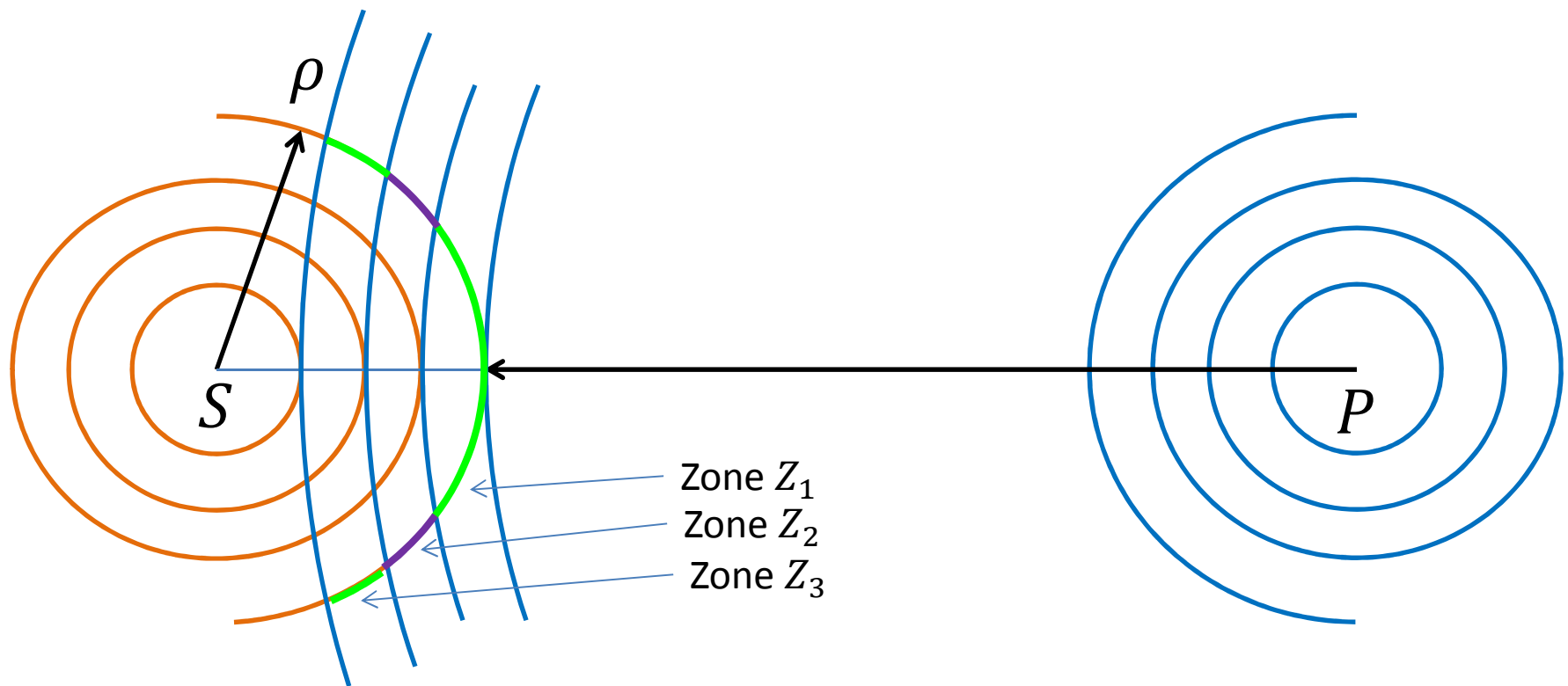
Propagation of Spherical Waves

- Consider a series of concentric spheres around another point P with radii $r_0, r_0 + \lambda/2, r_0 + \lambda, \dots$

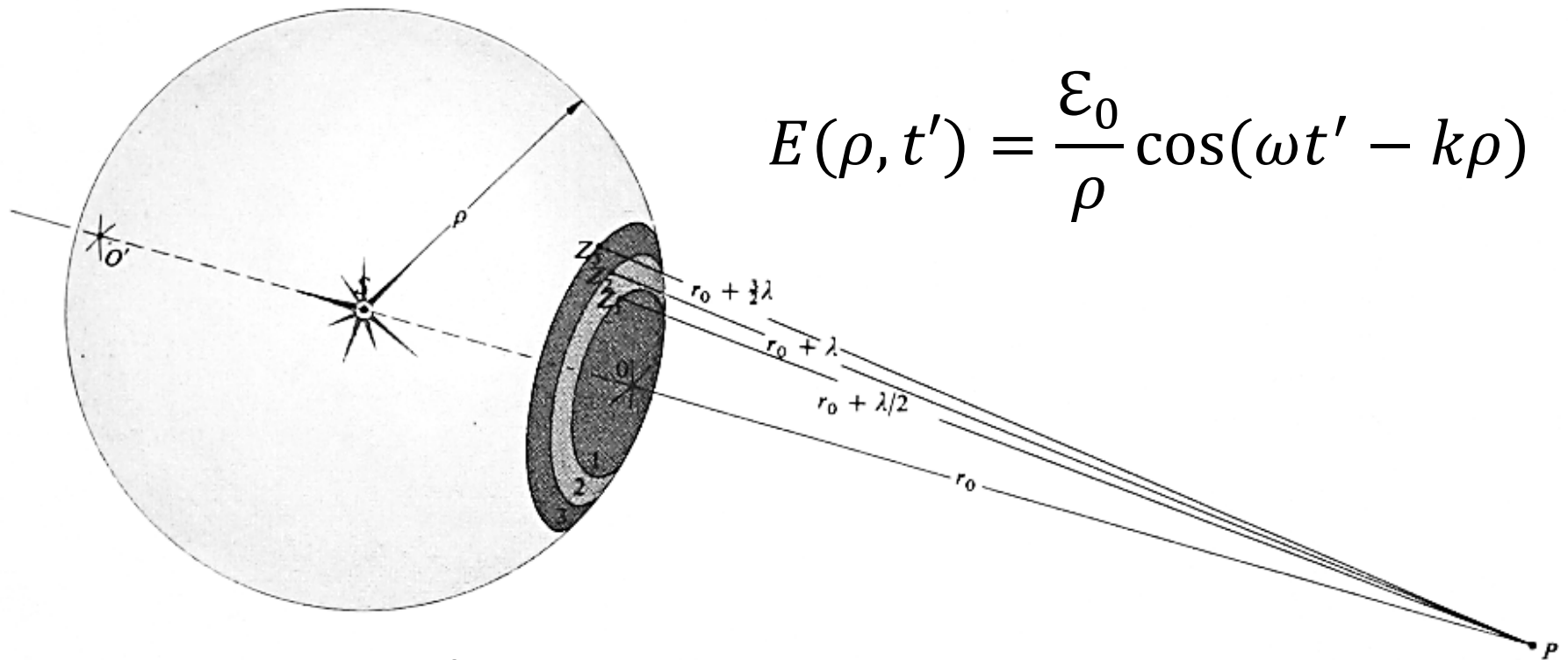


Propagation of Spherical Waves

- Consider a series of concentric spheres around another point P with radii $r_0, r_0 + \lambda/2, r_0 + \lambda, \dots$



Fresnel Zones



- Source strength per unit area in any zone is

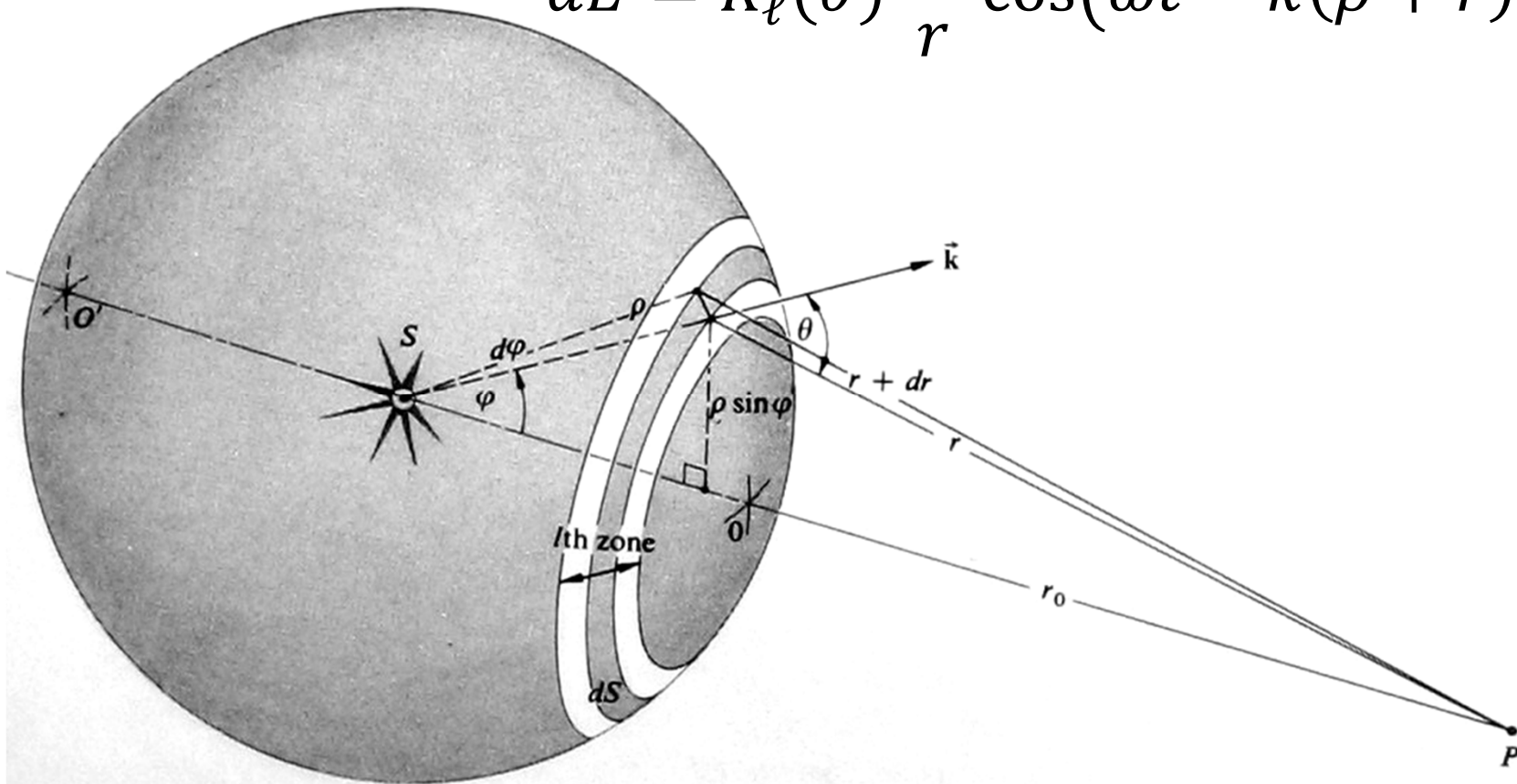
$$\varepsilon_A \propto \frac{\varepsilon_0}{\rho}$$

- Curvature of the surface in each zone is small

Fresnel Zones

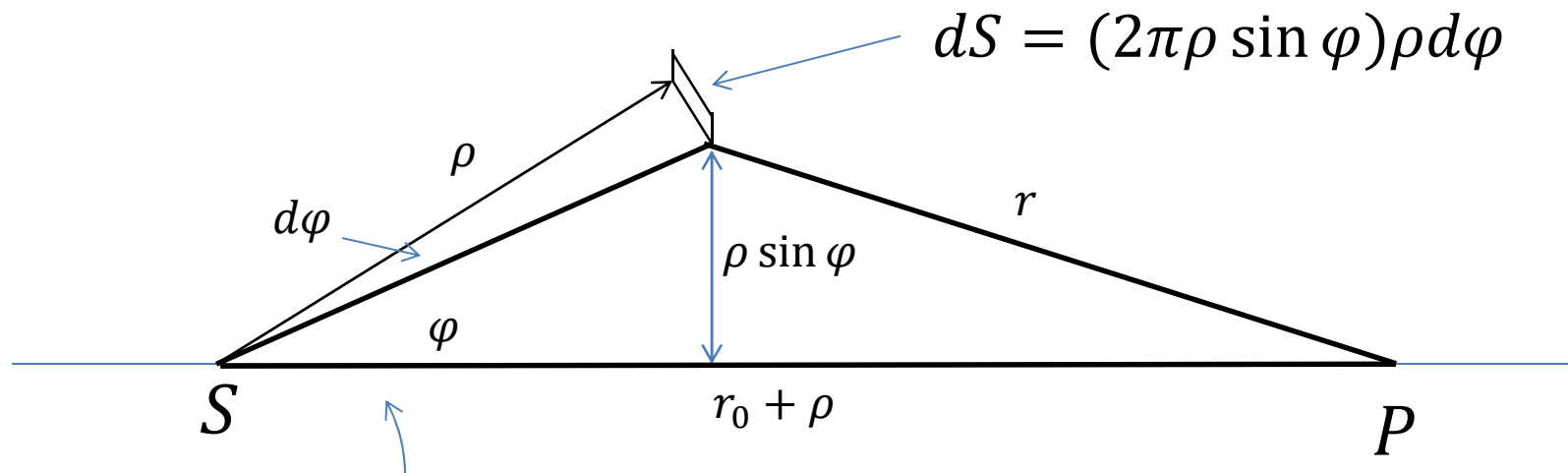
Electric field at point P due to secondary waves in zone ℓ :

$$dE = K_\ell(\theta) \frac{\varepsilon_A}{r} \cos(\omega t - k(\rho + r)) dS$$



Fresnel Zones

- How can we describe the element of area dS ?



Law of cosines: $r^2 = \rho^2 + (r_0 + \rho)^2 - 2\rho(r_0 + \rho) \cos \varphi$

$$2rdr = 2\rho(r_0 + \rho) \sin \varphi d\varphi$$

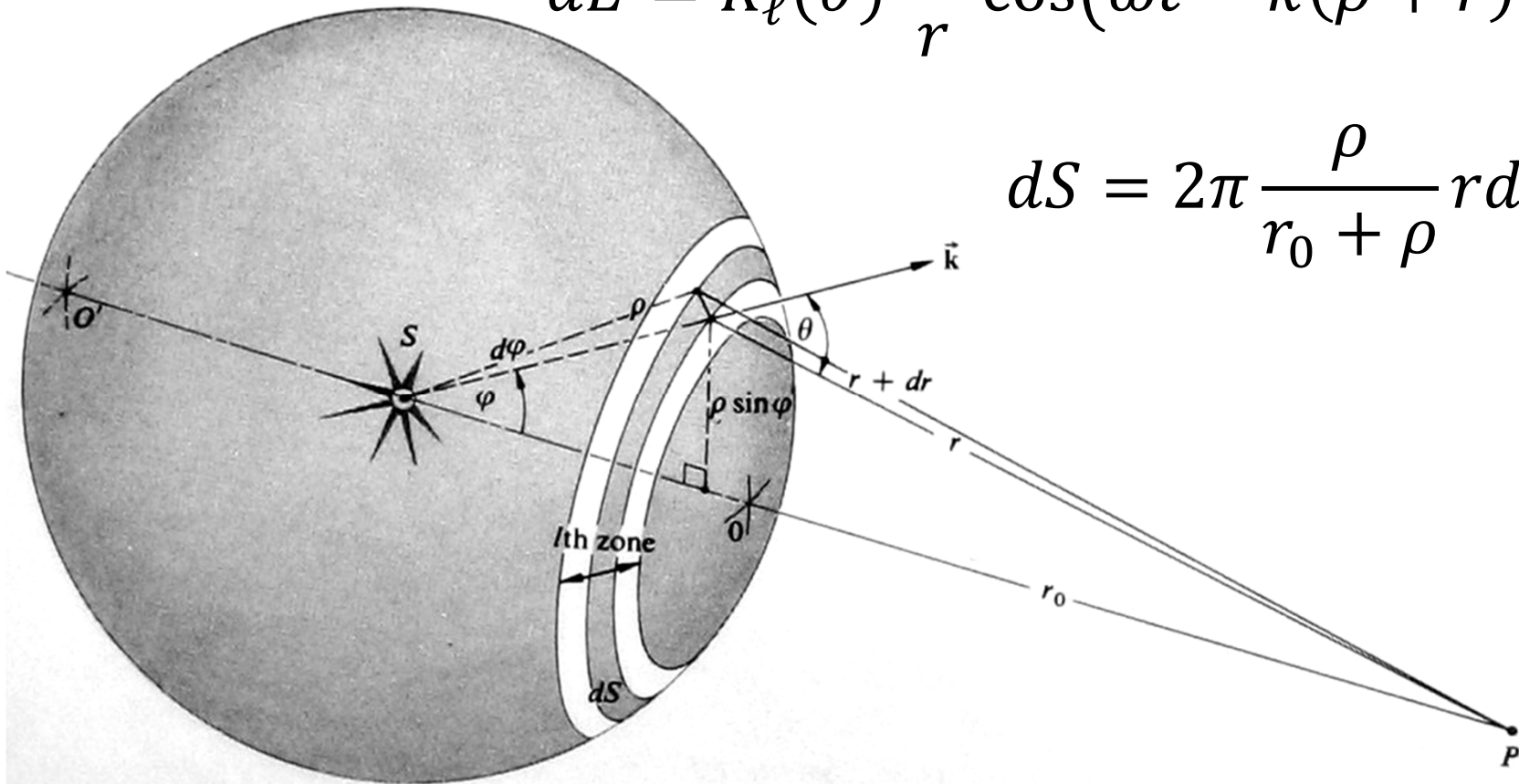
$$\rho \sin \varphi d\varphi = \frac{rdr}{r_0 + \rho}$$

Fresnel Zones

Electric field at point P due to secondary waves in zone ℓ :

$$dE = K_{\ell}(\theta) \frac{\varepsilon_A}{r} \cos(\omega t - k(\rho + r)) dS$$

$$dS = 2\pi \frac{\rho}{r_0 + \rho} r dr$$



Fresnel Zones

- Total electric field due to wavelets emitted in zone ℓ :

$$\begin{aligned} E_\ell &= 2\pi K_\ell(\theta) \frac{\mathcal{E}_A \rho}{\rho + r_0} \int_{r_{\ell-1}}^{r_\ell} \cos(\omega t - k(\rho + r)) dr \\ &= -\frac{2\pi}{k} K_\ell(\theta) \frac{\mathcal{E}_A \rho}{\rho + r_0} \left[\sin(\omega t - k(\rho + r)) \right]_{r_{\ell-1}}^{r_\ell} \end{aligned}$$

- But $r_\ell = r_0 + \ell\lambda/2$ and $r_{\ell-1} = r_0 + (\ell - 1)\lambda/2$ and $k\ell\lambda/2 = \ell\pi$, so

$$E_\ell = (-1)^{\ell+1} K_\ell(\theta) \frac{\mathcal{E}_A \rho \lambda}{\rho + r_0} \sin(\omega t - k(\rho + r_0))$$

- Even ℓ : $E_\ell < 0$, odd ℓ : $E_\ell > 0$

Fresnel Zones

- The total electric field at point P is the sum of all electric fields from each zone:

$$\begin{aligned} E &= E_1 + E_2 + E_3 + \cdots + E_m \\ &= |E_1| - |E_2| + |E_3| - |E_4| + \cdots \pm |E_m| \end{aligned}$$

- Most of the adjacent zones cancel:

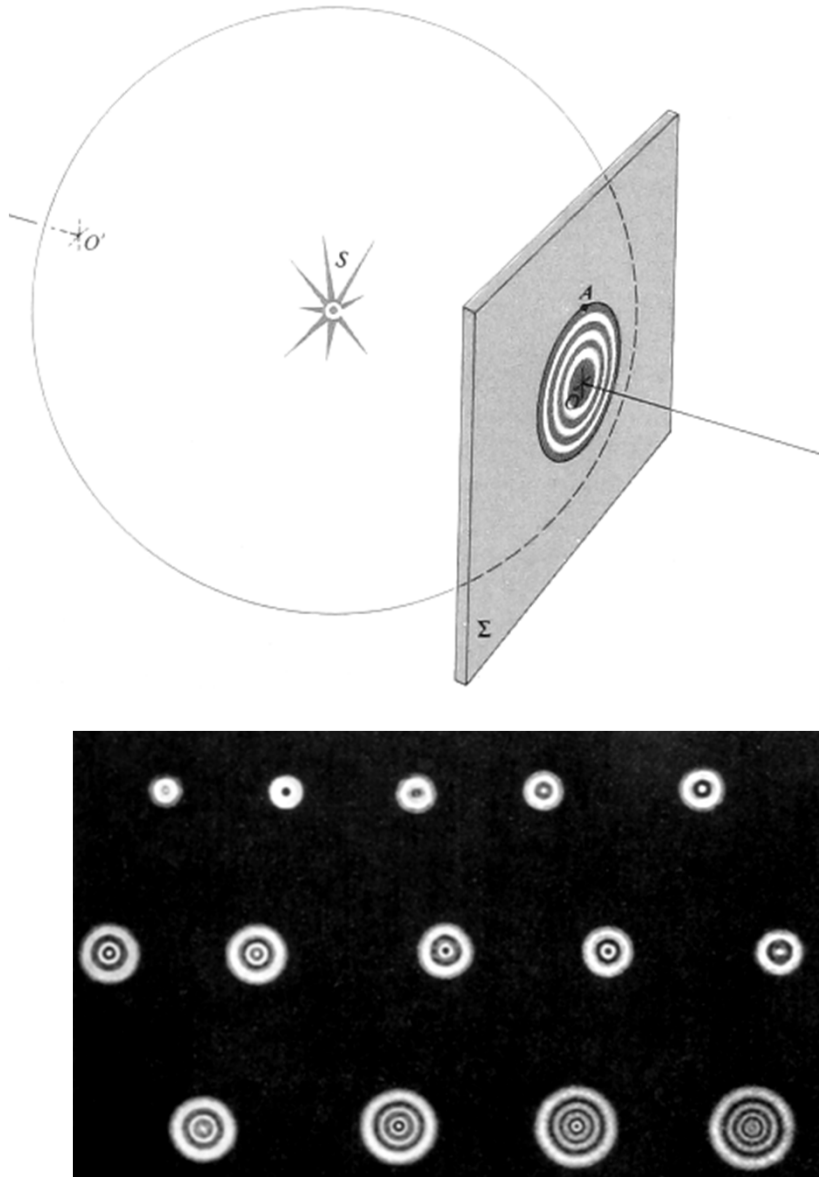
$$E = \frac{|E_1|}{2} + \left(\frac{|E_1|}{2} - |E_2| + \frac{|E_3|}{2} \right) + \cdots \pm \frac{|E_m|}{2}$$

- Two possibilities:

$$E \approx \frac{|E_1|}{2} + \frac{|E_m|}{2} \quad \text{or} \quad E \approx \frac{|E_1|}{2} - \frac{|E_m|}{2}$$

- Fresnel conjectured that $|E_m| \rightarrow 0$ so $E \approx |E_1|/2$

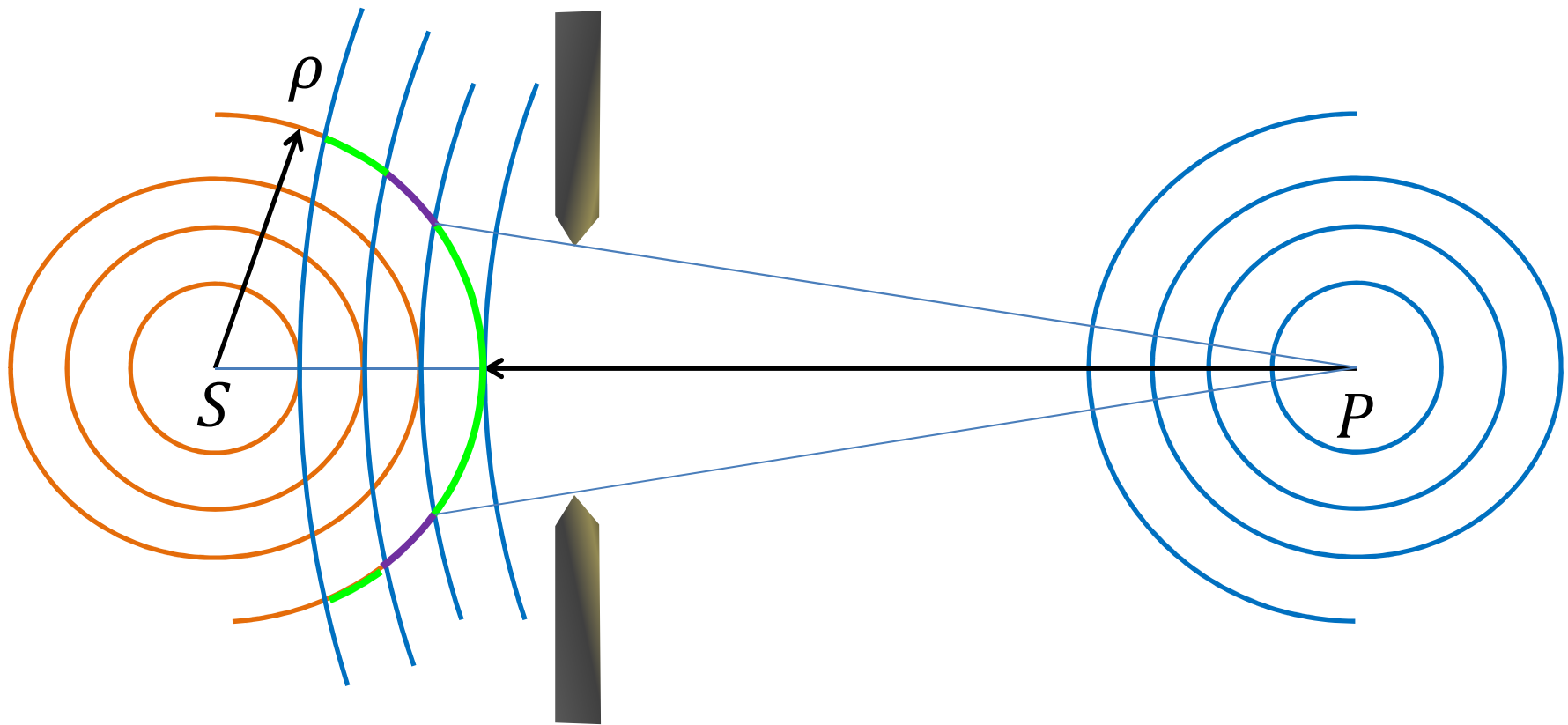
Circular Aperture: Fresnel Diffraction



- Suppose a circular aperture uncovers only the first m zones.
- If m is even, then the first two zones interfere destructively: $E = 0$
- If m is odd, then all but the first one cancel each other: $E = |E_1|$
- What about points off the central axis?

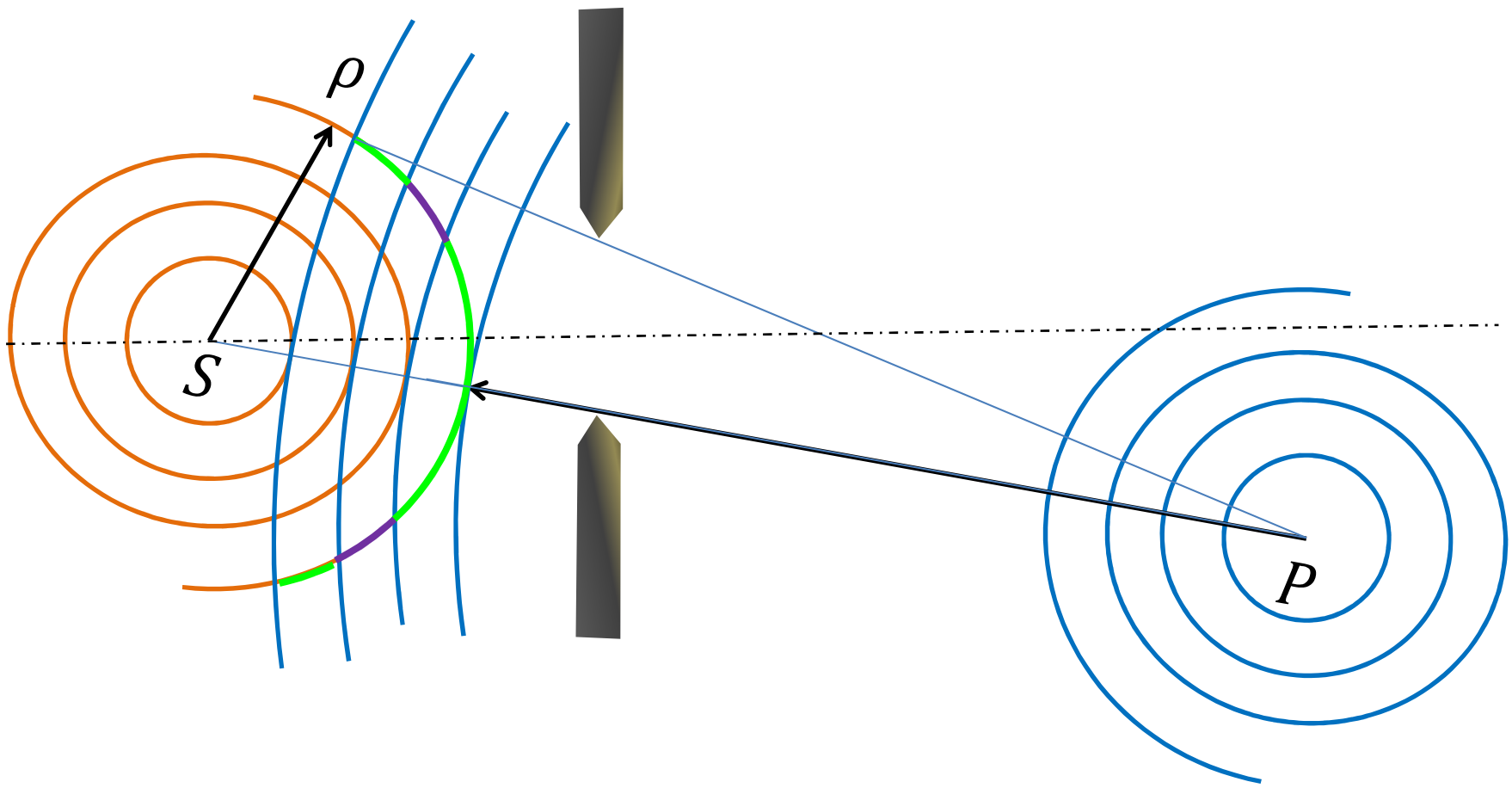
Circular Aperture: Fresnel Diffraction

- A point on the central axis:

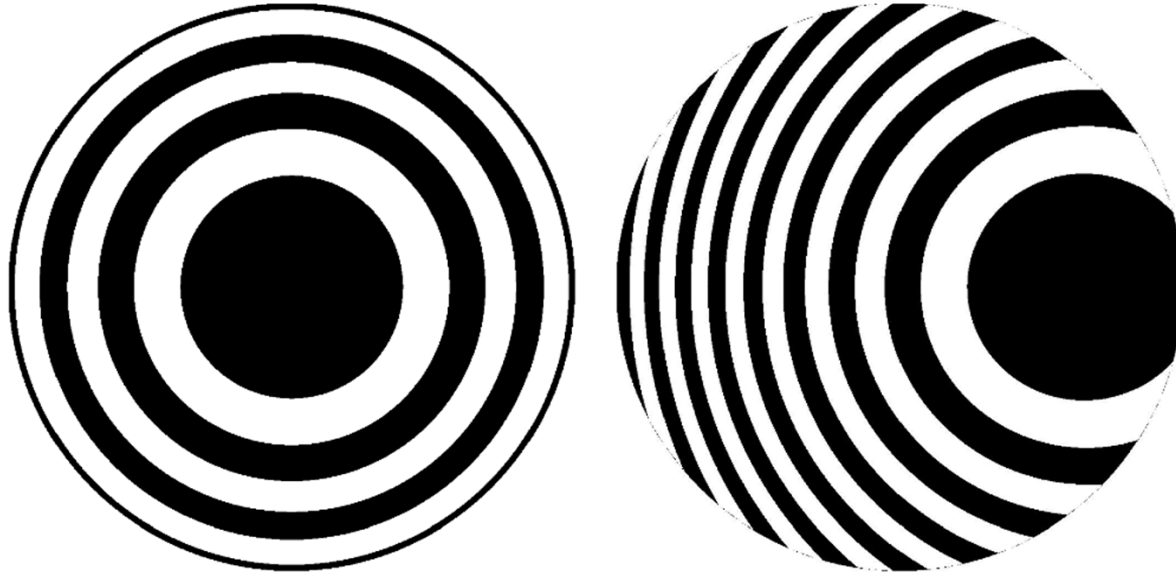


Circular Aperture: Fresnel Diffraction

- A point that is not on the central axis:

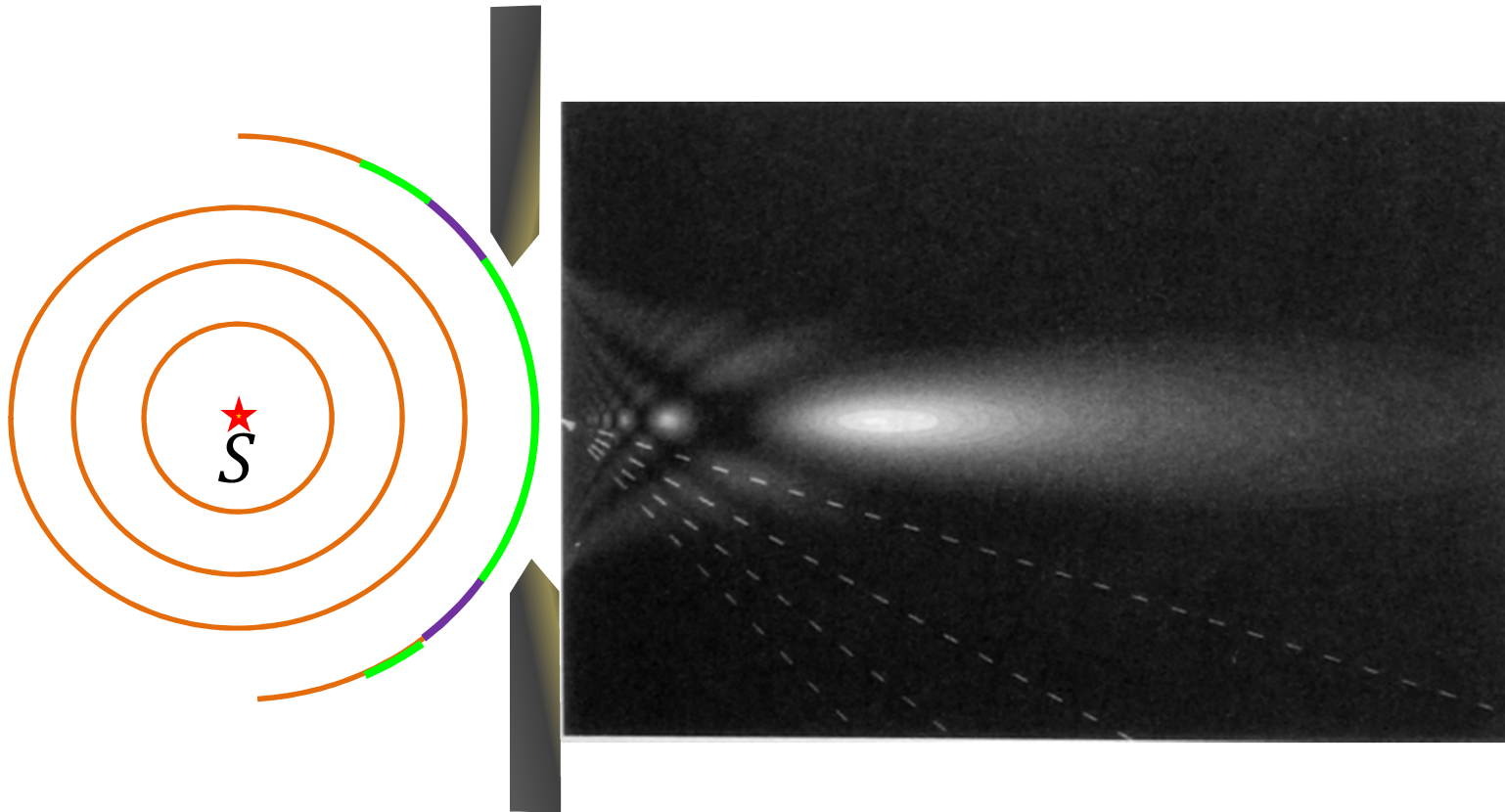


Circular Aperture: Fresnel Diffraction



- There will also be light and dark fringes off the central axis

Circular Aperture: Fresnel Diffraction



Circular Obstacle: Fresnel Diffraction

- A circular obstacle will remove the middle zones, but the remaining zones can interfere constructively and destructively

$$E = |E_{\ell+1}| - |E_{\ell+2}| + |E_{\ell+3}| \dots \pm |E_m|$$

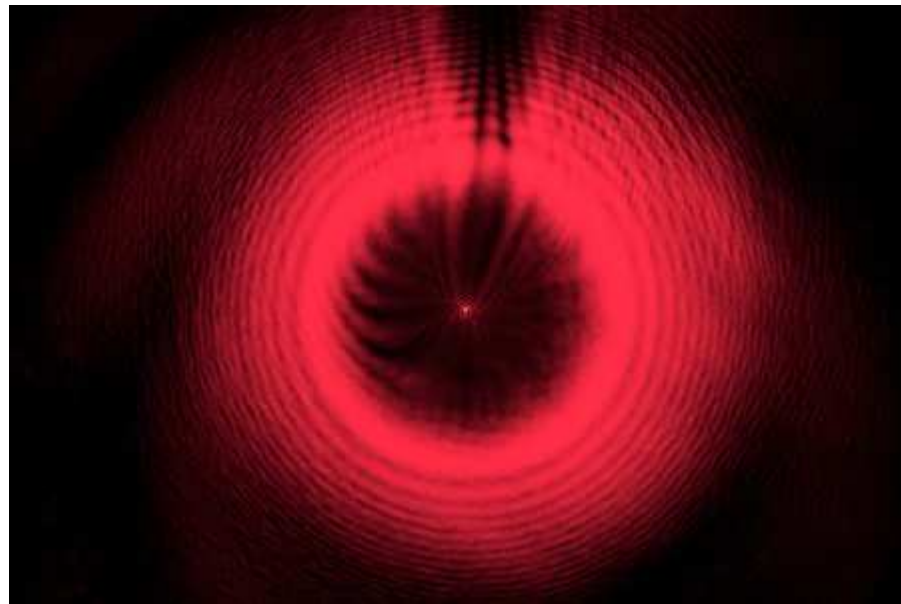
- As in the case of the unobstructed wave, only the first unobstructed zone contributes:

$$E \approx \frac{|E_{\ell+1}|}{2}$$

- There should be a bright spot on the central axis

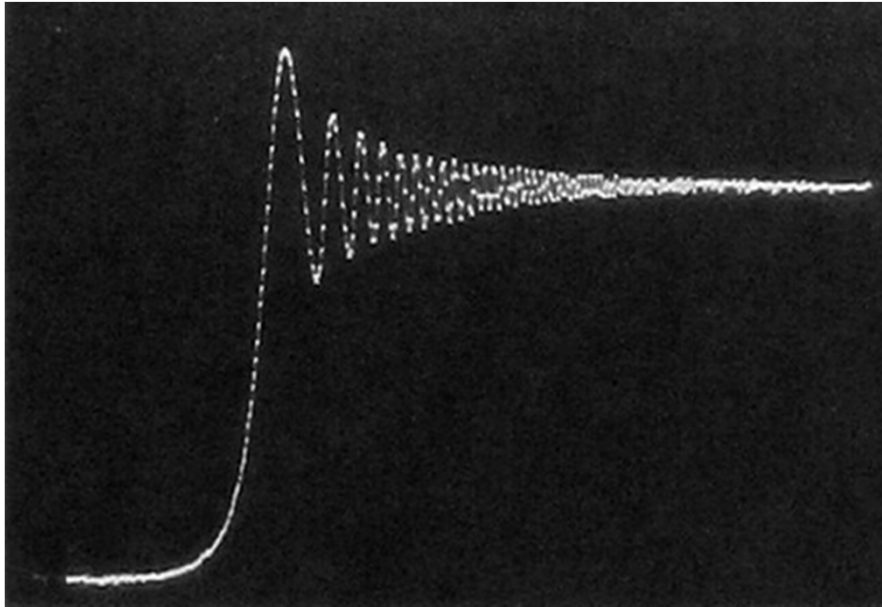
Poisson Bright Spot

- Poisson thought this result seemed absurd and dismissed Fresnel's paper
- Arago checked and found the bright spot:



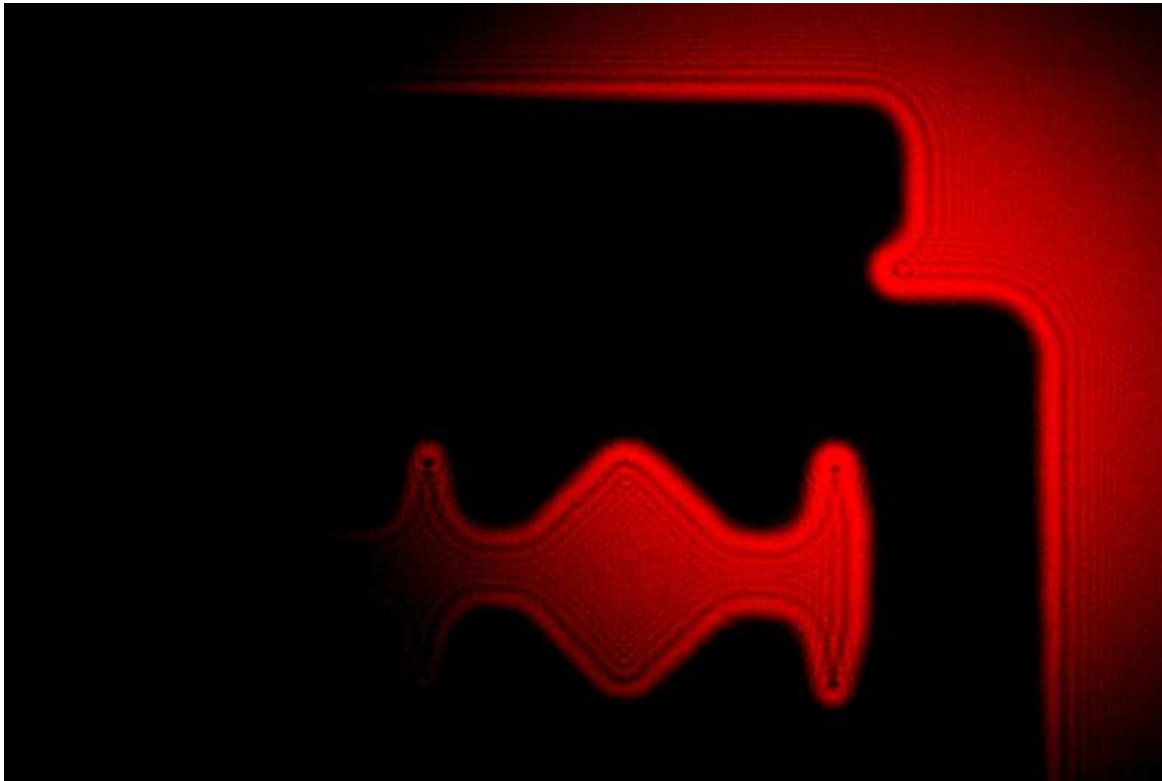
Diffraction around a 1/8" ball bearing

Diffraction from an Edge



- The edge does not form a distinct shadow

Diffraction from an Edge

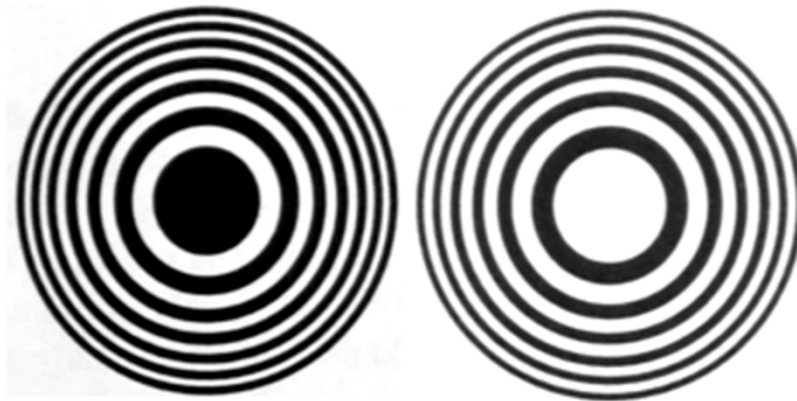


Fresnel Zone Plate

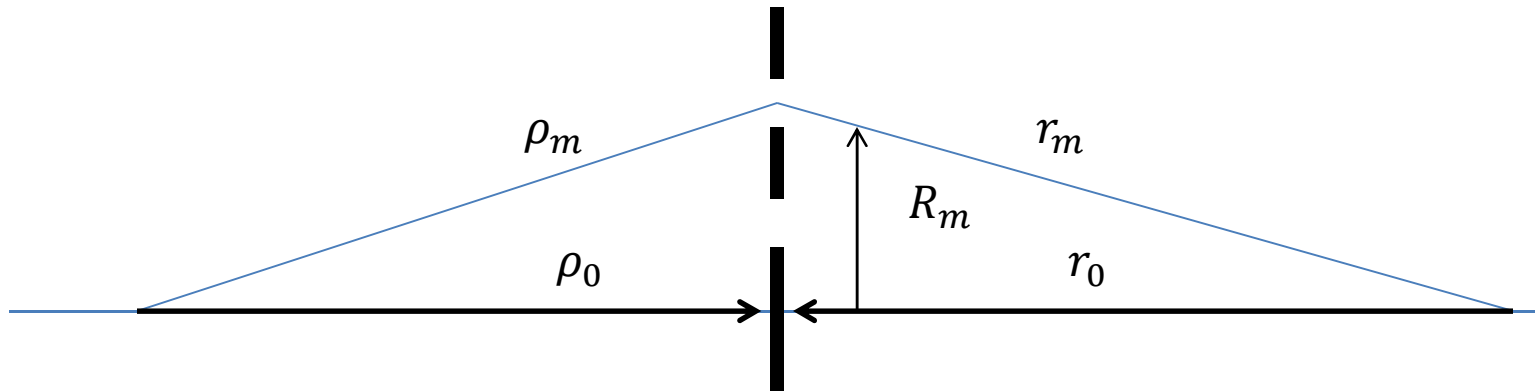
- Suppose we obscure only the even-numbered zones

$$E = |E_1| + |E_2| + |E_5| + \cdots + |E_m|$$

- The electric field at the origin is $2m$ times that of the unobstructed light
- What radii do we need to make some annular rings that block only the even-numbered zones?



Fresnel Zone Plate



$$(\rho_m + r_m) - (\rho_0 - r_0) = \frac{m\lambda}{2}$$

$$\rho_m = \sqrt{\rho_0^2 + R_m^2} \approx \rho_0 + \frac{R_m^2}{2\rho_0}$$

$$r_m = \sqrt{r_0^2 + R_m^2} \approx r_0 + \frac{R_m^2}{2r_0}$$

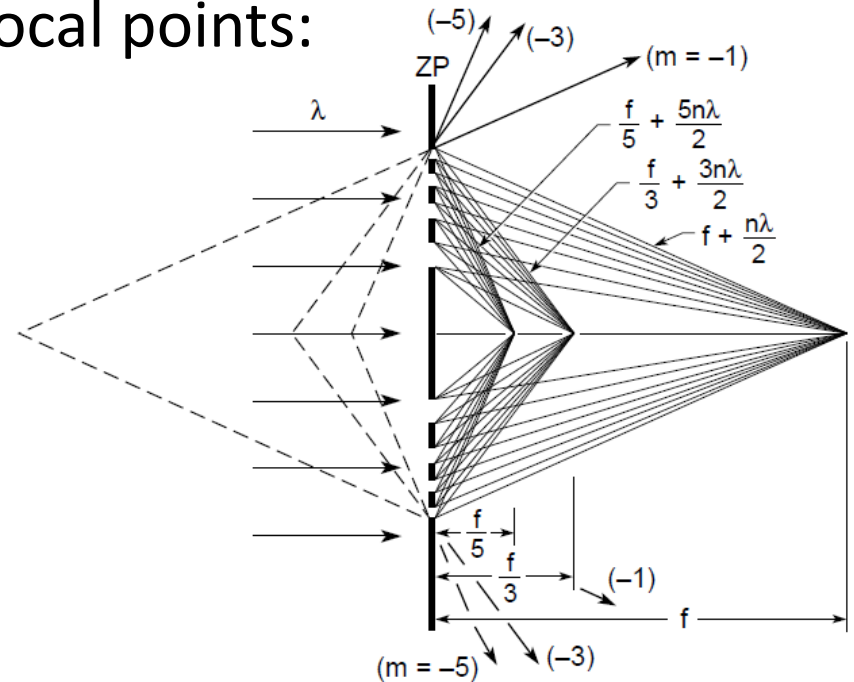
$$\frac{1}{\rho_0} + \frac{1}{r_0} \approx \frac{m\lambda}{R_m^2} = \frac{1}{f}$$

This looks like the
lens equation...

Fresnel Zone Plates

- There are also higher-order focal points:

$$f_m = \frac{1}{m} f_1$$



- Not an ideal lens
 - Works only for one wavelength (large chromatic aberration)
- But applicable to a wide range of wavelengths
 - Does not rely on weird atomic properties of transparent materials

Fresnel Zone Plate

$$R_m \approx \sqrt{mf\lambda}$$

- For green light, $\lambda = 500 \text{ nm}$
- Suppose $\rho_0 = r_0 = 10 \text{ cm}$
 - Then $R_1 = 0.223 \text{ mm}$, $R_2 = 0.316 \text{ mm}$, etc...
- But this also works for x-rays:
 $\lambda \sim 0.1 \text{ nm}$
 - Then $R_1 = 3.16 \text{ }\mu\text{m}$, $R_2 = 4.47 \text{ }\mu\text{m}$
- Challenges: very small spacing, but needs to be thick enough to absorb x-rays.

