PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 4 – French, Chapter 3

Spring 2013 Semester

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Announcements

1. Assignment #1 is due on Friday, January 18th
   - You can download it from the course web page
   - Ask questions in class if you are completely stuck
   - Make use of office hours if you are still completely stuck…
2. I have decided to change the course schedule from what was previously advertised
   - The following approach will be more logical…

Schedule for about the first half of the course

1. Free vibrations of physical systems
   - mass + spring, floating objects, torsion pendulum
   - simple pendulum, physical pendulum
   - reactive electronic circuits
   - damped oscillations
2. Forced oscillations of physical systems
   - steady state solutions
   - resonance phenomena
3. Coupled systems
   - normal modes of vibration
   - forced oscillations
4. Continuous systems
   - wave equation
   - elastic string, transmission lines

Free Vibrations of Physical Systems

- Yet again, consider a mass and a spring:

- Two ways to think about this…
  1. Hooke’s law + Newton’s second law:

$$m\ddot{x} = -kx$$

$$\ddot{x} + \omega^2 x = 0, \omega = \sqrt{k/m}$$

$$x(t) = A\ \cos(\omega t + \varphi)$$

  2. Energy conservation…

Simple Harmonic Motion

- Start from Newton's second law:

$$m\ddot{x} + kx = 0$$

- Multiply by $\dot{x}$:

$$m\dot{x}\ddot{x} + k\dot{x}x = 0$$

- Notice that

$$\frac{d}{dt}\dot{x}^2 = 2\dot{x}\ddot{x}$$

$$\frac{d}{dt}x^2 = 2x\dot{x}$$

- So we can write

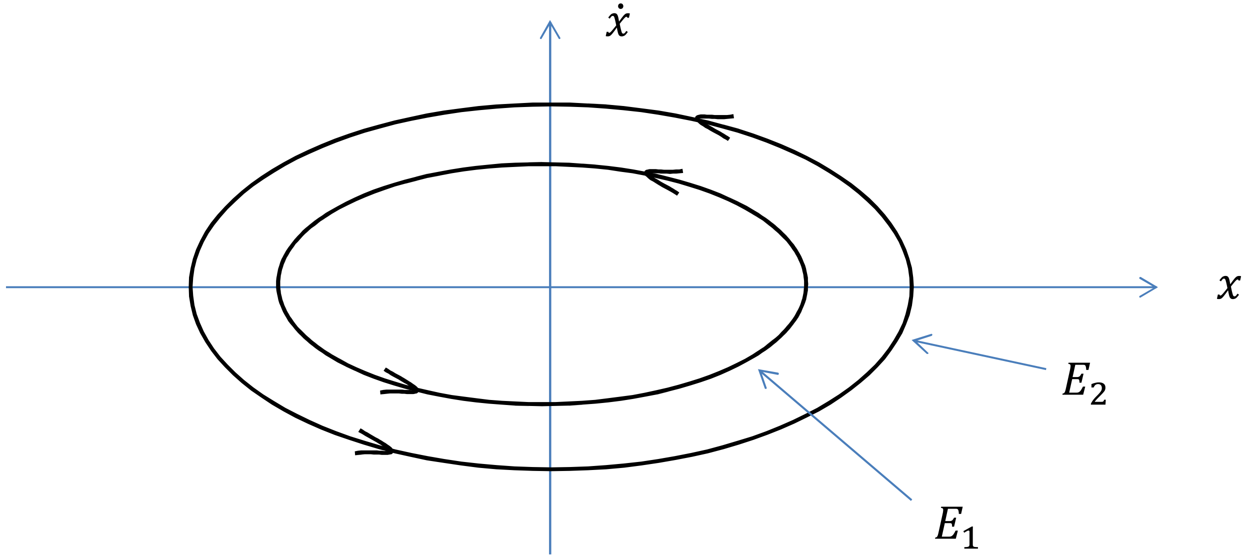
$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

- Which implies that

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E = const.$$

Simple Harmonic Motion

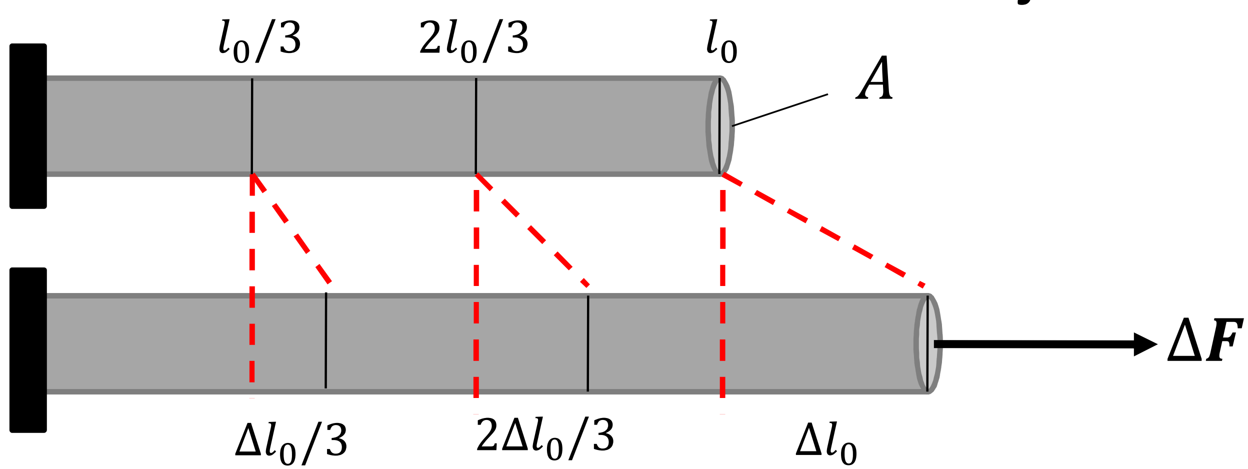
- Newton's law: $\ddot{x} + \omega^2 x = 0, x(t) = A\ \cos(\omega t + \varphi)$
- Energy conservation: $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$
- The energy conservation relation can tell us a lot about the motion even when we can't solve for $x(t)$.



This is called a "phase plot" or "phase diagram", not to be confused with something by the same name from thermodynamics…

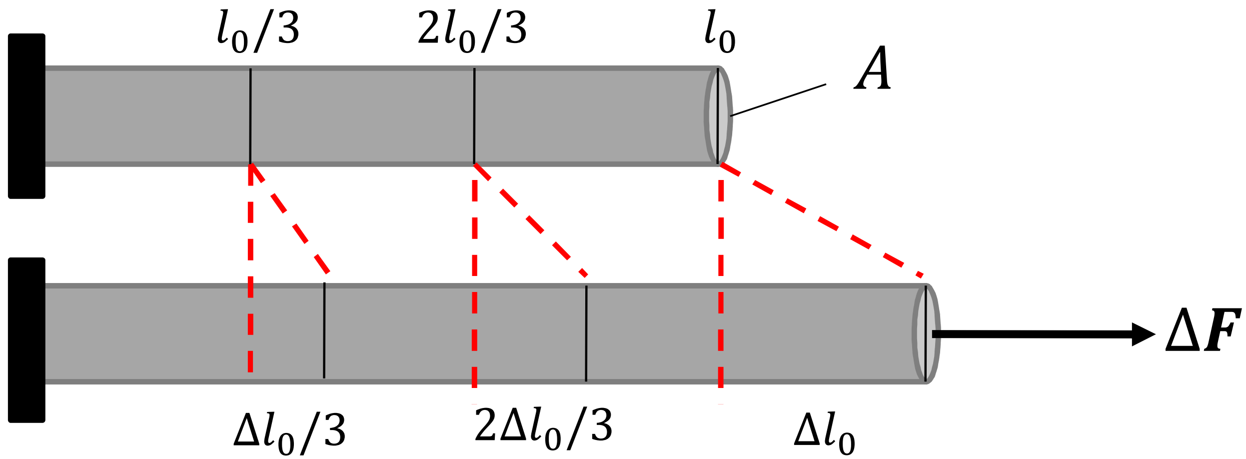
Oscillating Systems: Elastic Bodies

- Rigid bodies are usually elastic although we may not normally notice.
- What characterizes how elastic an object is?



- The extension under the force ΔF is proportional to the original length, l_0.
- Constant of proportionality: $strain \equiv \Delta l_0/l_0$

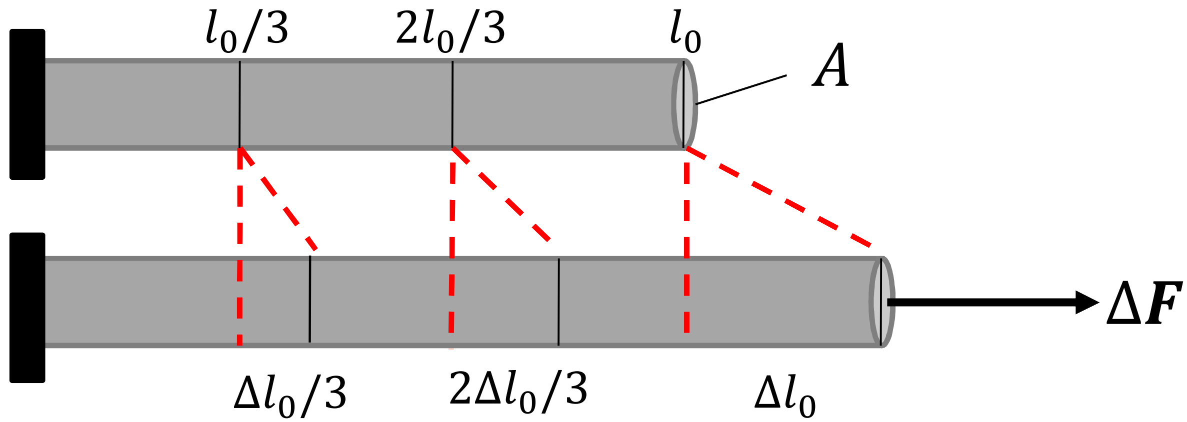
Oscillating Systems: Elastic Bodies



- The same deformation would result if ΔF were increased provided A also increased by the same amount.
- Stress is defined: $stress = \Delta F/A$
- When the strain is small (eg, $\Delta l_0/l_0 < 1\%$), the stress is proportional to the strain:

$$stress \propto strain$$

Oscillating Systems: Elastic Bodies



$$stress \propto strain$$

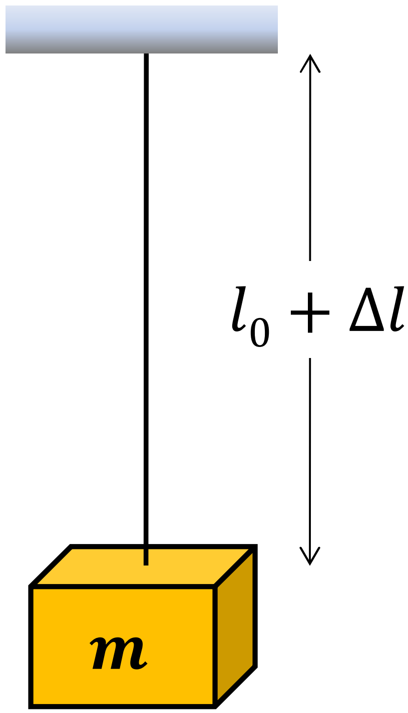
- Constant of proportionality is called Young's modulus

$$\frac{\Delta F}{A} = Y\frac{\Delta l_0}{l_0}$$

- Newton's third law: when the material is stretched by a distance x, the material will exert a reaction force

$$F = -\frac{YAx}{l_0} = -kx \text{ where } k = YA/l_0.$$

Example



- Steel has $Y = 20 \times 10^{10}\ N/m^2$
- Suppose that $m = 1\ kg,\ l_0 = 2\ m$ and has a diameter of $d = 0.5\ mm$ (24 AWG)
- Cross sectional area is

$$A = \pi \left(\frac{d}{2}\right)^2$$

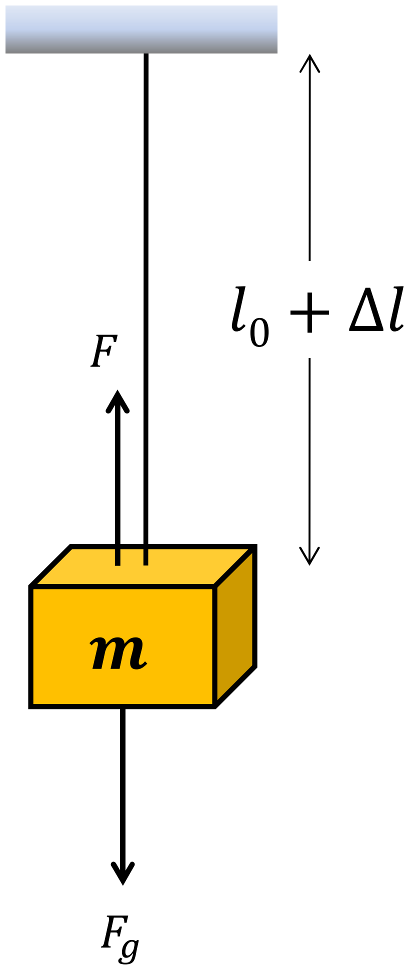
- Restoring force:

$$F = -\frac{YA\Delta l}{l_0} = -\frac{\pi}{4}\frac{Yd^2}{l_0}\Delta l = -k\Delta l$$

$$k = \frac{\pi \cdot (20 \times 10^{10}\ N/m^2) \cdot (0.0005\ m)^2}{4 \cdot (2\ m)}$$

$$= 1.96 \times 10^4\ N/m$$

Example



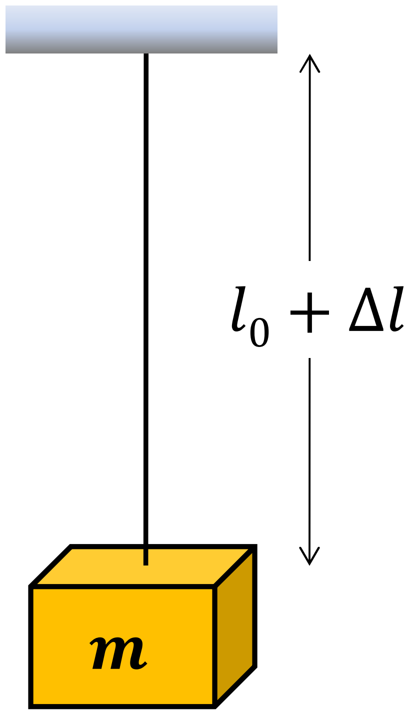
- How much will the wire stretch under the weight of the mass, m?

$$F_g = mg = k\Delta l$$

$$\Delta l = \frac{mg}{k} = \frac{(1\ kg)\cdot(9.81\ N/kg)}{1.96\times 10^4\ N/m}$$

$$= 5.00\times 10^{-4}\ m$$

Example



- Newton's second law:

$$m\frac{d^2}{dt^2}\Delta l = -k\Delta l$$

$$\frac{d^2}{dt^2}\Delta l + \frac{k}{m}\Delta l = 0$$

$$\frac{d^2}{dt^2}\Delta l + \omega^2\Delta l = 0$$

- Solutions can be written

$$\Delta l(t) = A\ \cos(\omega t + \varphi)$$

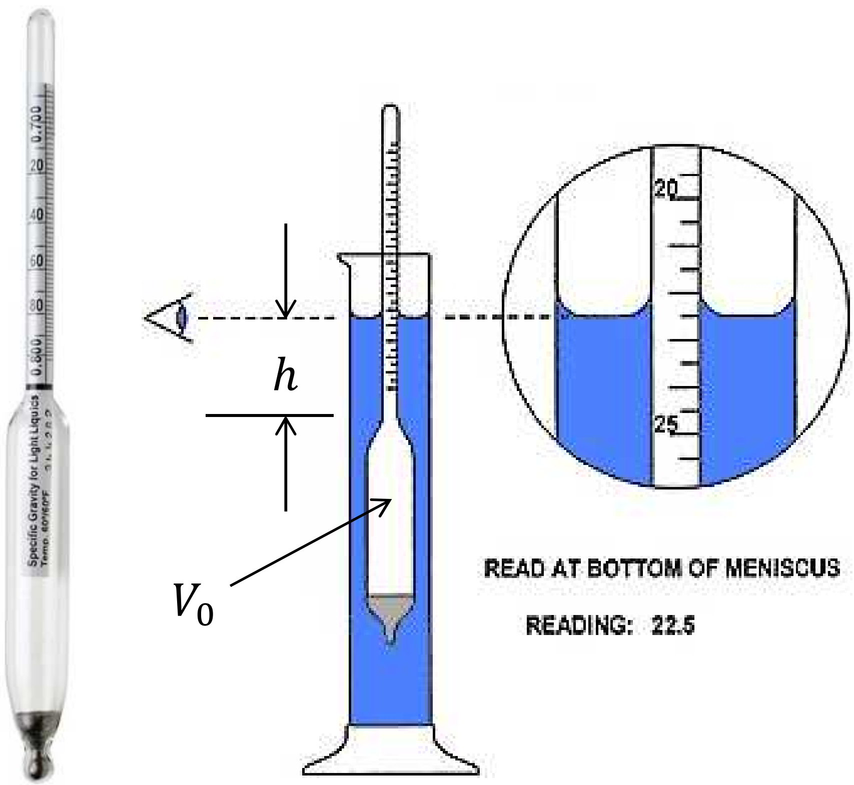
- Oscillation frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{k/m} = \frac{1}{2\pi}\sqrt{\frac{1.96 \times 10^4\ N/m^2}{1\ kg}}$$

$$= 22.3\ Hz$$

Floating Objects

- Hygrometer: measures density of liquids



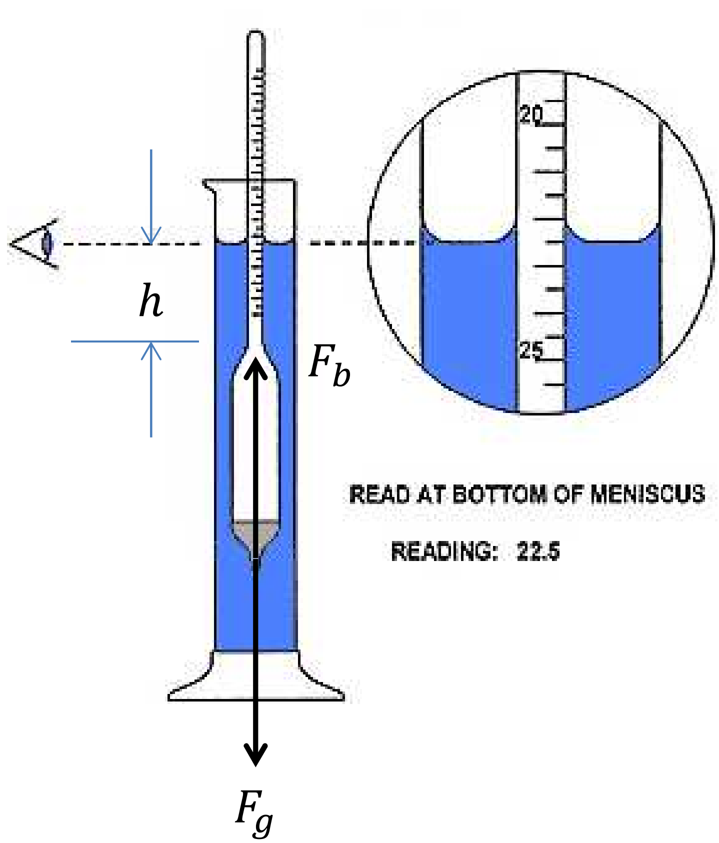
Archimedes’ principle: Buoyant force is equal to the weight of the volume of liquid displaced.

If the stem has a diameter of d then the displaced volume is

$$V = V_0 + \pi h \left(\frac{d}{2}\right)^2$$

$$F_b = \rho g \left(V_0 + \pi h \left(\frac{d}{2}\right)^2 \right)$$

Floating Objects



When in static equilibrium,

$$F_b = F_g$$

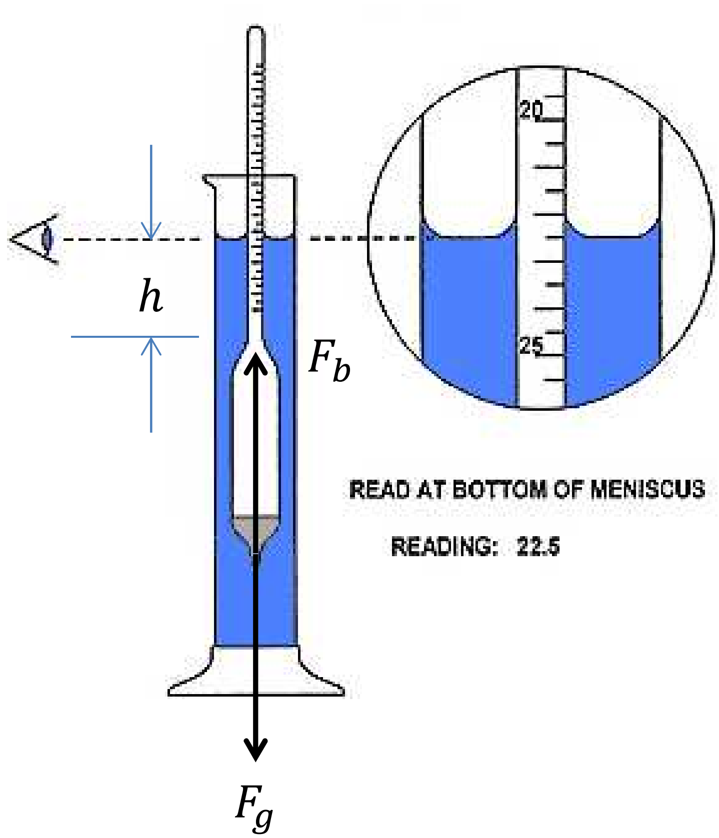
$$\rho g \left(V_0 + \pi h \left(\frac{d}{2} \right)^2 \right) = mg$$

$$h = \frac{m/\rho - V_0}{\pi d^2/4}$$

When the hydrometer is displaced by an additional distance Δh, the net force is

$$F = -\pi \rho g \left(\frac{d}{2} \right)^2 \Delta h$$

Floating Objects

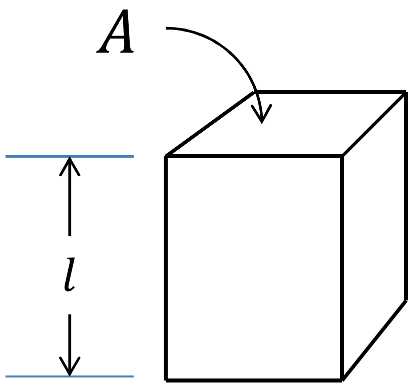


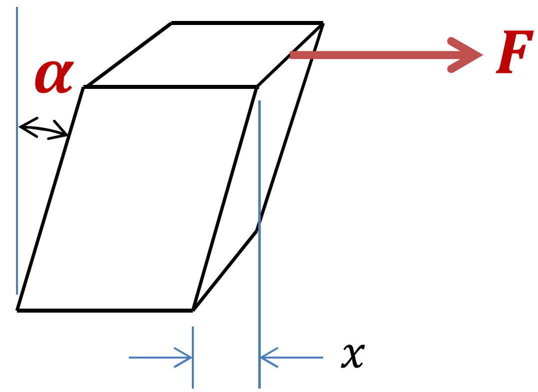
$$m\frac{d^2}{dt^2}\Delta h = -\pi\rho g\left(\frac{d}{2}\right)^2 \Delta h$$

$$\frac{d^2}{dt^2}\Delta h + \omega^2\Delta h = 0$$

where $\omega = \frac{d}{2}\sqrt{\frac{\pi\rho g}{m}}$

Shear Forces





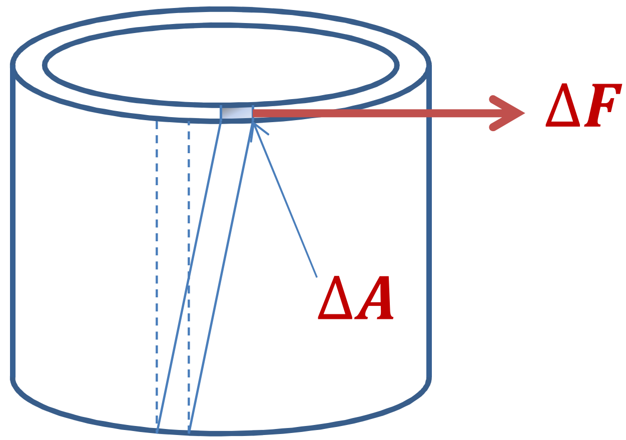
- Angle α is proportional to F and inversely proportional to A:

$$\frac{F}{A} = n\alpha \ \approx n\frac{x}{l}$$

- The constant of proportionality is called the *shear modulus*, denoted n.
- For example, steel has $n = 8 \times 10^{10}\ N/m^2$

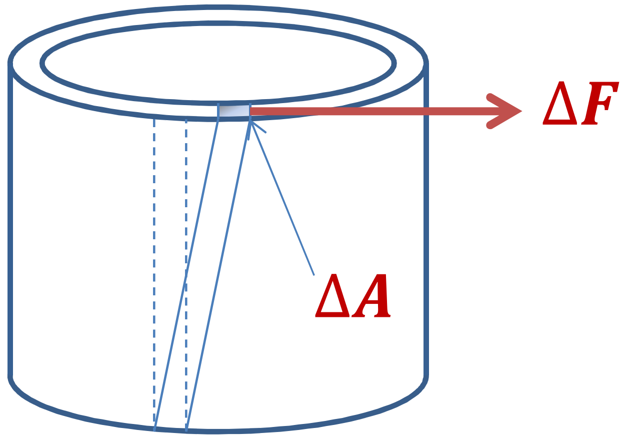
Shear Forces

- Torsion of a thin-walled tube of radius r and length l twisted through an angle θ:



- Angle of deflection: $\frac{x}{l} = \frac{r\theta}{l}$
- Shear force: $\Delta F = \frac{nr\theta\Delta A}{l}$

Shear Forces



- Differential element of torque:

$$\Delta M = r\Delta F$$

- Differential element of area:

$$\Delta A = r\Delta r\Delta\varphi$$

- Integrate around the circle…

$$M = \int dM = r\int dF = \frac{nr^2\theta}{l}\int dA = \frac{nr^2\theta}{l}\int_0^{2\pi}(r\Delta r)d\varphi = \frac{2\pi nr^3\Delta r\ \theta}{l}$$

- Total torque on a solid cylinder of radius R: integrate over r from 0 to R.

$$M = \frac{2\pi n\theta}{l}\int_0^R r^3 dr = \frac{\pi nR^4\theta}{2l}$$

Torsion Pendulum

- Suppose an object with moment of inertia $I = 0.00167\ kg \cdot m^2$ is suspended from a steel wire of length $\ell = 2\ m$ with a diameter of $d = 0.5\ mm$ (24 AWG).

$$I\ddot{\theta} = -\frac{\pi n R^4 \theta}{2\ell}$$

$$\ddot{\theta} + \omega^2 \theta = 0$$

where $\omega = \sqrt{\frac{\pi n R^4}{2I\ell}} = \sqrt{\frac{\pi n d^4}{32I\ell}}$

- Frequency of oscillation:

$$f = \frac{1}{2\pi}\sqrt{\frac{\pi \cdot (8 \times 10^{10}\ N/m^2) \cdot (0.0005\ m)^4}{32 \cdot (0.00167\ kg \cdot m^2) \cdot (2\ m)}} = 0.061\ \text{Hz}$$