Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 39 – Diffraction

Spring 2013 Semester

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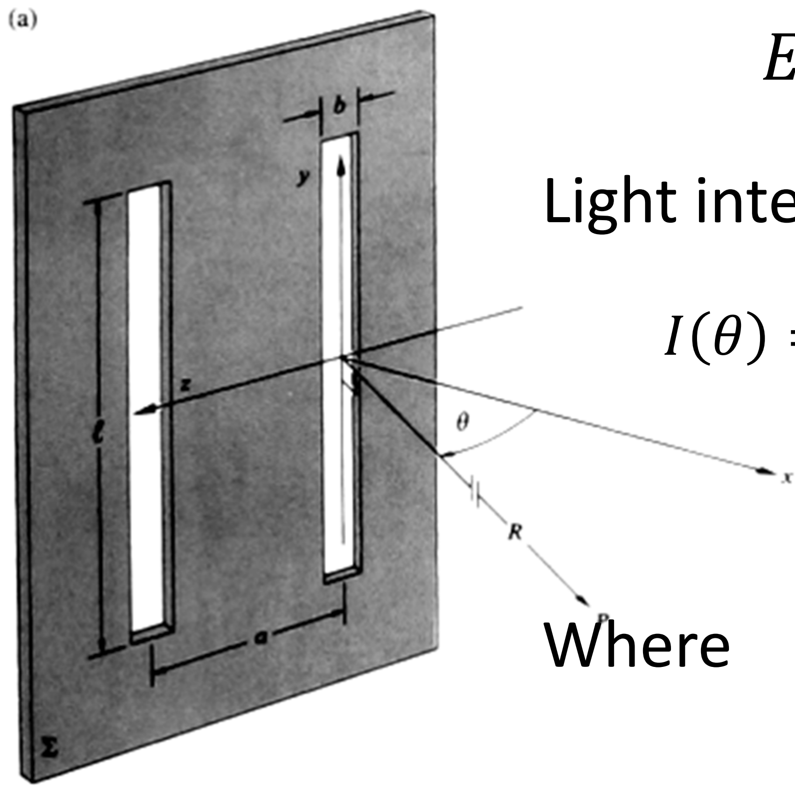
Double-Slit Fraunhofer Diffraction



$$E = \frac{\mathcal{E}_L b}{R} \frac{\sin\beta}{\beta}\left(1 + e^{ika\, sin\, \theta}\right)$$

Light intensity:

$$I(\theta) = 2I(0)\left(\frac{\sin\beta}{\beta}\right)^2 (1 + \cos(ka\sin\theta)\,)$$

$$= 4I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha$$

Where

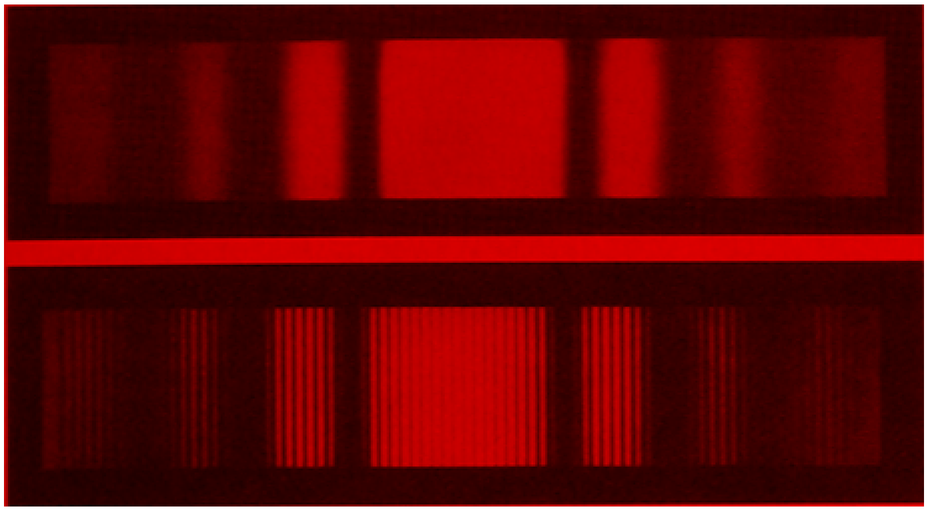
$$\alpha = \frac{1}{2} ka\sin\theta$$

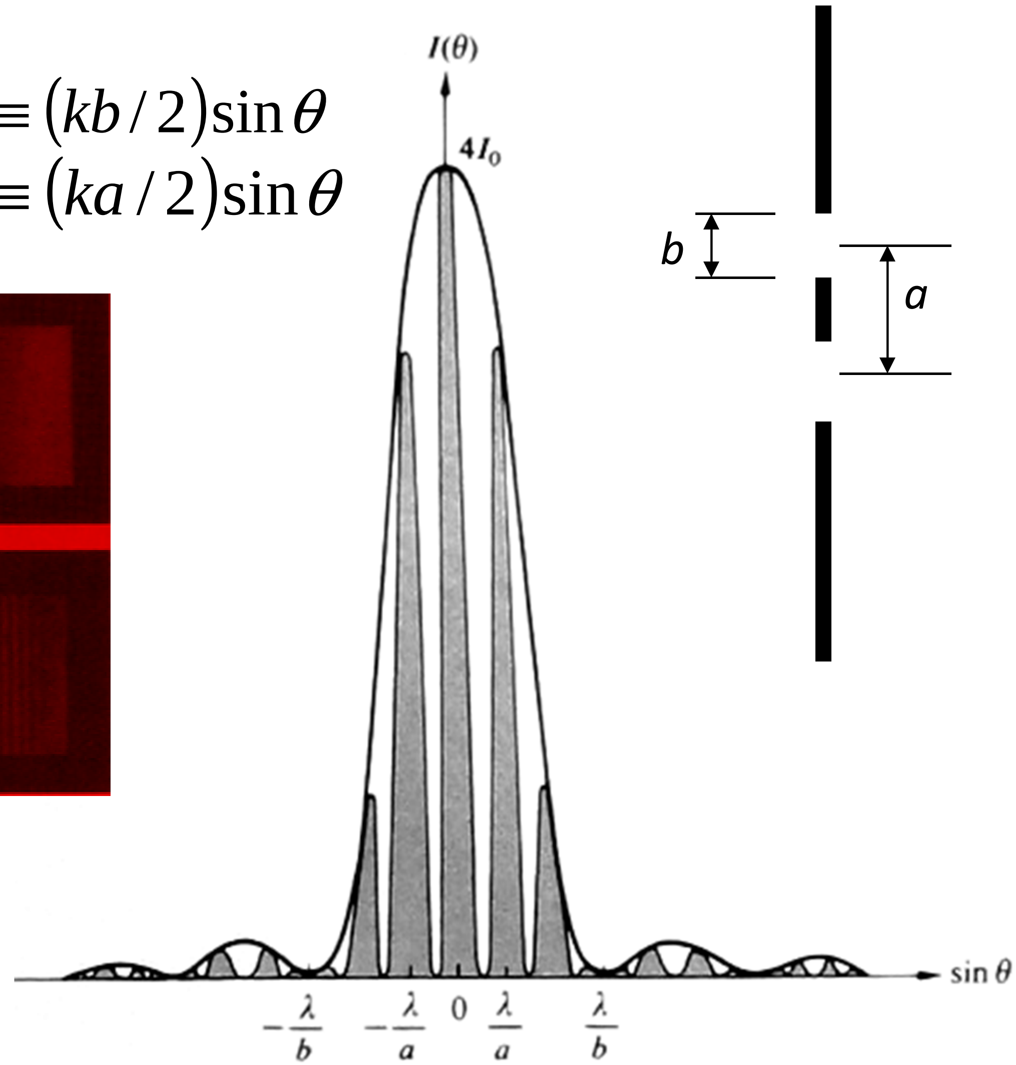
Since $a > b$, $\cos\alpha$ oscillates more rapidly than $\sin\beta$

Double Slit: Fraunhofer Diffraction

$$I(\theta) = 4I_0\left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha$$

$$\beta \equiv (kb/2)\sin\theta$$
$$\alpha \equiv (ka/2)\sin\theta$$





Minima: $\alpha = \pm\pi/2, \pm 3\pi/2$

$$a\sin\theta = (m + 1/2)\lambda$$

m=0, ±1, ±2…

or: $\beta = m\pi$, where m=±1, ±2…

$$b\sin\theta = m\lambda$$

Three-Slit Fraunhofer Diffraction

- Three slits, spaced by a distance a

$$E = \frac{\mathcal{E}_L}{R}\int_{-b/2}^{+b/2} e^{iky\sin\theta}\,dy + \frac{\mathcal{E}_L}{R}\int_{a-b/2}^{a+b/2} e^{iky\sin\theta}\,dy + \frac{\mathcal{E}_L}{R}\int_{2a-b/2}^{2a+b/2} e^{iky\sin\theta}\,dy$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}\left(1 + e^{ika\sin\theta} + e^{2ika\sin\theta}\right)$$

- A useful identity:

$$\left(1 + \left(e^{i\delta}\right) + \left(e^{i\delta}\right)^2 + \cdots + \left(e^{i\delta}\right)^{N-1}\right) = (e^{i\delta N} - 1)/(e^{i\delta} - 1)$$

Three-Slit Fraunhofer Diffraction

$$E = \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}\frac{e^{3i\delta}-1}{e^{i\delta}-1}$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}\frac{e^{3i\delta/2}}{e^{i\delta/2}}\frac{e^{3i\delta/2}-e^{-3i\delta/2}}{e^{i\delta/2}-e^{-i\delta/2}}$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}e^{i\delta}\frac{\sin 3\delta/2}{\sin\delta/2}$$

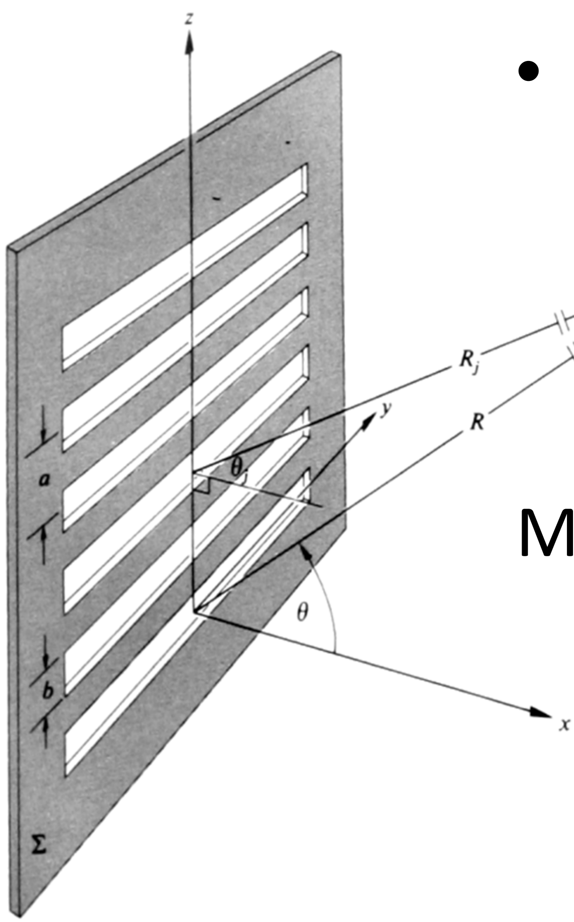
$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}e^{ika\sin\theta}\frac{\sin\left(\frac{3}{2}ka\sin\theta\right)}{\sin\left(\frac{1}{2}ka\sin\theta\right)}$$

$$= \frac{\mathcal{E}_L b}{R}\frac{\sin\beta}{\beta}e^{2i\alpha}\frac{\sin 3\alpha}{\sin\alpha}$$

Light intensity: $I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin 3\alpha}{\sin\alpha}\right)^2$

Three-Slit Fraunhofer Diffraction

- Light intensity: $I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin 3\alpha}{\alpha}\right)^2$

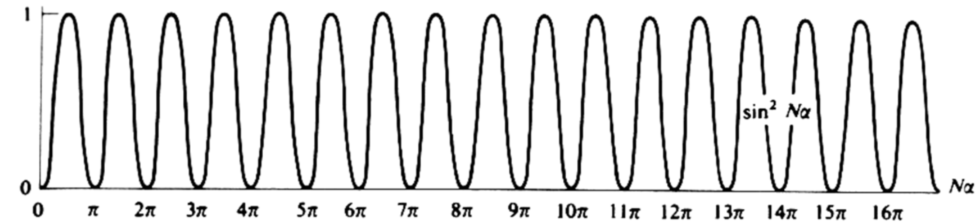


- In general, when there are N slits:

$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\alpha}\right)^2$$

Maxima occur when $\alpha = \frac{1}{2}ka\sin\theta = m\pi$

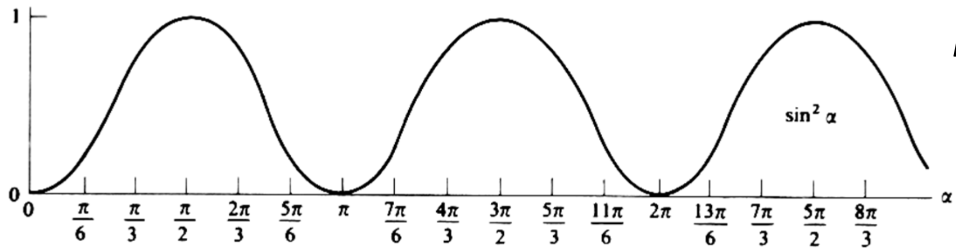
$$a\sin\theta = m\lambda$$

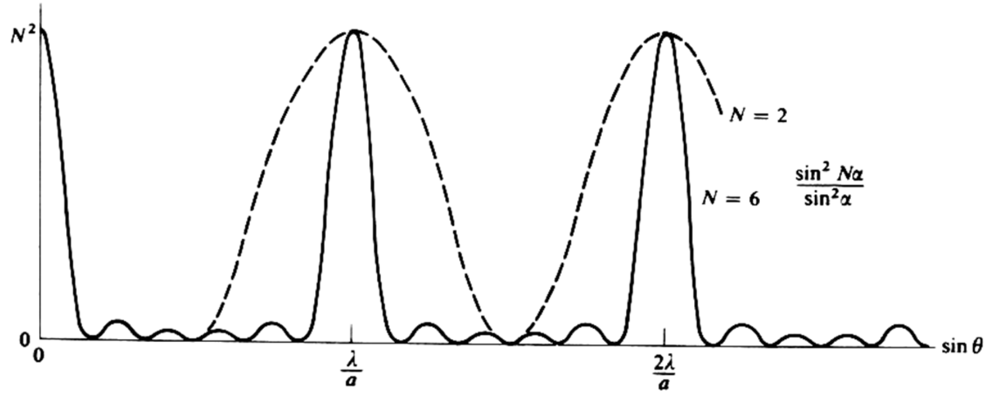


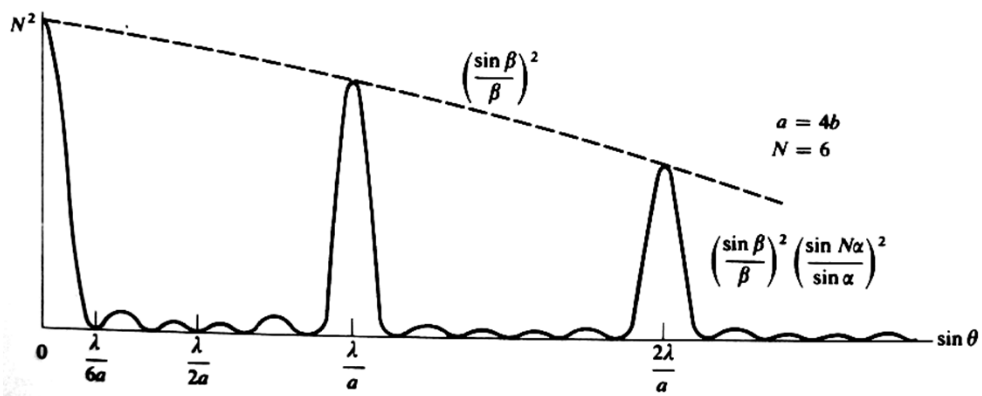
$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

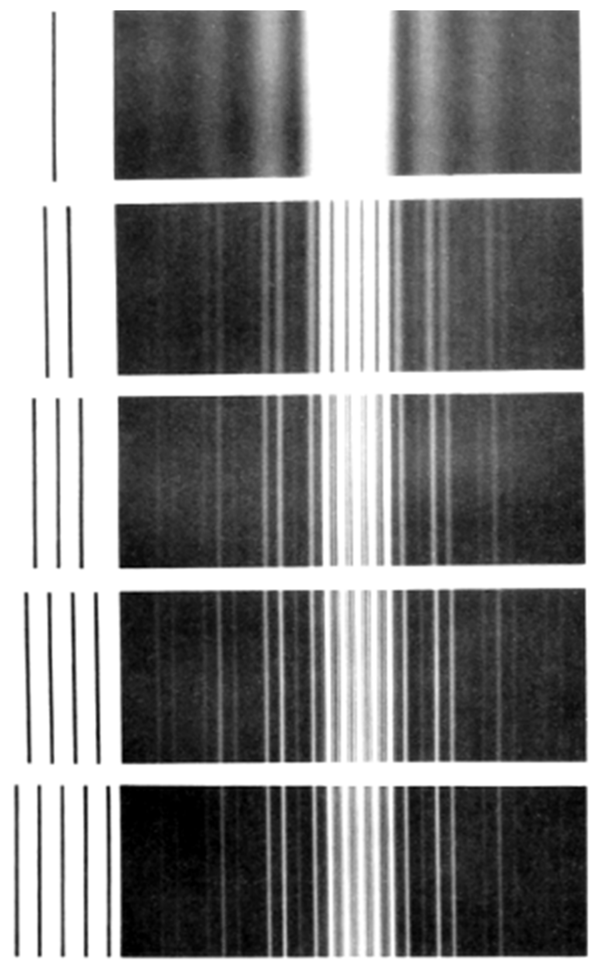
a = 4b

N=6



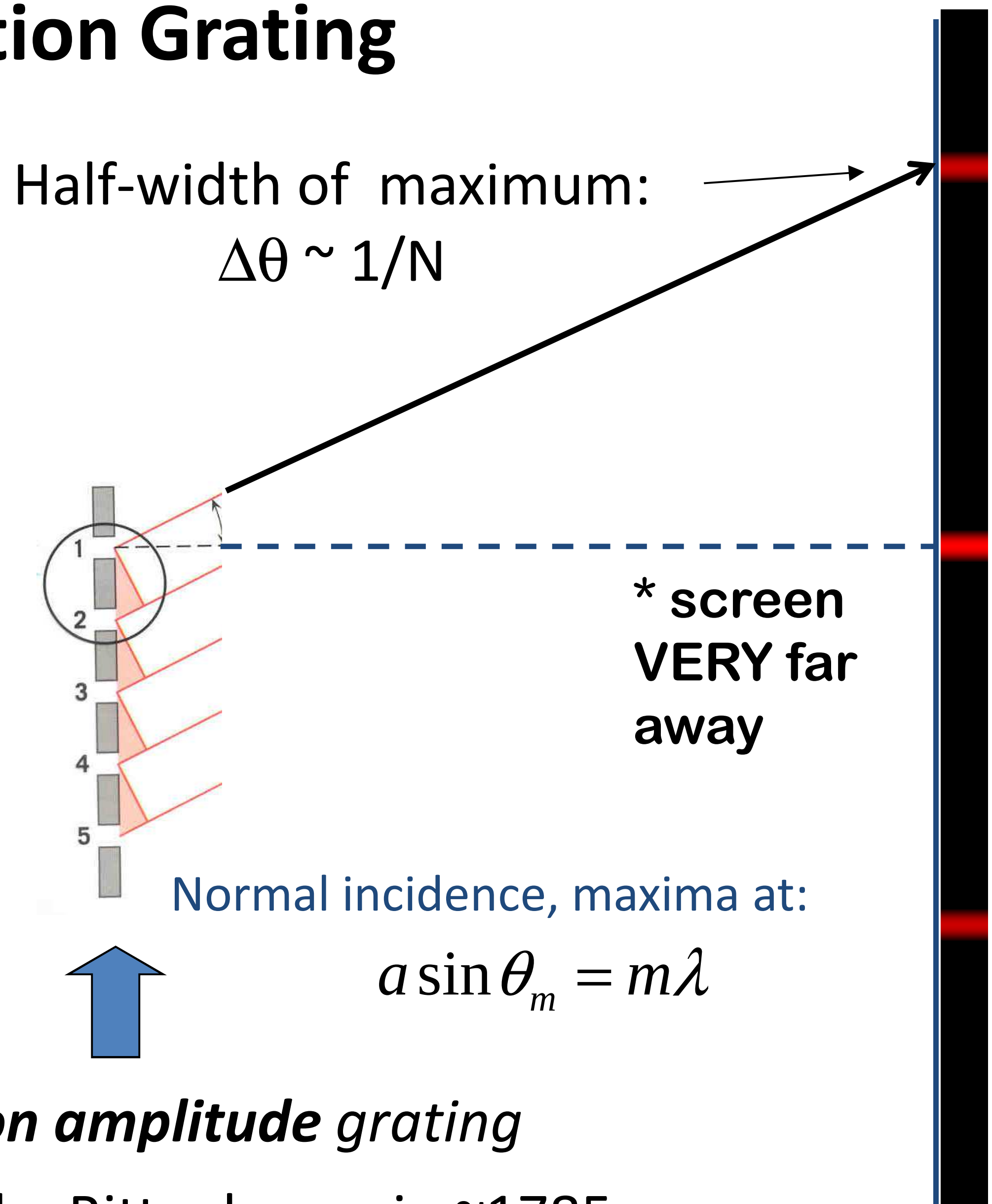






Diffraction Grating

Usually gratings have thousands of slits and are characterized by the number of slits per cm (for example: 6000 cm^{-1})

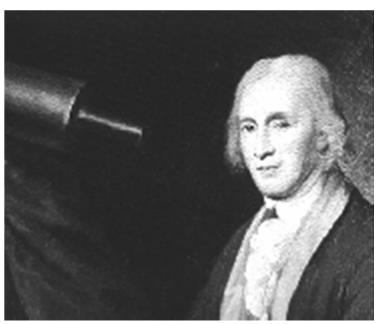


Normal incidence, maxima at:

$$a \sin\theta_m = m\lambda$$

Transmission amplitude *grating*

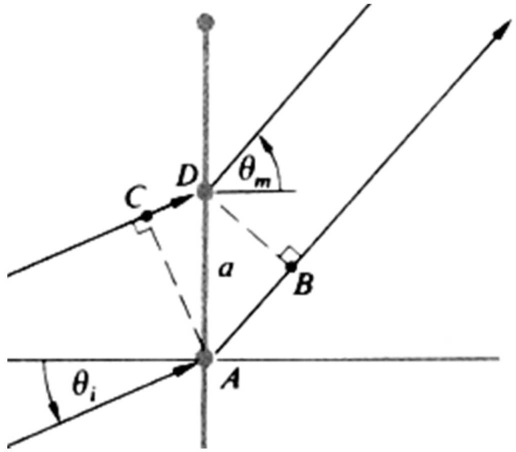
Introduced by Rittenhouse in ~1785

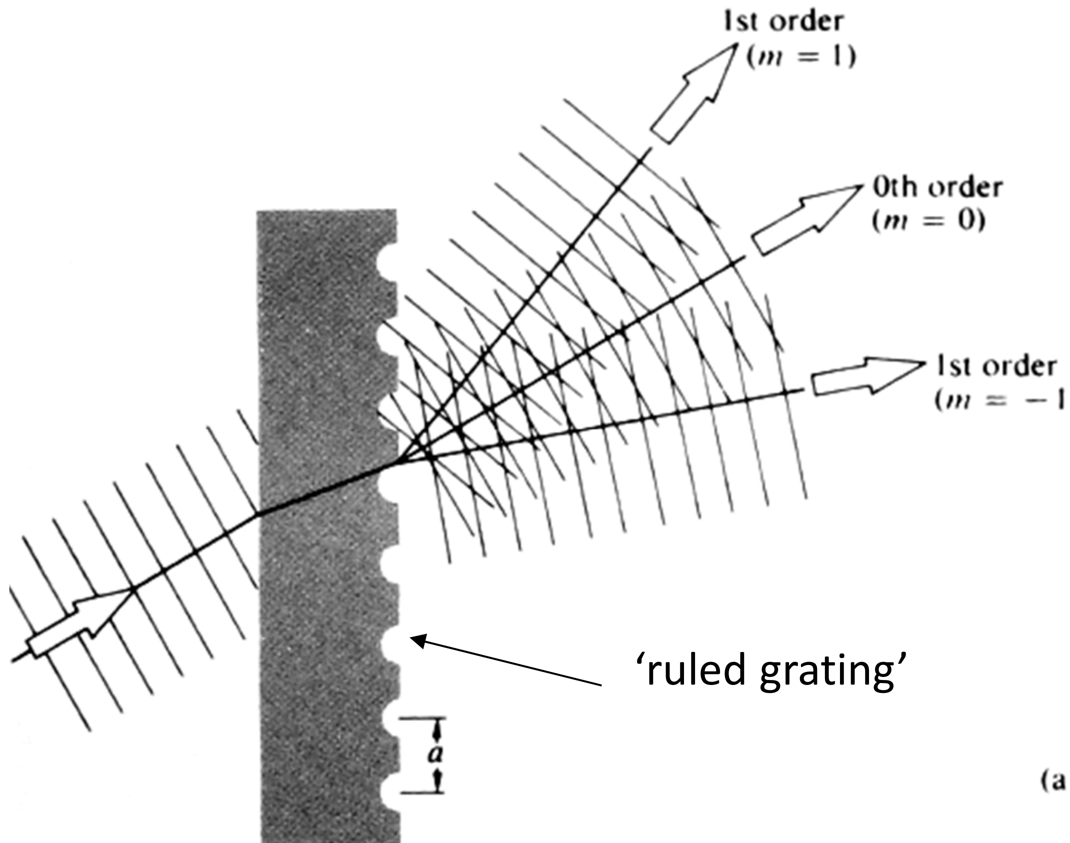


David Rittenhouse
1732 - 1796

Transmission Phase Grating

General case:
Incidence angle $\theta_i \neq 0$

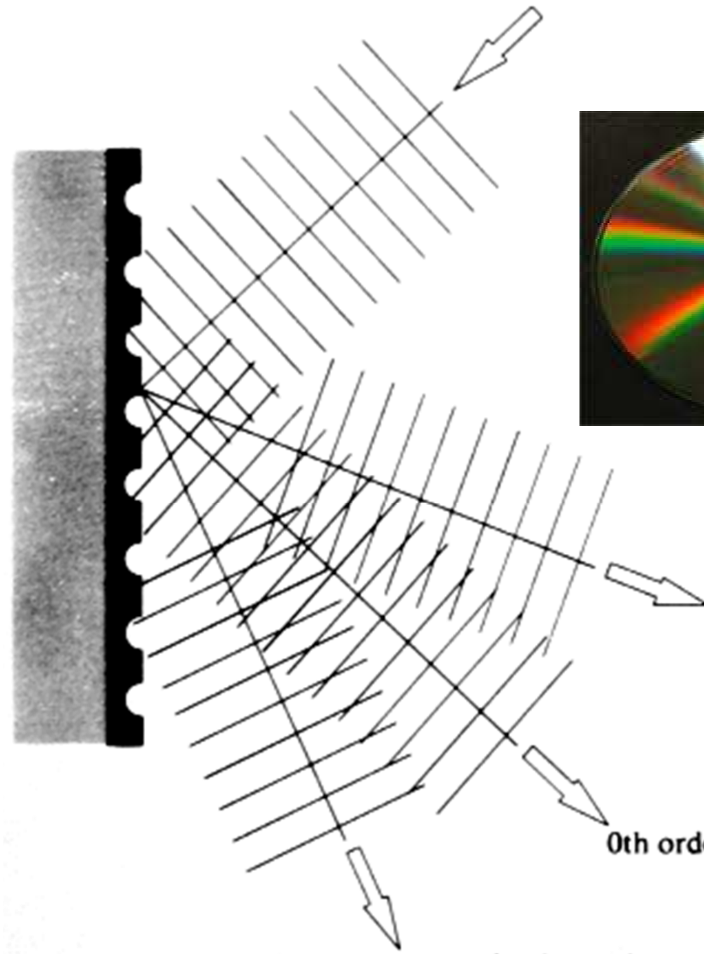




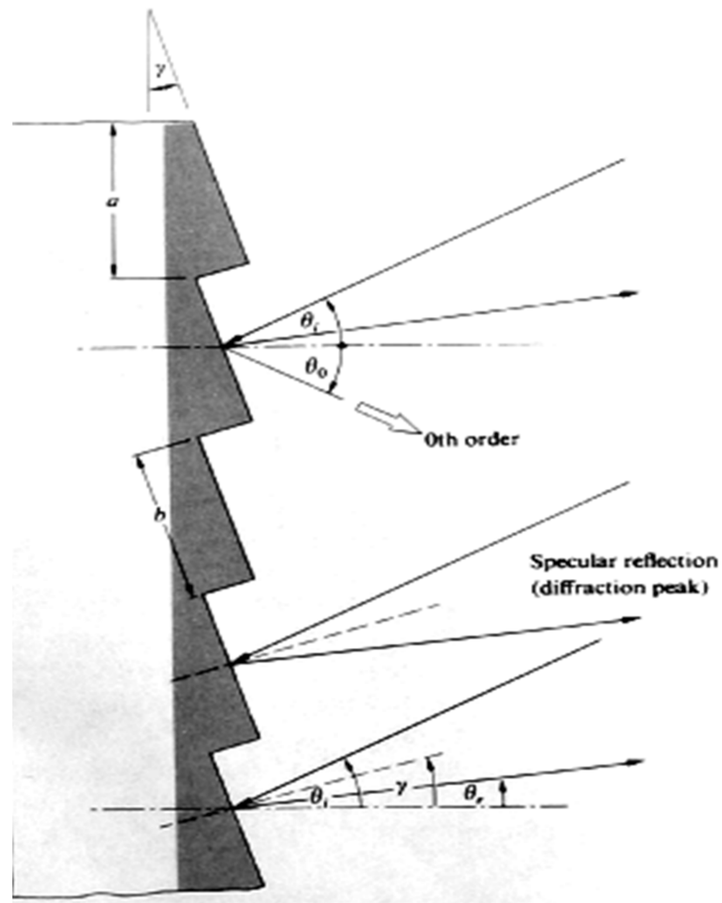
$$\overline{AB} - \overline{CD} = a\sin\theta_m - a\sin\theta_i$$

Maxima: $a(\sin\theta_m - \sin\theta_i) = m\lambda$ for arbitrary incidence angle

Reflection Phase Grating



Examples: CD disk
Finely machined surfaces

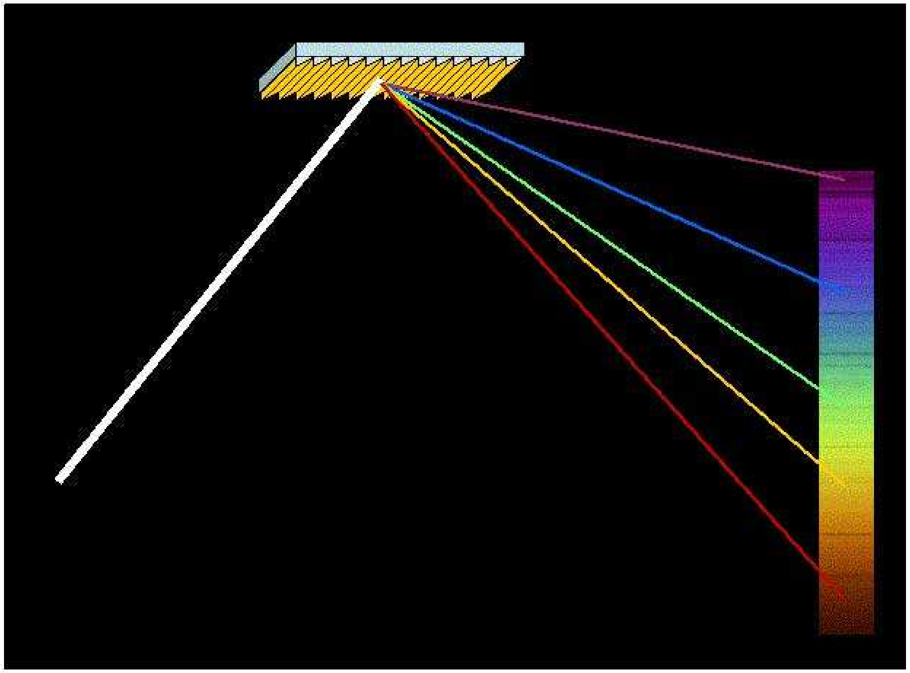


Diffraction Grating Spectrometers

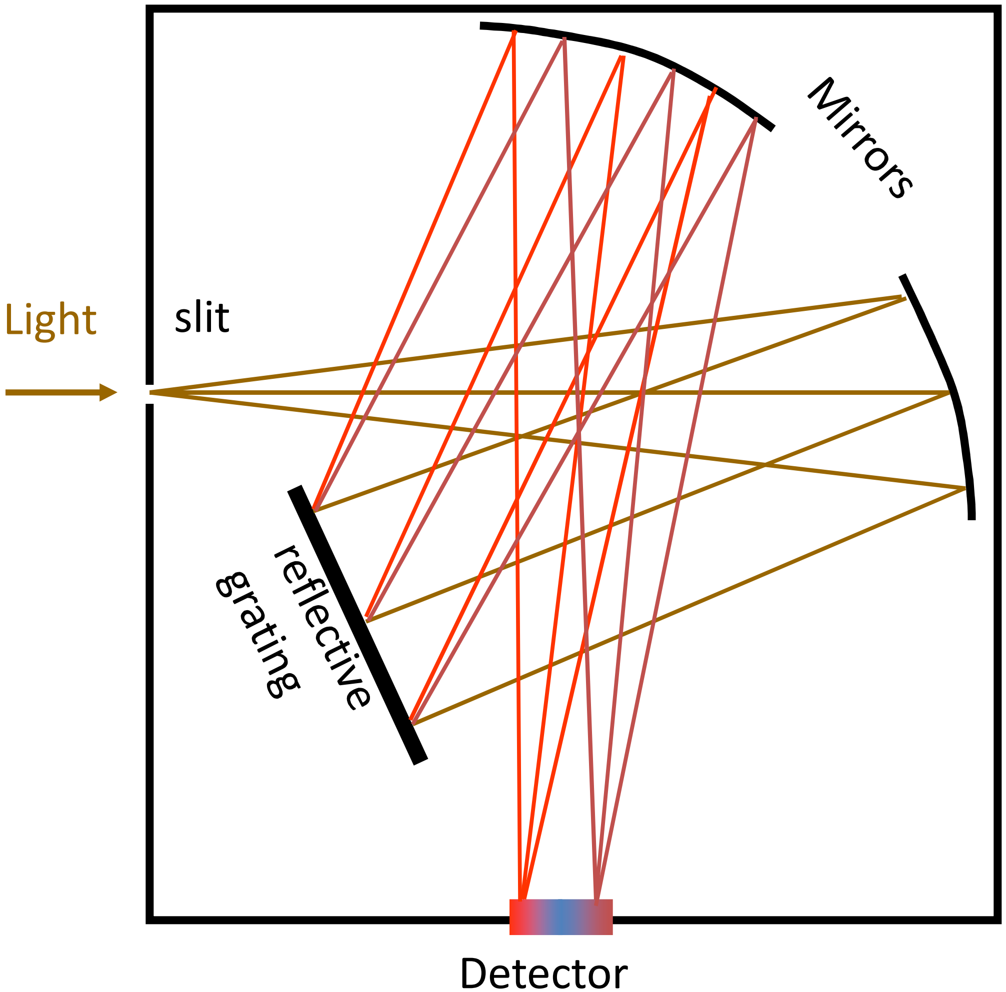
- Angle of maximum intensity depends on wavelength:

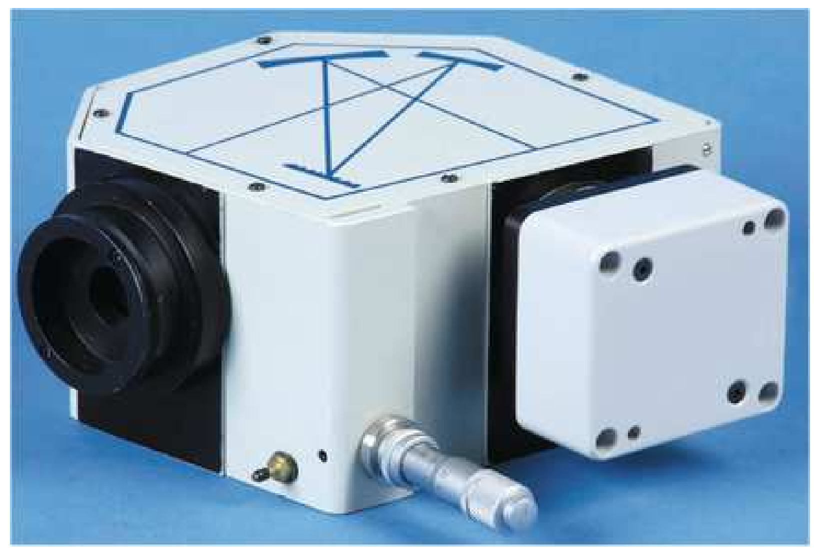
$$\sin\theta = \frac{m\lambda}{a}$$

- Diffraction gratings are used to separate and analyze the spectrum of light:

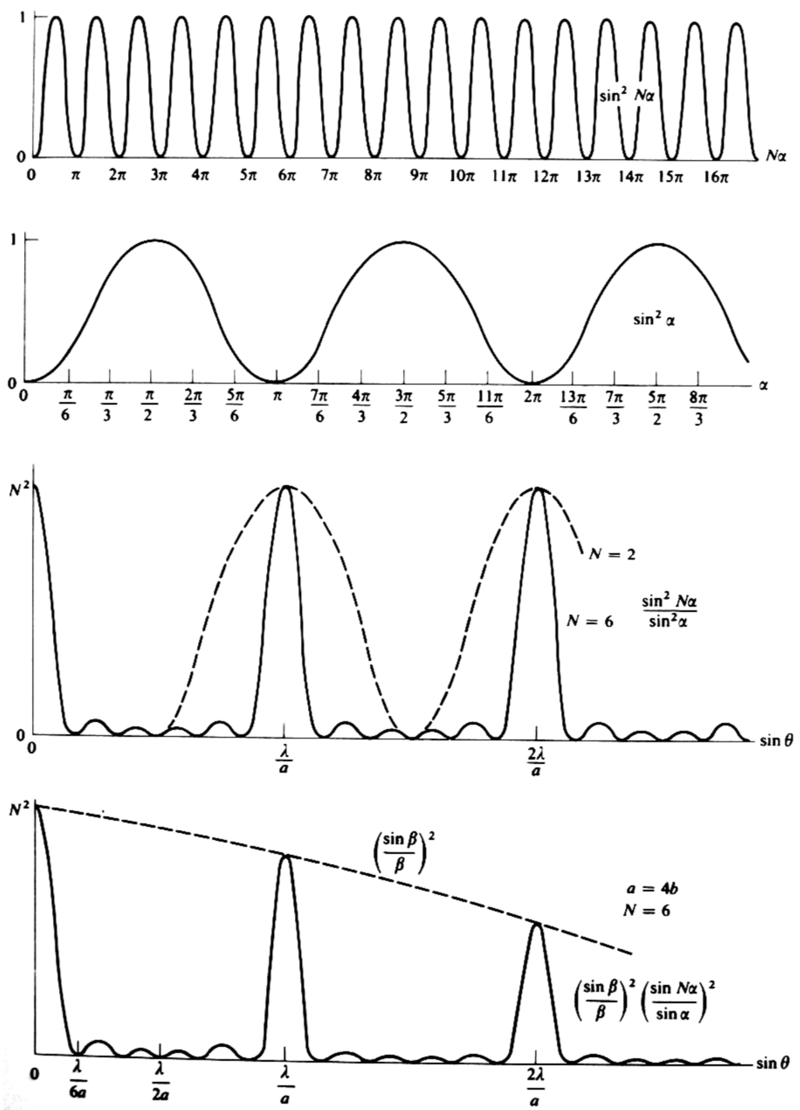


Diffraction Grating Spectrograph





Width of Spectral Lines



$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\alpha}{\alpha}\right)^2$$

- Maxima occur when

$$\alpha = m\pi$$

- Otherwise, zeros occur when

$$N\alpha = m'\pi$$

- Zeros on either side of a peak

$$N\alpha_{\pm} = (Nm \pm 1)\pi$$

- Width of peak:

$$\Delta\alpha = \alpha_+ - \alpha_- = \frac{2\pi}{N}$$

Width of Spectral Lines

$$\Delta\alpha = \frac{2\pi}{N}$$

$$\alpha = \frac{1}{2} ak \sin\theta = \frac{\pi a}{\lambda} \sin\theta$$

$$\Delta\alpha = \frac{\pi a}{\lambda} \cos\theta \, \Delta\theta$$

- Angular resolution:

$$\Delta\theta = \frac{2\lambda}{Na\cos\theta}$$

$$(\Delta\theta)_{min} = \frac{1}{2}\Delta\theta = \frac{\lambda}{Na\cos\theta}$$

Angular Dispersion

- The angle depends on the wavelength:

$$a \sin\theta = m\lambda$$

$$a \cos\theta \, \Delta\theta = m\Delta\lambda$$

$$\frac{d\theta}{d\lambda} = \frac{m}{a \cos\theta}$$

- Chromatic resolving power is defined:

$$\mathcal{R} \equiv \frac{\lambda}{(\Delta\lambda)_{min}}$$

$$(\Delta\lambda)_{min} = \frac{a\cos\theta}{m}(\Delta\theta)_{min} = \frac{a\cos\theta}{m}\,\frac{2\lambda}{Na\cos\theta} = \frac{\lambda}{Nm}$$

$$\mathcal{R} = Nm = \frac{Na\sin\theta}{\lambda}$$

Resolving Power

- The chromatic resolving power is proportional to Na
- Example: 6000 lines per cm, 15 cm width

$$N = (6000\, lines/cm) \times (15\ cm) = 90{,}000$$

$$a = 1/(6000\, lines/cm) = 1.667\ \mu m$$

$$\left.\begin{aligned} \lambda &= 588.991\, nm \\ \lambda' &= 589.595\, nm \end{aligned}\right\} \Delta\lambda = 0.604\, nm$$

$m = 2$ (second order)

$$\sin\theta = \frac{m\lambda}{a} =$$

$$2 \times (589\ \times 10^{-7} cm) \times (6000\, lines/cm)$$

$$= 0.707$$

$$\mathcal{R} = mN = 2 \times 90{,}000 = 180{,}000$$

$$(\Delta\lambda)_{min} = \frac{\lambda}{\mathcal{R}} = \frac{(589\, nm)}{180{,}000} = 0.00327\, nm$$

Overlapping Orders

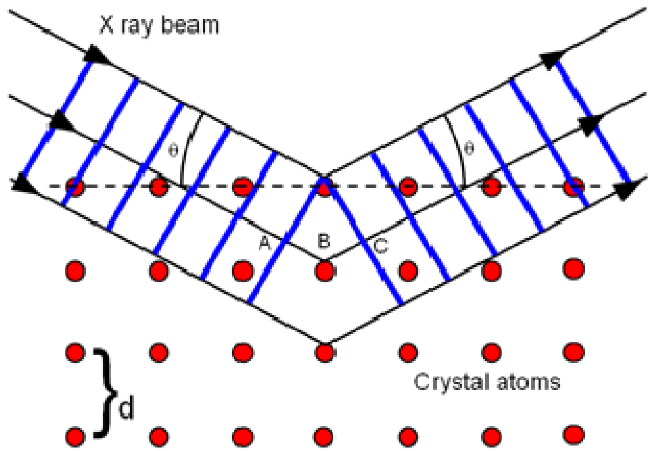
- Confusion can arise when a spectral line at one order overlaps with a different spectral line at a different order:

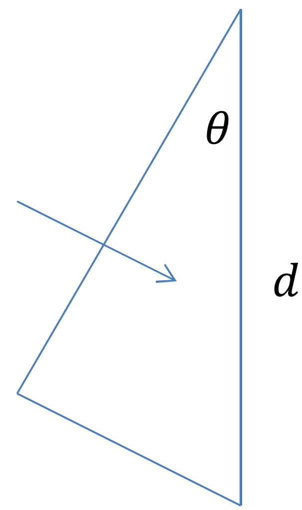
$$\sin\theta = \frac{(m+1)\lambda}{a} = \frac{m\lambda'}{a} = \frac{m(\lambda + \Delta\lambda)}{a}$$

$$\Delta\lambda = \frac{\lambda}{m} \equiv (\Delta\lambda)_{fsr} \text{ (free spectral range)}$$

X-ray Diffraction

- With short enough wavelengths, the atoms in a crystal lattice form the diffraction grating:



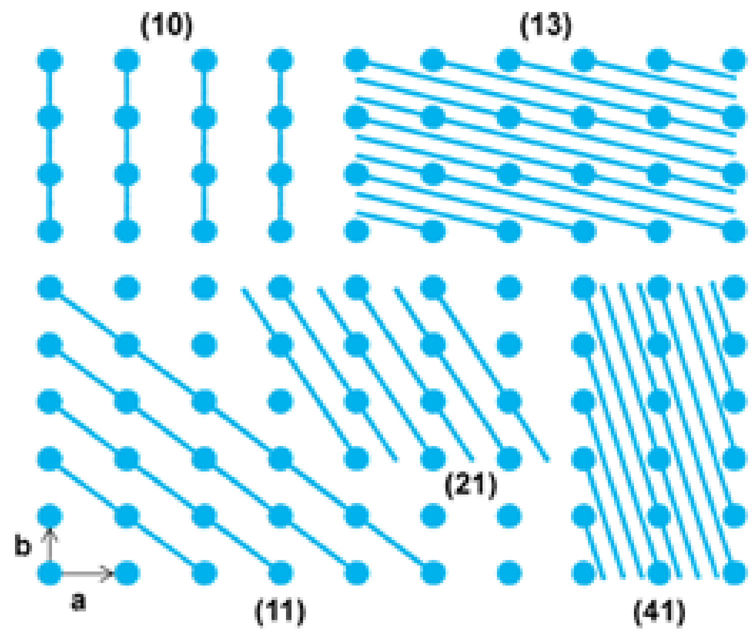


$$2d \sin \theta = n\lambda$$

(Bragg's Law)

X-ray Diffraction

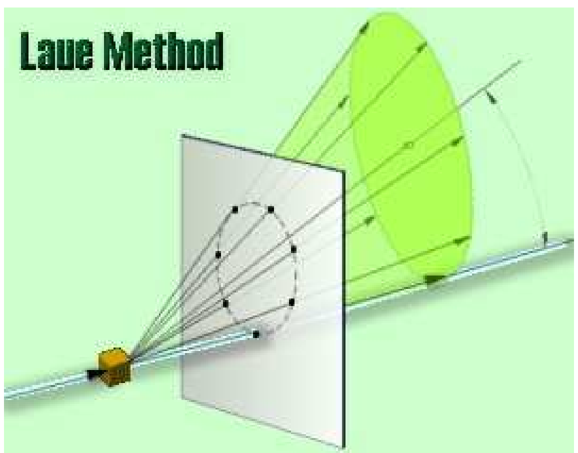
- Regular crystal lattices have many “planes”:

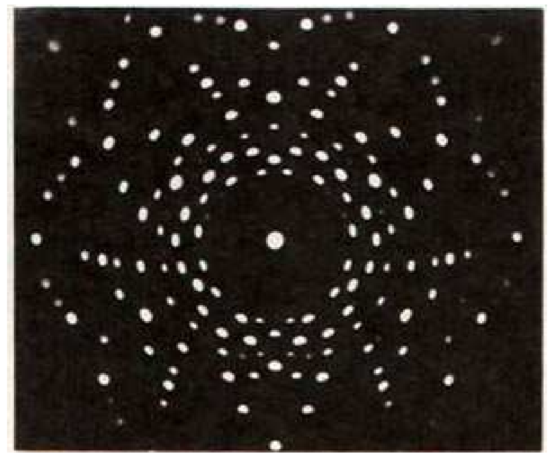


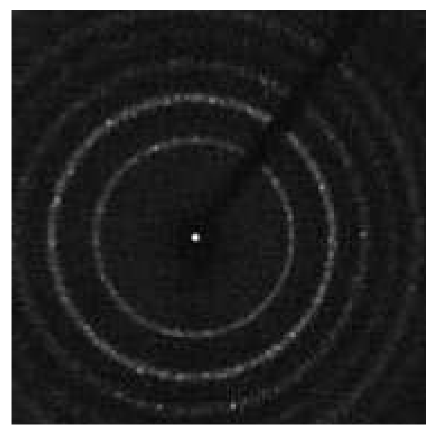
$$2d_{hkl} \sin \theta_{hkl} = n\lambda$$

X-ray Diffraction

- Max von Laue exposed crystals to a continuous x-ray spectrum:







X-ray Diffraction