Simple Harmonic Motion

• The time dependence of a single dynamical variable that satisfies the differential equation

\[ \ddot{x} + \omega^2 x = 0 \]

can be written in various ways:

a) \( x(t) = A \cos(\omega t + \phi) \)
b) \( x(t) = A \sin \omega t + B \cos \omega t \)
c) \( x(t) = re^{i(\omega t+\phi)} = (re^{i\phi})e^{i\omega t} = ce^{i\omega t} \)

• Waves are closely related, but also quite different...
Wave Motion

- The motion is still periodic
- *No single dynamical variable*
Wave Motion in One Dimension

- The deviation from equilibrium is a function of position and time
- Examples:

  **LONGITUDINAL**
  - Springs:
  - Sound: air pressure

  **TRANSVERSE**
  - Springs:
  - Water: surface height
Wave Motion in One Dimension

- The shape of the disturbance at one instance in time is called the wave profile

\[ y(x, t) = f(x - vt) \]

- Positive \( v \), the wave moves to the right
- Negative \( v \), the wave moves to the left
- Sometimes we will write \( y(x, t) = f(x \pm vt) \) when it is understood that \( v \) is positive
Wave Motion in One Dimension

- The shape remains unchanged
- The profile moves with constant velocity
- What differential equation describes this?
The Wave Equation

• Let $y(x, t) = f(x \pm vt) \equiv f(u)$

• Chain rule:

\[
\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}
\]

\[
\frac{\partial y}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} = \pm v \frac{\partial f}{\partial u}
\]

• Second derivatives:

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}
\]

\[
\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial u^2} = v^2 \frac{\partial^2 y}{\partial x^2}
\]
The Wave Equation

\[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \]

General solution: \( y(x, t) = f(x \pm vt) \)

Some particular solutions are of special interest:

• Suppose the disturbance is created by simple harmonic motion at one point:
  \[ y(0, t) = A \ cos(\omega t + \varphi) \]

• Then the wave equation tells us how this disturbance will propagate to other points in space.

• This form is called a harmonic wave.
The Wave Equation

• One way to describe a harmonic wave:
  \[ y(x, t) = A \cos(kx - \omega t + \varphi) \]

• What is the speed of wave propagation?
  – Write this in terms of \( x \pm vt \):
    \[ y(x, t) = A \cos(k(x \pm vt) + \varphi) \]
  – Equate the coefficients to the term in \( t \):
    \[ \omega t = kv t \]
    \[ \text{So } v = \omega/k \]

• What do these parameters represent?
Harmonic Waves

- Wavelength, $\lambda$:
  - When $x = \lambda$ then $kx = k\lambda = 2\pi$

- Wavenumber, $k = \frac{2\pi}{\lambda}$
  - The phase advances by “$k$” radians per unit length
Harmonic Waves

Functional form: \( y(x, t) = A \cos(kx - \omega t + \phi) \)

Notation:
- Amplitude: \( A \)
- Initial phase: \( \phi \)
- Angular frequency: \( \omega \)
- Frequency: \( f = \omega / 2\pi \)
- Period: \( T = 1/f = 2\pi/\omega \)
- Wave number: \( k \)
- Wavelength: \( \lambda = 2\pi/k \)
- Speed of propagation: \( v \)

Be careful! Sometimes people use “wavenumber” to mean \( 1/\lambda \)…
Harmonic Waves

• Elementary relationships:
  \[ \nu = \omega/k \]
  \[ T = \lambda/\nu \]
  \[ f = \nu = 1/T \]
  \[ \nu = \lambda \nu \]
  \[ \omega = 2\pi/T = 2\pi\nu \]
  \[ k = 2\pi/\lambda \]

• You should be able to work these out using dimensional analysis.
Examples:

• Green light emitting diodes emit light with a wavelength of \( \lambda = 530 \text{ nm} \). What is the frequency? Write the harmonic function that describes the propagation of this light in the \(+x\) direction.

• Purdue’s wireless service uses radio frequencies in the range 2.412 to 2.472 GHz. What is the range of wavelengths?

Both light and radio waves travel with speed
\[
c = 2.998 \times 10^8 \text{ m/s} = 29.98 \text{ cm/ns}
\]
Periodic Waves

The same parameters can be used to describe arbitrary periodic waveforms:

• Wavelength of one profile-element: $\lambda$
• Period in time of one profile-element: $T$
• The whole waveform moves with velocity $v = \pm \lambda/T$

profile-elements - when repeated can reproduce the whole waveform
Periodic Waves

• Why are harmonic waves special?
  \[ y(x, t) = A \cos(kx - \omega t) + B \sin(kx - \omega t) \]

• Any periodic wave with period \( T \) can be expressed as the linear superposition of harmonic waves with periods \( T, 2T, 3T, \ldots \) (Fourier’s Theorem)

• In fact, an arbitrary waveform can be expressed as a linear superposition of harmonic waves.

• It is sufficient to understand how harmonic waves propagate to describe the propagation of an arbitrary disturbance.
Superposition of Waves

• The wave equation is linear:
  – Suppose \( y_1(x, t) \) and \( y_2(x, t) \) are both solutions
  – Then the function \( y(x, t) = a \ y_1(x, t) + b \ y_2(x, t) \) is also a solution for any real numbers \( a \) and \( b \).

• The resulting disturbance at any point in a region where waves overlap is the algebraic sum of the constituent waves at that point.
  – The constituent waves do not interact with each other.
Example

\[ \psi(x, 0) \]

\[ \psi_1(x_0) \]

\[ \psi_2(x_0) \]

\[ \psi_1(x_0) \]

\[ \psi_1 = 1.0 \sin kx \]

\[ \psi_2 = 0.9 \sin (kx + 1.0 \text{ rad}) \]

\[ \psi = \psi_1 + \psi_2 \]
Interference

Two waves are “in phase”:

\[ y_1(x, t) = A_1 \sin(kx - \omega t) \]
\[ y_2(x, t) = A_2 \sin(kx - \omega t) \]

Amplitude of resulting wave increases: **constructive** interference

Two waves are “out of phase”:

\[ y_1(x, t) = A_1 \sin(kx - \omega t) \]
\[ y_2(x, t) = A_2 \sin(kx - \omega t + \pi) \]

Amplitude of resulting wave decreases: **destructive** interference
Interference

• Consider two interfering waves with different frequencies:
  \[ y_1(x, t) = A e^{i(k_1x-\omega_1 t)} \]
  \[ y_2(x, t) = A e^{i(k_2x-\omega_2 t)} \]

• The frequencies are not independent:
  \[ \omega_1 = k_1 v \]
  \[ \omega_2 = k_2 v \]

• Average wavenumber: \( k = \frac{1}{2}(k_1 + k_2) \)

• Then, \( k_1 = k - \Delta k \) and \( k_2 = k + \Delta k \)

• Written in terms of \( k \) and \( \Delta k \):
  \[ y_1(x, t) = A e^{i((k-\Delta k)(x-vt))} \]
  \[ y_2(x, t) = A e^{i((k+\Delta k)(x-vt))} \]
Interference

- Superposition of the two waveforms:

\[
y(x, t) = y_1(x, t) + y_2(x, t) = A e^{i(k(x-vt))} \left( e^{-i(\Delta k(x-vt))} + e^{i(\Delta k(x-vt))} \right) = 2A e^{i(k(x-vt))} \cos(\Delta k(x - vt))
\]