PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 3 – French, Chapter 1

Spring 2013 Semester

Matthew Jones

Simple Harmonic Motion

- The time dependence of a single *dynamical variable* that satisfies the differential equation

$$\ddot{x} + \omega^2 x = 0$$

can be written in various ways:

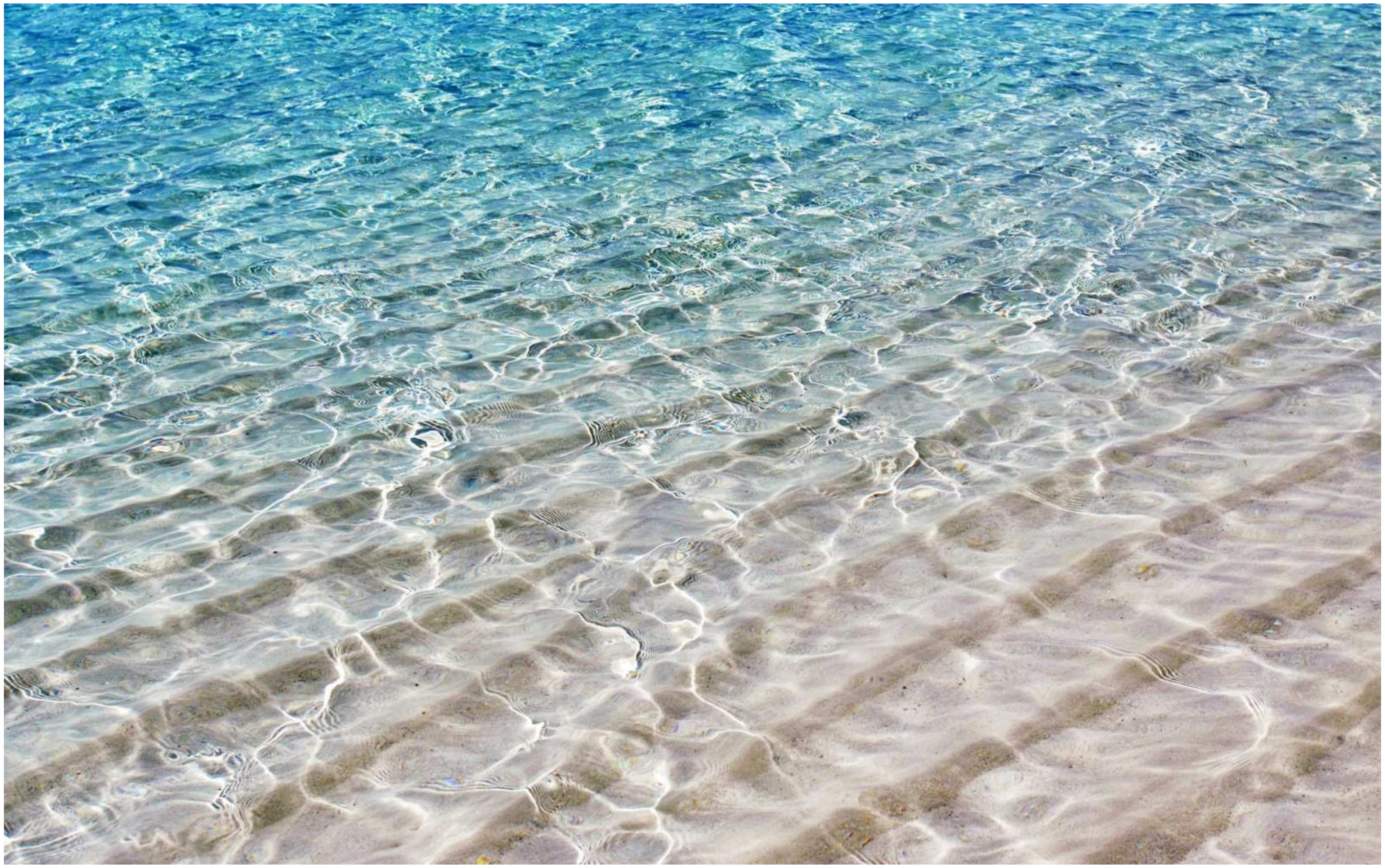
a) $x(t) = \boldsymbol{A}\ \cos(\omega t + \boldsymbol{\varphi})$

b) $x(t) = \boldsymbol{A}\, sin\, \omega t + \boldsymbol{B}\ \ cos\, \omega t$

c) $x(t) = \boldsymbol{r} e^{i(\omega t + \boldsymbol{\varphi})} = \left(\boldsymbol{r} e^{i\boldsymbol{\varphi}}\right) e^{i\omega t} = \boldsymbol{c} e^{i\omega t}$

- Waves are closely related, but also quite different…

Wave Motion

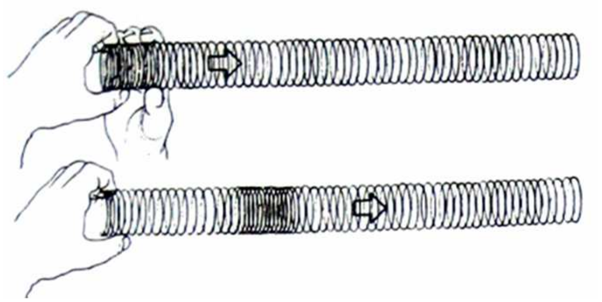


- The motion is still periodic
- *No single dynamical variable*

Wave Motion in One Dimension

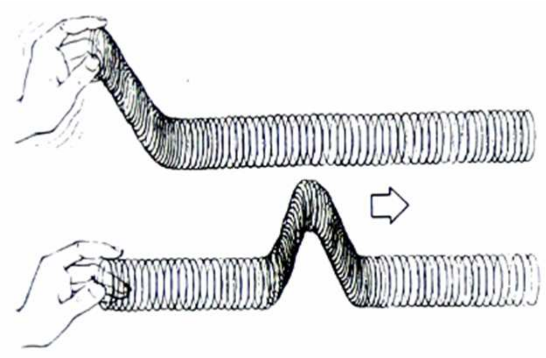
- The *deviation from equilibrium* is a *function* of position and time
- Examples:

LONGITUDINAL
Springs:



Sound: air pressure

TRANSVERSE
Springs:



Water: surface height

Wave Motion in One Dimension

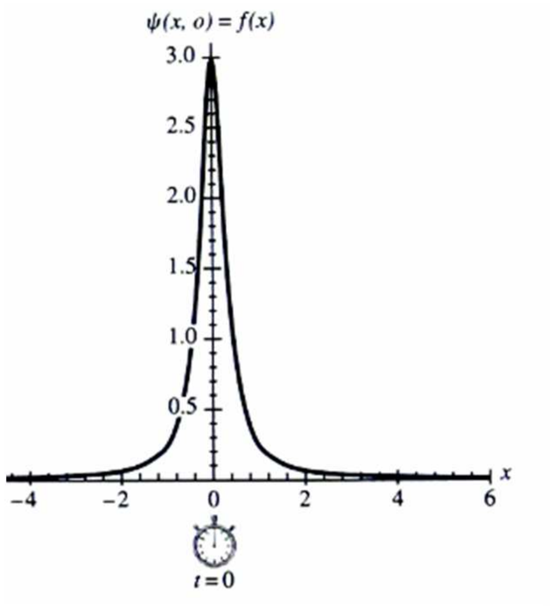
- The shape of the disturbance at one instance in time is called the wave profile

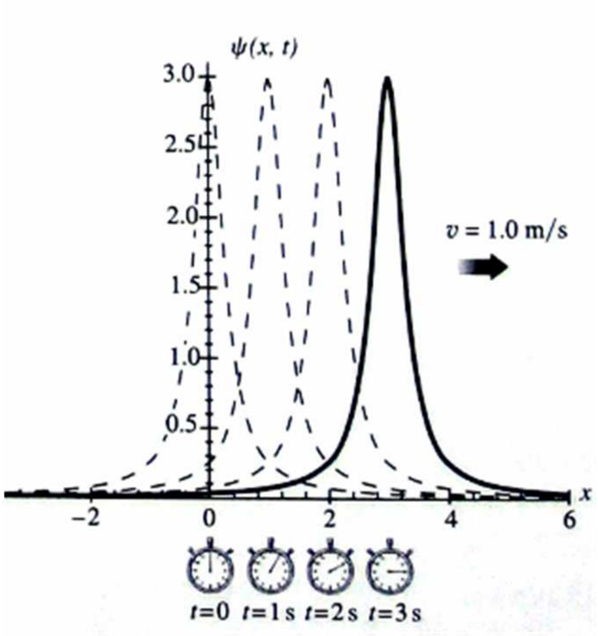
- If the wave moves with constant velocity, then

$$y(x,t) = f(x - vt)$$

  - Positive v, the wave moves to the right
  - Negative v, the wave moves to the left
  - Sometimes we will write $y(x,t) = f(x \pm vt)$ when it is understood that v is positive

Wave Motion in One Dimension





- The shape remains unchanged
- The profile moves with constant velocity
- What differential equation describes this?

The Wave Equation

- Let $y(x,t) = f(x \pm vt) \equiv f(u)$
- Chain rule:

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}$$

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = \pm v\frac{\partial f}{\partial u}$$

- Second derivatives:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 f}{\partial u^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

The Wave Equation

$$\frac{\partial^2 y}{dx^2} = \frac{1}{v^2}\frac{\partial^2 y}{dt^2}$$

General solution: $y(x,t) = f(x \pm vt)$

Some particular solutions are of special interest:

- Suppose the disturbance is created by simple harmonic motion at one point:

$$y(0,t) = A\ \cos(\omega t + \varphi)$$

- Then the wave equation tells us how this disturbance will propagate to other points in space.
- This form is called a harmonic wave.

The Wave Equation

- One way to describe a harmonic wave:

$$y(x,t) = A\ \cos(kx - \omega t + \varphi)$$

- What is the speed of wave propagation?
  - Write this in terms of $x \pm vt$:

$$y(x,t) = A\ \cos(k(\boldsymbol{x} \pm \boldsymbol{vt}) + \varphi)$$

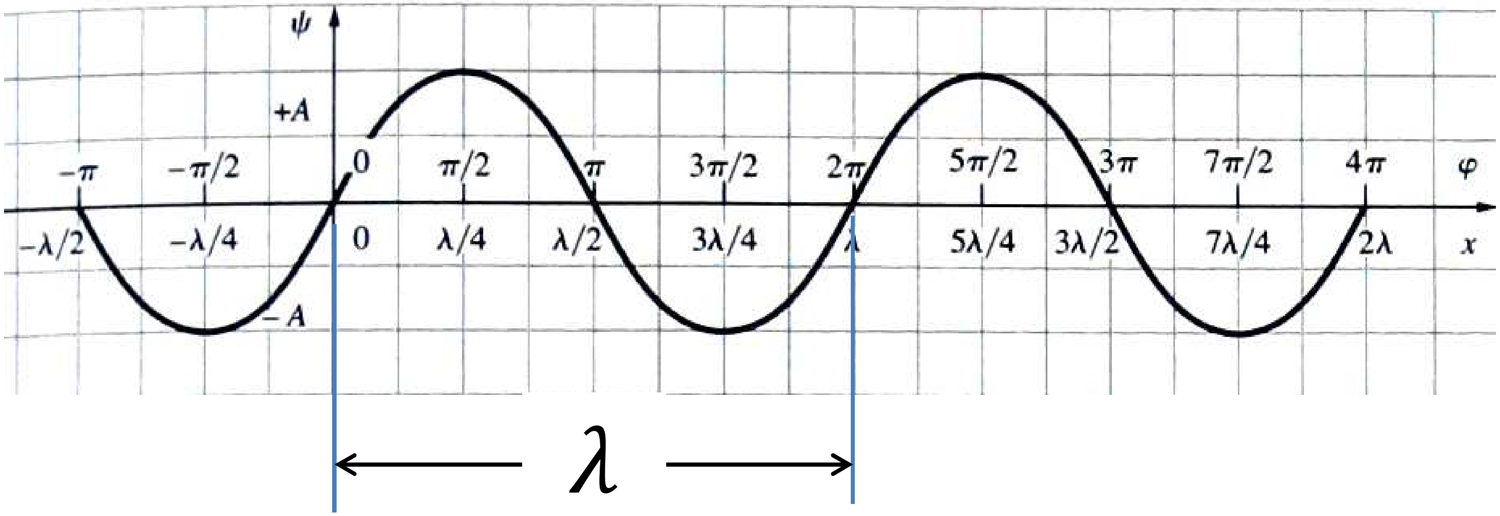
  - Equate the coefficients to the term in t:

$$\omega t = kvt$$

So $v = \omega/k$

- What do these parameters represent?

Harmonic Waves



- Wavelength, λ:
  - When $x = \lambda$ then $kx = k\lambda = 2\pi$
- Wavenumber, $k = 2\pi/\lambda$
  - The phase advances by “k” radians per unit length

Harmonic Waves

Functional form: $y(x,t) = \boldsymbol{A}\cos(\boldsymbol{k}x - \boldsymbol{\omega}t + \boldsymbol{\varphi})$

Notation:

Amplitude: $\boldsymbol{A}$

Initial phase: $\boldsymbol{\varphi}$

Angular frequency: $\boldsymbol{\omega}$

Frequency: $\boldsymbol{f = \omega/2\pi}$

Often expressed $\boldsymbol{\nu}$…

Period: $\boldsymbol{T = 1/f = 2\pi/\omega}$

Wave number: $\boldsymbol{k}$

Wavelength: $\boldsymbol{\lambda = 2\pi/k}$

Speed of propagation: $\boldsymbol{v}$

Be careful! Sometimes people use “wavenumber” to mean $1/\lambda$…

Harmonic Waves

- Elementary relationships:

$$v = \omega/k$$

$$T = \lambda/v$$

$$f = \nu = 1/T$$

$$v = \lambda\,\nu$$

$$\omega = 2\pi/T = 2\pi\nu$$

$$k = 2\pi/\lambda$$

- You should be able to work these out using dimensional analysis.

Examples:

- Green light emitting diodes emit light with a wavelength of $\lambda = 530\ \mathrm{nm}$. What is the frequency? Write the harmonic function that describes the propagation of this light in the $+x$ direction.
- Purdue's wireless service uses radio frequencies in the range 2.412 to 2.472 GHz. What is the range of wavelengths?

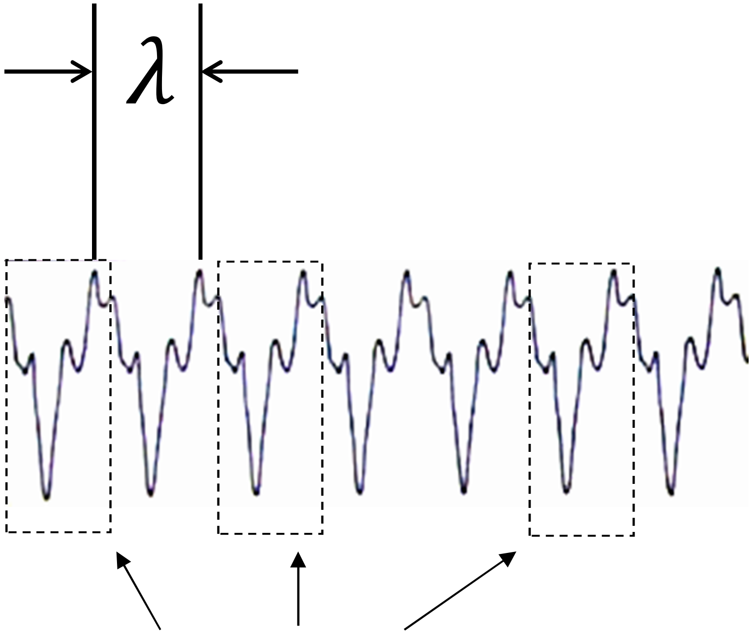
Both light and radio waves travel with speed

$$c = 2.998 \times 10^8\ \mathrm{m}/s = 29.98\ \mathrm{cm/ns}$$

Periodic Waves

The same parameters can be used to describe arbitrary periodic waveforms:

- Wavelength of one profile-element: λ
- Period in time of one profile-element: T
- The whole waveform moves with velocity $v = \pm\lambda/T$



profile-elements - when repeated can reproduce the whole waveform

Periodic Waves

- Why are harmonic waves special?

$$y(x,t) = \boldsymbol{A}\cos(kx - \omega t) + \boldsymbol{B}\sin(kx - \omega t)$$

- Any periodic wave with period T can be expressed as the linear superposition of harmonic waves with periods $T, 2T, 3T, \ldots$

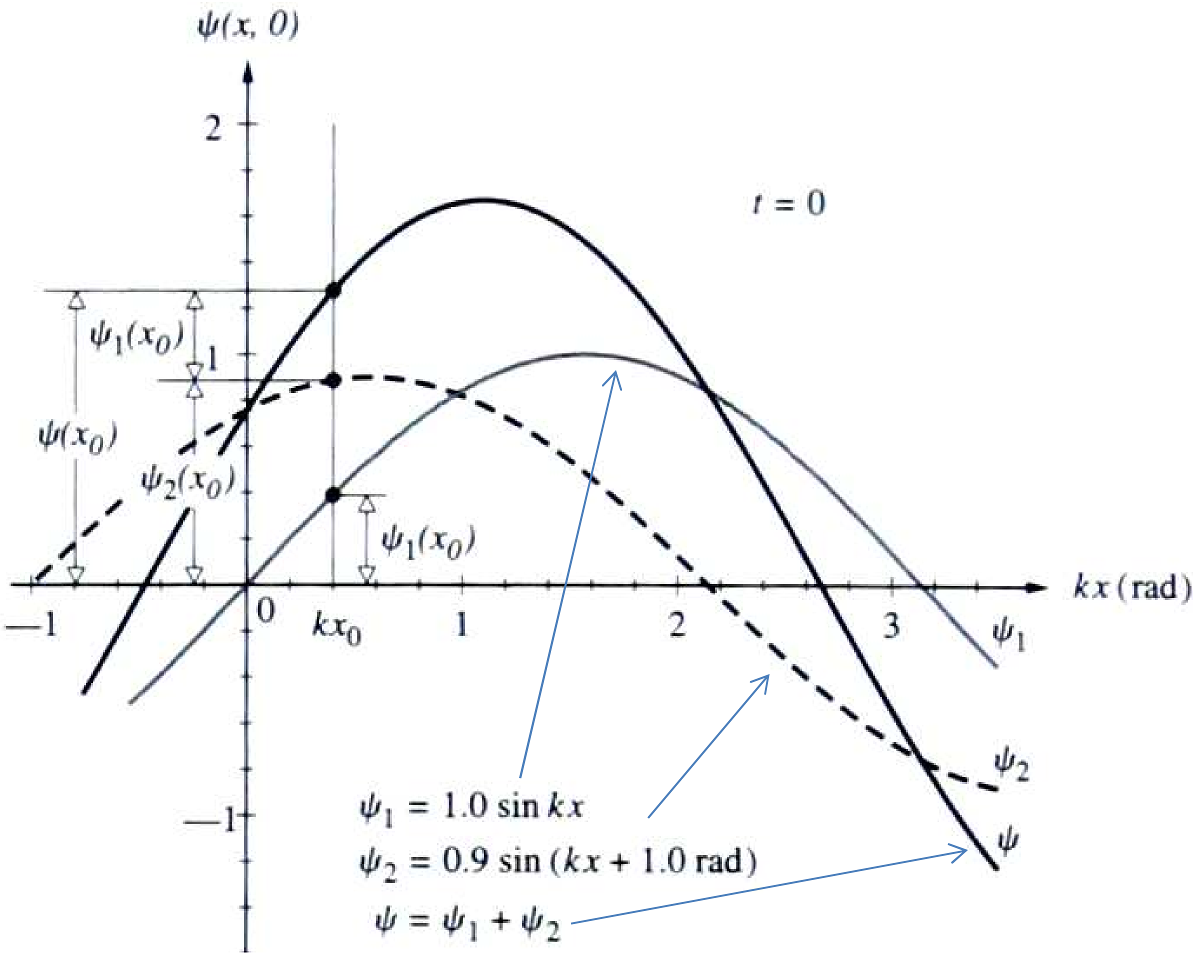
(Fourier's Theorem)

- In fact, an arbitrary waveform can be expressed as a linear superposition of harmonic waves.
- It is sufficient to understand how harmonic waves propagate to describe the propagation of an arbitrary disturbance.

Superposition of Waves

- The wave equation is linear:
  - Suppose $y_1(x,t)$ and $y_2(x,t)$ are both solutions
  - Then the function $y(x,t) = a\, y_1(x,t) + b\, y_2(x,t)$ is also a solution for any real numbers a and b.
- The resulting disturbance at any point in a region where waves overlap is the algebraic sum of the constituent waves at that point.
  - The constituent waves do not interact with each other.

Example



$\psi_1 = 1.0 \sin kx$

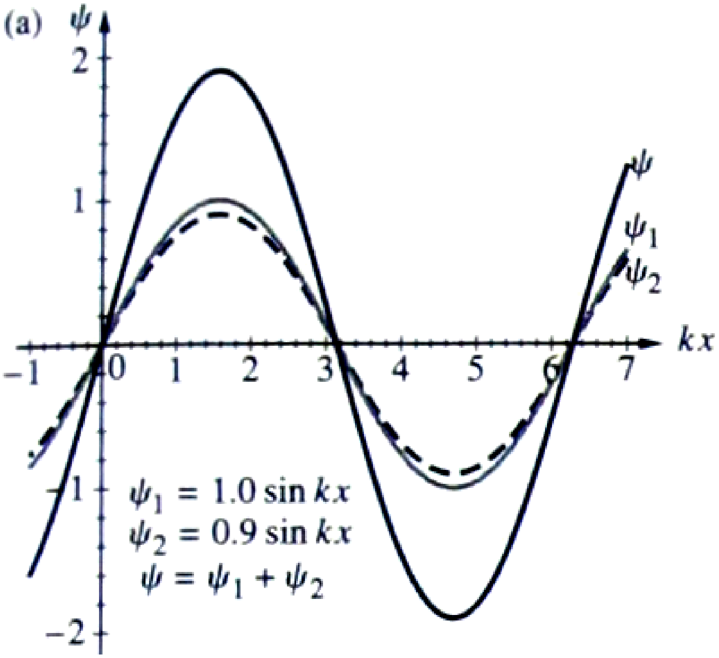
$\psi_2 = 0.9 \sin(kx + 1.0 \text{ rad})$

$\psi = \psi_1 + \psi_2$

Interference

Two waves are "*in phase*":

$$y_1(x,t) = A_1 \sin(kx - \omega t)$$
$$y_2(x,t) = A_2 \sin(kx - \omega t)$$

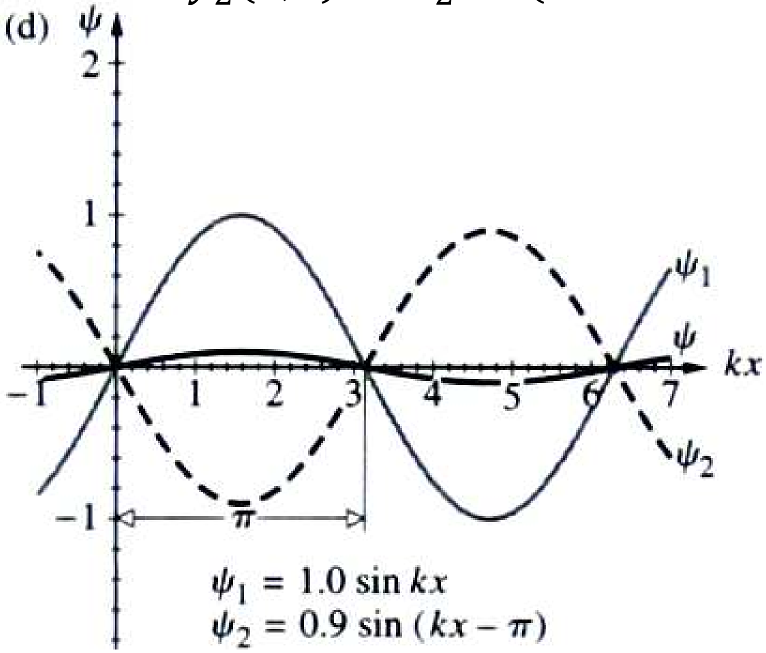


$$y(x,t) = (A_1 + A_2) \sin(kx - \omega t)$$

Amplitude of resulting wave increases: **constructive** interference

Two waves are "*out of phase*":

$$y_1(x,t) = A_1 \sin(kx - \omega t)$$
$$y_2(x,t) = A_2 \sin(kx - \omega t + \pi)$$



$$y(x,t) = (A_1 - A_2) \sin(kx - \omega t)$$

Amplitude of resulting wave decreases: **destructive** interference

Interference

- Consider two interfering waves with different frequencies:

$$y_1(x,t) = A\ e^{i(k_1 x - \omega_1 t)}$$
$$y_2(x,t) = A\ e^{i(k_2 x - \omega_2 t)}$$

- The frequencies are not independent:

$$\omega_1 = k_1 v$$
$$\omega_2 = k_2 v$$

- Average wavenumber: $k = \frac{1}{2}(k_1 + k_2)$
- Then, $k_1 = k - \Delta k$ and $k_2 = k + \Delta k$
- Written in terms of k and Δk:

$$y_1(x,t) = A\ e^{i((k-\Delta k)(x-vt))}$$
$$y_2(x,t) = A\ e^{i((k+\Delta k)(x-vt))}$$

Interference

- Superposition of the two waveforms:

$$y(x,t) = y_1(x,t) + y_2(x,t)$$
$$= A\,e^{i(k(x-vt))}\left(e^{-i(\Delta k(x-vt))} + e^{i(\Delta k(x-vt))}\right)$$
$$= 2Ae^{i(k(x-vt))}\cos(\Delta k(x-vt))$$