Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 22 – Review

Spring 2013 Semester

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Midterm Exam:

Date:	Wednesday, March 6th
Time:	8:00 – 10:00 pm
Room:	PHYS 203
Material:	French, chapters 1-8

Review

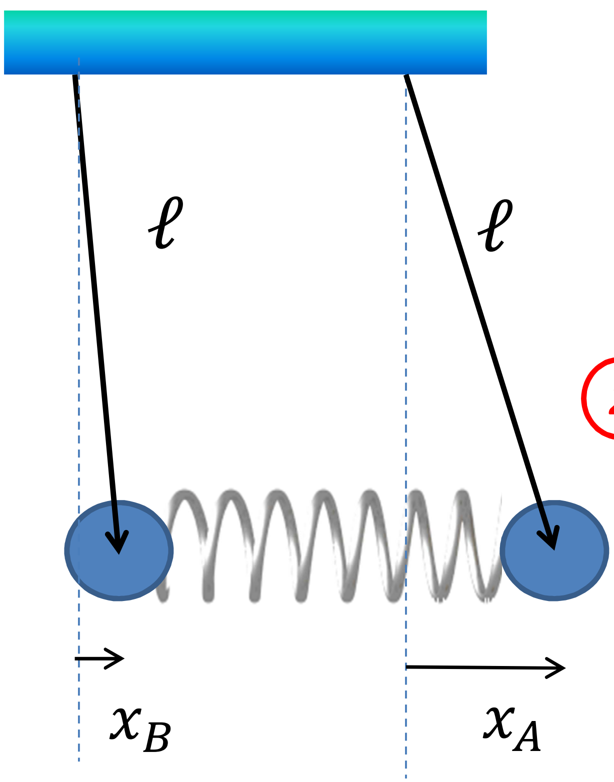
1. Simple harmonic motion (one degree of freedom)
   - mass/spring, pendulum, water in pipes, RLC circuits
   - damped harmonic motion
2. Forced harmonic oscillators
   - amplitude/phase of steady state oscillations
   - transient phenomena
3. Coupled harmonic oscillators
   - masses/springs, coupled pendula, RLC circuits
   - forced oscillations
4. Uniformly distributed discrete systems
   - masses on string fixed at both ends
   - lots of masses/springs

Review

5. Continuously distributed systems (standing waves)
   - string fixed at both ends
   - sound waves in pipes (open end/closed end)
   - transmission lines
   - Fourier analysis
6. Progressive waves in continuous systems
   - dispersion, phase velocity/group velocity
   - reflection/transmission coefficients
7. Waves in two and three dimensions
   - Laplacian operator
   - Rotationally symmetric solutions in 2d and 3d

Coupled Discrete Systems

- ***The general method of calculating eigenvalues will always work, but for simple systems you should be able to decouple the equations by a change of variables.***



1.
$$\begin{cases} m\ddot{x}_A + \dfrac{mg}{\ell}x_A + k(x_A - x_B) = 0 \\ m\ddot{x}_B + \dfrac{mg}{\ell}x_B - k(x_A - x_B) = 0 \end{cases}$$

2.
$$\begin{cases} \ddot{x}_A + [(\omega_0)^2 + (\omega_c)^2]x_A - (\omega_c)^2 x_B = 0 \\ \ddot{x}_B + [(\omega_0)^2 + (\omega_c)^2]x_B - (\omega_c)^2 x_A = 0 \\ \omega_0 = \sqrt{g/\ell},\ \omega_c = \sqrt{k/m} \end{cases}$$

3.
$$\begin{cases} q_1 = x_A + x_B \\ q_2 = x_A - x_B \end{cases}$$

4.
$$\begin{cases} \ddot{q}_1 + (\omega_0)^2 q_1 = 0 \\ \ddot{q}_2 + (\omega')^2 q_2 = 0 \end{cases}$$

Forced Oscillations

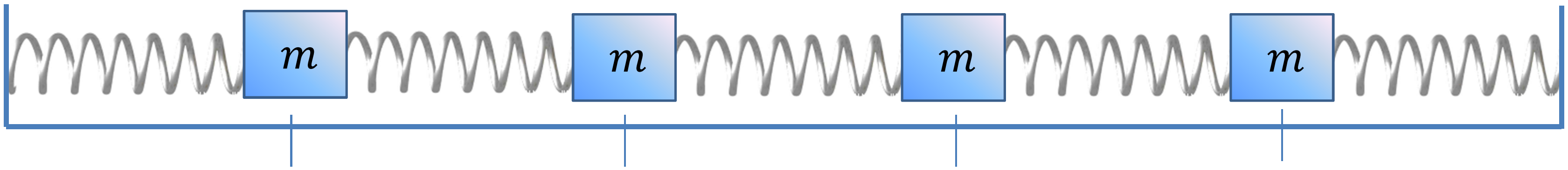
- We mainly considered the qualitative aspects
  - We did not analyze the behavior when damping forces were significant
- Main features:
  - Resonance occurs at each normal mode frequency
  - Phase difference is $\delta = \pi/2$ at resonance
- Example: x_A driven by the force $F(\omega) = F_0\, cos\, \omega t$
  - Calculate force term applied to normal coordinates

$$F_1(\omega) = F_2(\omega) = F_0 \cos \omega t$$

  - Reduced to two one-dimensional forced oscillators:

$$\ddot{q}_1 + (\omega_0)^2 q_1 = F_0/m \cos \omega t$$
$$\ddot{q}_2 + (\omega')^2 q_2 = F_0/m \cos \omega t$$

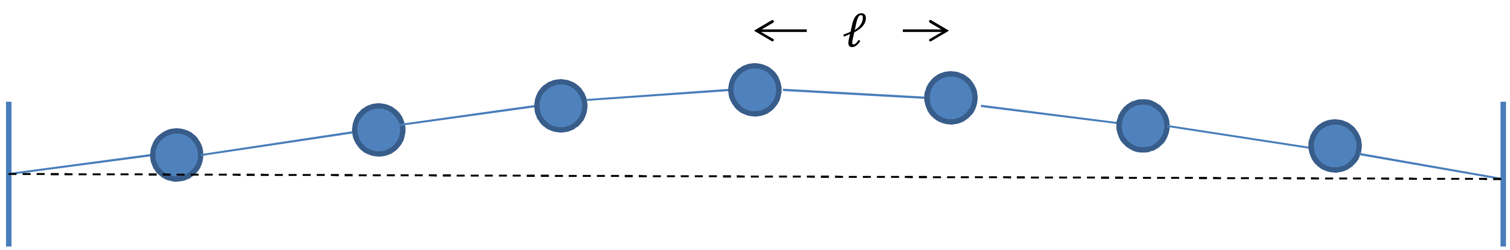
Uniformly Distributed Discrete Systems



Equations of motion for masses in the middle:

$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0$$

$$(\omega_0)^2 = k/m$$



$$\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0$$

$$(\omega_0)^2 = T/m\ell$$

Uniformly Distributed Discrete Masses

- Proposed solution:

$$x_n(t) = A_n \cos \omega t$$

$$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- We solved this to determine A_n and ω_k:

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

Amplitude of mass n oscillating in normal mode k

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

Frequency of normal mode k

- General solution:

$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

Vibrations of Continuous Systems

- Amplitude of mass n for normal mode k:

$$A_{n,k} = C \sin\left(\frac{nk\pi}{N+1}\right)$$

- Frequency of normal mode k:

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

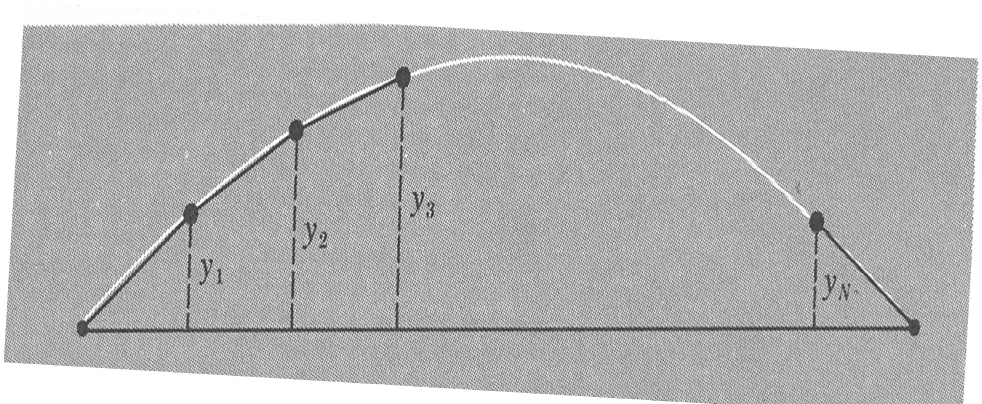
- Solution for normal modes:

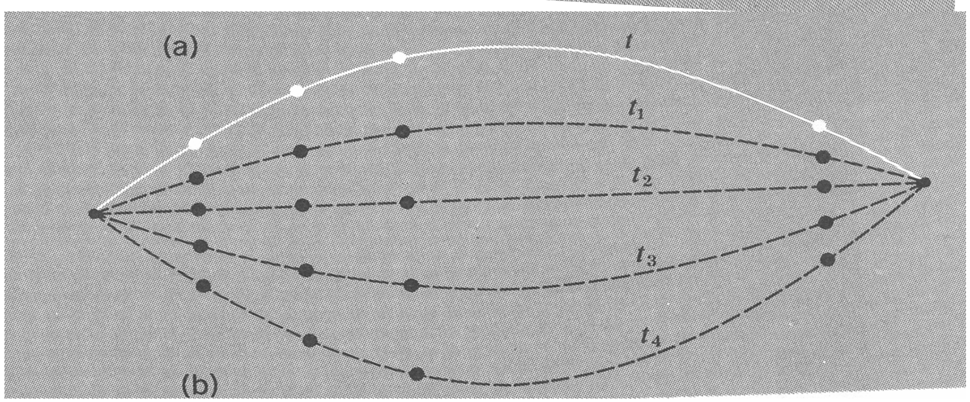
$$x_n(t) = A_{n,k} \cos \omega_k t$$

- General solution:

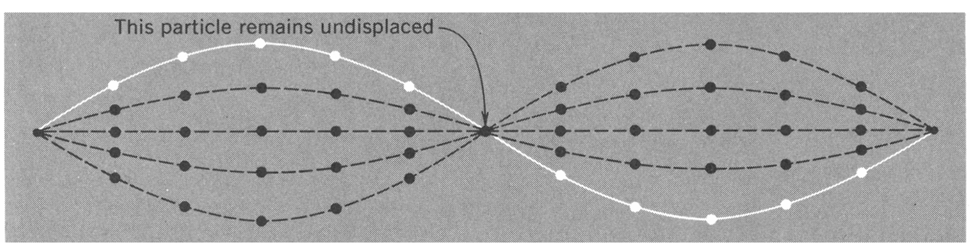
$$x_n(t) = \sum_{k=1}^{N} a_k \sin\left(\frac{nk\pi}{N+1}\right) \cos(\omega_k t - \delta_k)$$

Masses on a String



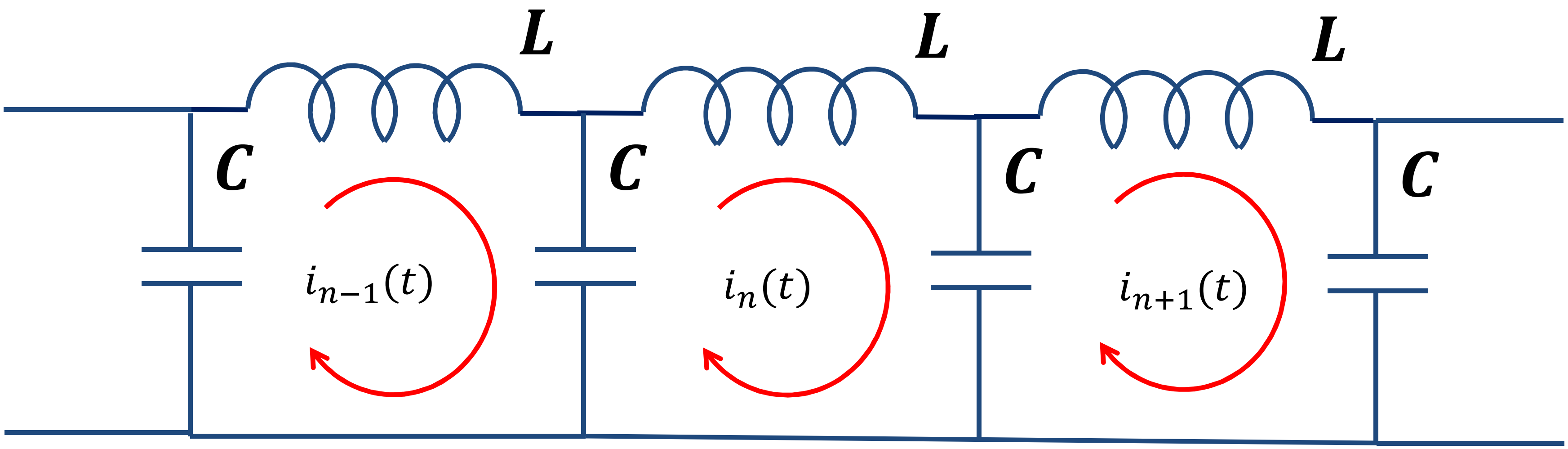


First normal mode



Second normal mode

Lumped LC Circuit



$$-L\frac{di_n}{dt} - \frac{1}{C}\int (i_n - i_{n+1})dt \; - \frac{1}{C}\int (i_n - i_{n-1})dt = 0$$

$$\frac{d^2 i_n}{dt^2} + 2\omega_0^2 i_n - \omega_0^2(i_{n-1} + i_{n+1}) = 0$$

This is the exact same problem as the previous two examples.

Forced Coupled Oscillators

- Qualitative features are the same:
  - Motion can be decoupled into a set of N independent oscillator equations (normal modes)
  - Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
  - Phase difference approaches $\pi/2$ at resonance
- ***You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.***

Continuous Distributions

Limit as $N \to \infty$ and $m/\ell \to \mu$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Boundary conditions specified at $x = 0$ and $x = L$:

- Fixed ends: $y(0) = y(L) = 0$
- Maximal motion at ends: $\dot{y}(0) = \dot{y}(L) = 0$
- Mixed boundary conditions

Normal modes will be of the form

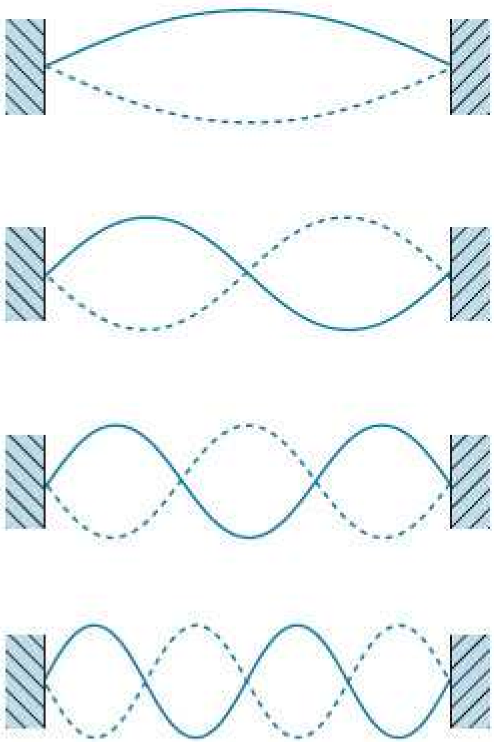
$$y_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t - \delta_n)$$

or

$$y_n(x,t) = A_n \cos(k_n x)\cos(\omega_n t - \delta_n)$$

Properties of the Solutions

$$y(L,t) \sim \sin k_n L = 0 \quad \Rightarrow \quad k_n L = n\pi$$



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

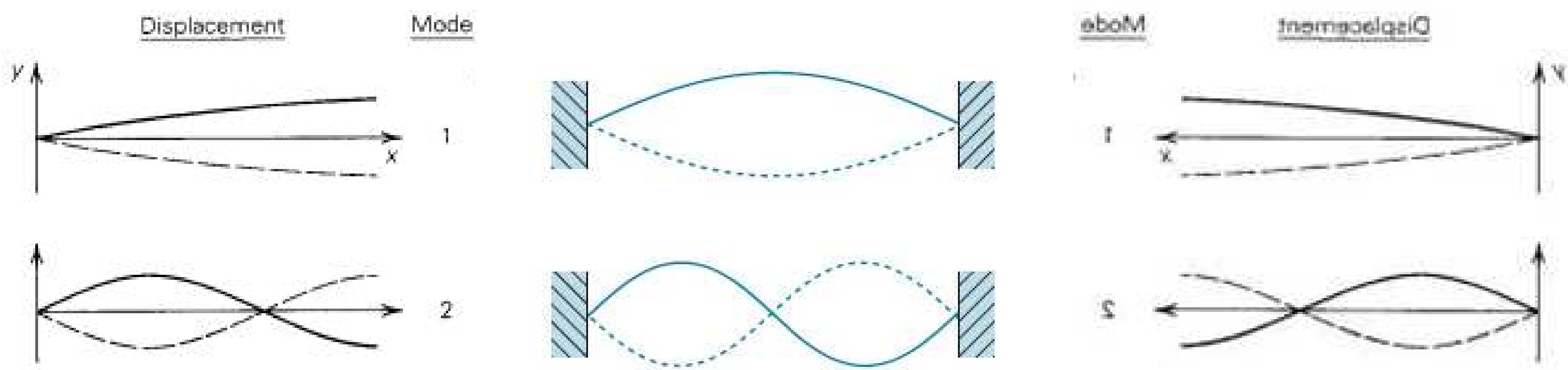
$$\lambda_n = \frac{2L}{n}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$f_n = \frac{nv}{2L}$$

Boundary Conditions

- Examples:
  - String fixed at both ends: $y(0) = y(L) = 0$
  - Organ pipe open at one end: $\dot{y}(0) = \dot{y}(L) = 0$
    - Driving end has maximal pressure amplitude
  - Organ pipe closed at one end: $\dot{y}(0) = 0, y(L) = 0$
  - Transmission line open at one end: $i(L) = 0$
  - Transmission line shorted at one end: $v(L) \propto \frac{di(L)}{dt} = 0$



Fourier Analysis

- Normal modes satisfying $y(0) = y(L) = 0$:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

- General solution:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

- Initial conditions:

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\delta_n) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} A_n \omega_n \sin\left(\frac{n\pi x}{L}\right)\sin(\delta_n) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier Analysis

- Fourier sine transform:

$$u(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L u(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

- Fourier cosine transform:

$$v(x) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L v(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Fourier Analysis

$$B_n = A_n \cos \delta_n$$
$$C_n = A_n \omega_n \sin \delta_n$$

Solve for amplitudes:

$$A_n = \sqrt{B_n^2 + \frac{C_n^2}{\omega_n^2}}$$

Solve for phase:

$$\tan \delta_n = \frac{C_n}{B_n \omega_n}$$

Fourier Analysis

- ***Suggestion: don't simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.***
- Example:
  - If you are given $\dot{y}(x,0) = 0$ and $y(0) = y(L) = 0$ then you know that solutions are of the form

$$y(x,t) = \sum A_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

  - If you are given $\dot{y}(x,0) = 0$ and $\dot{y}(0) = 0, y(L) = 0$ then solutions are of the form

$$y(x,t) = \sum_{odd\ n} A_n \cos\left(\frac{n\pi x}{L}\right) \cos \omega_n t$$

Progressive Waves

- Far from the boundaries, other descriptions are more transparent:

$$y(x,t) = f(x \pm vt)$$

- The Fourier transform gives the frequency components:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx)\, dx$$

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx)\, dx$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx)\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx)\, dk$$

- Narrow pulse in space ➔ wide range of frequencies
- Pulse spread out in space ➔ narrow range of frequencies

Properties of Progressive Waves

- Power carried by a wave:
  - String with tension T and mass per unit length μ

$$P = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2} Z\omega^2 A^2$$

- Impedance of the medium:

$$Z = \mu v = T/v$$

- Important properties:
  - ***Impedance is a property of the medium, not the wave***
  - ***Energy and power are proportional to the square of the amplitude***

Reflections

- Wave energy is reflected by discontinuities in the impedance of a system
- Reflection and transmission coefficients:
  - The wave is incident and reflected in medium 1
  - The wave is transmitted into medium 2

$$\boldsymbol{\rho} = \frac{\boldsymbol{Z_1} - \boldsymbol{Z_2}}{\boldsymbol{Z_1} + \boldsymbol{Z_2}}$$

Important: when is this negative?

$$\boldsymbol{\tau} = \frac{\boldsymbol{2Z_1}}{\boldsymbol{Z_1} + \boldsymbol{Z_2}}$$

Always positive

- Wave amplitudes:

$$A_r = \rho A_i$$
$$A_t = \tau A_i$$

Reflected and Transmitted Power

- Power is proportional to the square of the amplitude.
  - Reflected power: $P_r = \rho^2 P_i$
  - Transmitted power: $P_t = \tau^2 P_i$
- ***You should be able to demonstrate that energy is conserved:***

  ie, show that $\boldsymbol{P_i = P_r + P_t}$

Dispersion

- Wave speed is sometimes a function of frequency.
- Phase velocity: $v = \lambda f = \frac{\omega}{k}$ (constant)
- Group velocity: $v_g = \frac{d\omega}{dk}$ (function of frequency)
- Energy that is carried by a pulse propagates with the group velocity
- In optics, $v = c/n(k)$ and

$$v_g = v\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

(evaluated at the average wavenumber of the pulse)

Waves in Two and Three Dimensions

- Wave equation:

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

- When the function only depends on the radius,

(eg, $\partial\psi/d\theta = 0$) then this can be written:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Polar coordinates (2d)

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Spherical coordinates (3d)

Waves in Two Dimensions

- Wave equation in polar coordinates:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

But only when $\frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial\theta} = 0$!

- Bessel's equation:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\omega^2}{v^2}\psi = 0$$

Let $z = kr$
where $k = \omega/v$

$$\frac{\partial^2\psi}{\partial z^2} + \frac{1}{z}\frac{\partial\psi}{\partial z} + \psi(z) = 0$$

- Solutions: $J_0(z) \sim \sqrt{\frac{2}{\pi}}\frac{\cos(z - \pi/4)}{\sqrt{z}}$ and $Y_0(z) \sim \sqrt{\frac{2}{\pi}}\frac{\sin(z - \pi/4)}{\sqrt{z}}$

Waves in Three Dimensions

- Wave equation in spherical coordinates:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

But only when $\frac{\partial\psi}{\partial\varphi} = \frac{\partial\psi}{\partial\theta} = 0$!

- When $\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi$ this is

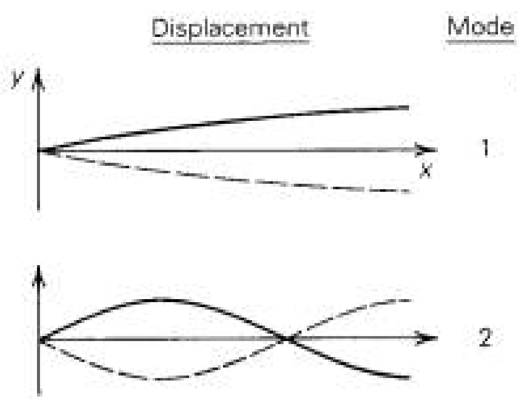
$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{\omega^2}{v^2}\psi = 0$$

- Solutions are of the form:

$$\psi(r,t) = A\frac{e^{ikr}}{r}\cos\omega t$$

Boundary Conditions in Two and Three Dimensions

- When a boundary condition imposes the restriction that $\psi(R, t) = 0$ then the function must have a node at $r = R$.
- Analogous to the 1-dimensional case:



This imposes the requirement that kR is a root of the equation $f(kR) = 0$ which implies that $k_n = \frac{\omega_n}{v} = z_n/R$ where z_n are roots of $f(z) = 0$.

That’s all for now…

- Study these topics – make sure you understand the examples and assignment questions.
- Send e-mail if you would like specific examples discussed before the exam next Wednesday.

- Next topics: ***waves applied to optics.***