Physics 42200

Waves & Oscillations

Lecture 21 – Review

Spring 2013 Semester

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Midterm Exam:

Date: Wednesday, March 6th
Time: 8:00 – 10:00 pm
Room: PHYS 203
Material: French, chapters 1-8
Review

1. Simple harmonic motion (one degree of freedom)
   - mass/spring, pendulum, water in pipes, RLC circuits
   - damped harmonic motion
2. Forced harmonic oscillators
   - amplitude/phase of steady state oscillations
   - transient phenomena
3. Coupled harmonic oscillators
   - masses/springs, coupled pendula, RLC circuits
4. Uniformly distributed discrete systems
   - masses on string fixed at both ends
   - lots of masses/springs
Review

5. Continuously distributed systems (standing waves)
   – string fixed at both ends
   – sound waves in pipes (open end/closed end)
   – transmission lines
   – Fourier analysis

6. Progressive waves in continuous systems
   – dispersion, phase velocity/group velocity
   – reflection/transmission coefficients

7. Waves in two and three dimensions
   – Laplacian operator
   – Rotationally symmetric solutions in 2d and 3d
Simple Harmonic Motion

• Any system in which the force is opposite the displacement will oscillate about a point of stable equilibrium
• If the force is proportional to the displacement it will undergo simple harmonic motion

Examples:
– Mass/massless spring
– Elastic rod (characterized by Young’s modulus)
– Floating objects
– Torsion pendulum (shear modulus)
– Simple pendulum
– Physical pendulum
– LC circuit
Simple Harmonic Motion

• You should be able to draw a free-body diagram and express the force in terms of the displacement.

• Use Newton’s law: \( m\ddot{x} = F \)

• Write it in standard form:
  \[
  \ddot{x} + \omega^2 x = 0
  \]

• Solutions are of the form:
  \[
  x(t) = A \cos(\omega t - \delta)
  \]
  \[
  x(t) = A \cos \omega t + B \sin \omega t
  \]

• You must be able to use the initial conditions to solve for the constants of integration
Examples

\[ m\ddot{x} = -mgx/\ell \]

\[ m\ddot{x} = -kx \]
Examples

\[ m\ddot{x} = ? \]
Damped Harmonic Motion

• Damping forces remove energy from the system
• We will only consider cases where the force is proportional to the velocity:  \( F = -bv \)
• You should be able to construct a free-body diagram and write the resulting equation of motion:
  \[
  m\ddot{x} + b\dot{x} + kx = 0
  \]
  – You should be able to write it in the standard form:
  \[
  \ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0
  \]
• You must be able to solve this differential equation!
Damped Harmonic Motion

\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \]

Let \( x(t) = Ae^{\alpha t} \)

- Characteristic polynomial:
  \[ \alpha^2 + \gamma \alpha + \omega_0^2 = 0 \]

- Roots (use the quadratic formula):
  \[ \alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2} \]

- Classification of solutions:
  - Over-damped: \( \gamma^2 / 4 - (\omega_0)^2 > 0 \) (distinct real roots)
  - Critically damped: \( \gamma^2 / 4 = (\omega_0)^2 \) (one root)
  - Under-damped: \( \gamma^2 / 4 - (\omega_0)^2 < 0 \) (complex roots)
Damped Harmonic Motion

- Over-damped motion: $\gamma^2/4 - (\omega_0)^2 > 0$
  
  $$x(t) = Ae^{-\frac{\gamma}{2}t} e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}} + Be^{-\frac{\gamma}{2}t} e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

- Under-damped motion: $\gamma^2/4 - (\omega_0)^2 < 0$
  
  $$x(t) = Ae^{-\frac{\gamma}{2}t} e^{it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}} + Be^{-\frac{\gamma}{2}t} e^{-it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}}$$

- Critically damped motion:
  
  $$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

- You must be able to use the initial conditions to solve for the constants of integration
Example

Sum of potential differences:

\[-L \frac{di}{dt} - i(t)R - \frac{1}{C} \left( Q_0 + \int_0^t i(t) dt \right) = 0\]

Initial charge, $Q_0$, defines the initial conditions.
Example

\[ L \frac{di}{dt} + i(t)R + \frac{1}{C} \left( Q_0 + \int_0^t i(t)dt \right) = 0 \]

Differentiate once with respect to time:

\[ L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i(t) = 0 \]

\[ \frac{d^2i}{dt^2} + \gamma \frac{di}{dt} + \omega_0^2 i(t) = 0 \]

Remember, the solution is \( i(t) \) but the initial conditions might be in terms of \( Q(t) = Q_0 + \int i(t)dt \)

(See examples from Lecture 7)
Forced Harmonic Motion

• Now the differential equation is
  \[ m\ddot{x} + b\dot{x} + kx = F(\omega) = F_0 \cos \omega t \]

• Driving function is not always given in terms of a force... remember Question #2 on Assignment #3:
  \[ \ddot{y} + \gamma \dot{y} + \omega_0^2 y = -\frac{d^2 \eta}{dt^2} = C \omega^2 \cos \omega t \]

• General properties:
  – Steady state properties: \( t \gg 1/\gamma \)
  – Solution is \( y(t) = A \cos(\omega t - \delta) \)
  – Amplitude, \( A \), and phase, \( \delta \), depend on \( \omega \)
Forced Harmonic Motion

“Q” quantifies the amount of damping:

\[
Q = \frac{\omega_0}{\gamma}
\]

(large Q means small damping force)

\[
A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}
\]

\[
\delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega}{\omega_0} - \frac{\omega}{\omega_0}}\right)
\]
Resonance

- Qualitative features: amplitude

\[ \frac{\omega_{\text{free}}}{\omega_0} = 1 \]

\[ \omega_{\text{free}} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \]
Average Power

- The rate at which the oscillator absorbs energy is:

\[
\bar{P}(\omega) = \frac{(F_0)^2 \omega_0}{2kQ} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}
\]

Full-Width-at-Half-Max:

\[
FWHM = \frac{\omega_0}{Q} = \gamma
\]
Resonance

• Qualitative features: phase shift

\[ \delta = \tan^{-1}\left( \frac{1/Q}{\omega_0 - \frac{\omega}{\omega_0}} \right) \]

\[ \delta \to 0 \text{ at low frequencies} \]

\[ \delta \to \pi \text{ at high frequencies} \]

\[ \delta = \frac{\pi}{2} \text{ when } \omega = \omega_0 \]
Coupled Oscillators

• Restoring force on pendulum A:
  \[ F_A = -k(x_A - x_B) \]

• Restoring force on pendulum B:
  \[ F_B = k(x_A - x_B) \]

\[
\begin{align*}
  m\ddot{x}_A + \frac{mg}{\ell} x_A + k(x_A - x_B) &= 0 \\
  m\ddot{x}_B + \frac{mg}{\ell} x_B - k(x_A - x_B) &= 0
\end{align*}
\]
Coupled Oscillators

• You must be able to draw the free-body diagram and set up the system of equations.

\[ m\ddot{x}_A + \frac{mg}{\ell} x_A + k(x_A - x_B) = 0 \]
\[ m\ddot{x}_B + \frac{mg}{\ell} x_B - k(x_A - x_B) = 0 \]

• You must be able to write this system as a matrix equation.

\[
\begin{pmatrix} \ddot{x}_A \\ \ddot{x}_B \end{pmatrix} + \begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = 0
\]
Coupled Oscillators

- Assume solutions are of the form
  \[
  \begin{pmatrix}
  x_A(t) \\
  x_B(t)
  \end{pmatrix}
  = \begin{pmatrix}
  x_A \\
  x_B
  \end{pmatrix} \cos(\omega t - \delta)
  \]

- Then,
  \[
  \begin{pmatrix}
  (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\
  -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2
  \end{pmatrix}
  \begin{pmatrix}
  x_A \\
  x_B
  \end{pmatrix} = 0
  \]

- You must be able to calculate the eigenvalues of a 2x2 or 3x3 matrix.
  - Calculate the determinant
  - Calculate the roots by factoring the determinant or using the quadratic formula.

- These are the frequencies of the normal modes of oscillation.
Coupled Oscillators

• You must be able to calculate the eigenvectors of a 2x2 or 3x3 matrix

• General solution:
\[ \ddot{x}(t) = A\dot{x}_1 \cos(\omega_1 t - \alpha) + B\dot{x}_2 \cos(\omega_2 t - \beta) + \cdots \]

• You must be able to solve for the constants of integration using the initial conditions.
Example
Example
Example

\[ m\ddot{x}_1 = -mg \frac{x_1}{\ell} + mg \frac{x_2 - x_1}{\ell} \]

\[ m\ddot{x}_2 = -mg \frac{x_2 - x_1}{\ell} \]
Example

Equations of motion:

\[
\begin{align*}
    m\ddot{x}_1 &= -mg \frac{x_1}{\ell} + mg \frac{x_2 - x_1}{\ell} \\
    m\ddot{x}_2 &= -mg \frac{x_2 - x_1}{\ell}
\end{align*}
\]

Write as a matrix:

\[
\begin{align*}
    \ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 &= 0 \\
    \ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 &= 0 \\
    \begin{pmatrix}
        \ddot{x}_1 \\
        \ddot{x}_2
    \end{pmatrix} + \begin{pmatrix}
        2\omega_0^2 & -\omega_0^2 \\
        -\omega_0^2 & \omega_0^2
    \end{pmatrix} \begin{pmatrix}
        x_1 \\
        x_2
    \end{pmatrix} &= 0
\end{align*}
\]
Example

Normal modes:

\[ \ddot{x}_i(t) = \ddot{x}_i \cos(\omega t - \delta) \]

Eigenvalue problem:

\[
\begin{pmatrix}
-\omega^2 + 2\omega_0^2 & -\omega_0^2 \\
-\omega_0^2 & -\omega^2 + \omega_0^2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 0
\]

\[
\begin{vmatrix}
-\omega^2 + 2\omega_0^2 & -\omega_0^2 \\
-\omega_0^2 & -\omega^2 + \omega_0^2
\end{vmatrix}
= 0
\]

\[
(\omega^2 + 2\omega_0^2)(\omega^2 + \omega_0^2) - \omega_0^4 = 0
\]

\[
(\lambda - 2\omega_0^2)(\lambda - \omega_0^2) - \omega_0^4 = 0
\]

\[
\lambda = \omega^2 = \frac{3}{2} \omega_0^2 \pm \frac{1}{2} \sqrt{\omega_0^4 + 4\omega_0^4}
\]
Example

• Eigenvector for $\omega_1^2 = \omega_0^2 \left( \frac{3-\sqrt{5}}{2} \right)$

$$\omega_0^2 \begin{pmatrix} \frac{1 + \sqrt{5}}{2} & -1 \\ -1 & \frac{-1 + \sqrt{5}}{2} \end{pmatrix} (x_1) = 0$$

$$x_2 = \frac{1 + \sqrt{5}}{2} x_1$$

• Normal mode:

$$x_1(t) = A \begin{pmatrix} 1 \\ \frac{1 + \sqrt{5}}{2} \end{pmatrix} \cos(\omega_1 t - \alpha)$$
Example

• Eigenvector for $\omega_1^2 = \omega_0^2 \left(\frac{3+\sqrt{5}}{2}\right)$

$$\omega_0^2 \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_2 = \frac{1-\sqrt{5}}{2} x_1$$

• Normal mode:

$$x_2(t) = B \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \cos(\omega_2 t - \beta)$$