Purdue University Department of Physics

Physics 42200

Waves & Oscillations

Lecture 21 – Review

Spring 2013 Semester

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Midterm Exam:

Date:	Wednesday, March 6th
Time:	8:00 – 10:00 pm
Room:	PHYS 203
Material:	French, chapters 1-8

Review

1. Simple harmonic motion (one degree of freedom)
   - mass/spring, pendulum, water in pipes, RLC circuits
   - damped harmonic motion
2. Forced harmonic oscillators
   - amplitude/phase of steady state oscillations
   - transient phenomena
3. Coupled harmonic oscillators
   - masses/springs, coupled pendula, RLC circuits
4. Uniformly distributed discrete systems
   - masses on string fixed at both ends
   - lots of masses/springs

Review

5. Continuously distributed systems (standing waves)
   - string fixed at both ends
   - sound waves in pipes (open end/closed end)
   - transmission lines
   - Fourier analysis
6. Progressive waves in continuous systems
   - dispersion, phase velocity/group velocity
   - reflection/transmission coefficients
7. Waves in two and three dimensions
   - Laplacian operator
   - Rotationally symmetric solutions in 2d and 3d

Simple Harmonic Motion

- Any system in which the force is opposite the displacement will oscillate about a point of stable equilibrium
- If the force is proportional to the displacement it will undergo simple harmonic motion
- Examples:
  - Mass/massless spring
  - Elastic rod (characterized by Young's modulus)
  - Floating objects
  - Torsion pendulum (shear modulus)
  - Simple pendulum
  - Physical pendulum
  - LC circuit

Simple Harmonic Motion

- You should be able to draw a free-body diagram and express the force in terms of the displacement.
- Use Newton's law: $m\ddot{x} = F$
- Write it in standard form:

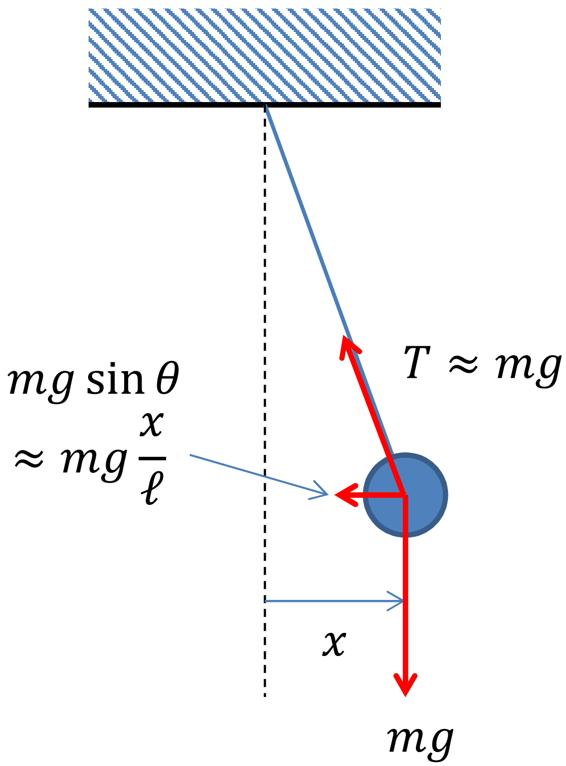
$$\ddot{x} + \omega^2 x = 0$$

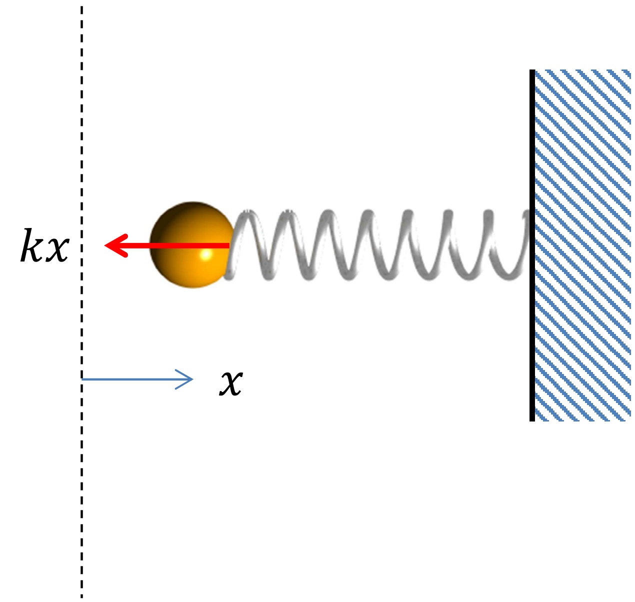
- Solutions are of the form:

$$x(t) = A\cos(\omega t - \delta)$$
$$x(t) = A\cos\omega t + B\sin\omega t$$

- ***You must be able to use the initial conditions to solve for the constants of integration***

Examples

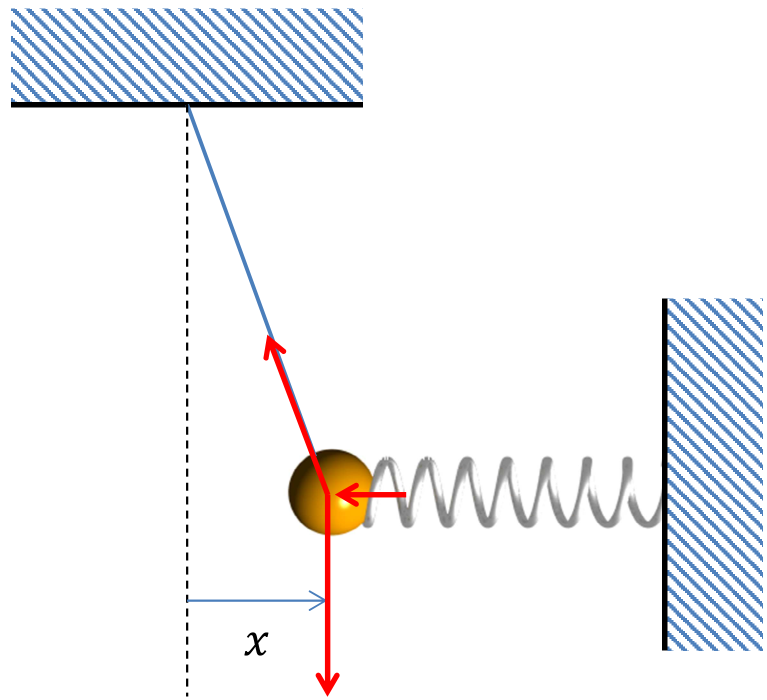




$$m\ddot{x} = -mgx/\ell$$

$$m\ddot{x} = -kx$$

Examples



$$m\ddot{x} = ?$$

Damped Harmonic Motion

- Damping forces remove energy from the system
- We will only consider cases where the force is proportional to the velocity: $F = -bv$
- You should be able to construct a free-body diagram and write the resulting equation of motion:

$$m\ddot{x} + b\dot{x} + kx = 0$$

  – You should be able to write it in the standard form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

- ***You must be able to solve this differential equation!***

Damped Harmonic Motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Let $x(t) = Ae^{\alpha t}$

- Characteristic polynomial:

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

- Roots (use the quadratic formula):

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (\omega_0)^2}$$

- Classification of solutions:
  - Over-damped: $\gamma^2/4 - (\omega_0)^2 > 0$ (distinct real roots)
  - Critically damped: $\gamma^2/4 = (\omega_0)^2$ (one root)
  - Under-damped: $\gamma^2/4 - (\omega_0)^2 < 0$ (complex roots)

Damped Harmonic Motion

- Over-damped motion: $\gamma^2/4-(\omega_0)^2>0$

$$x(t)=Ae^{-\frac{\gamma}{2}t}e^{t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}+Be^{-\frac{\gamma}{2}t}e^{-t\sqrt{\frac{\gamma^2}{4}-(\omega_0)^2}}$$

- Under-damped motion: $\gamma^2/4-(\omega_0)^2<0$

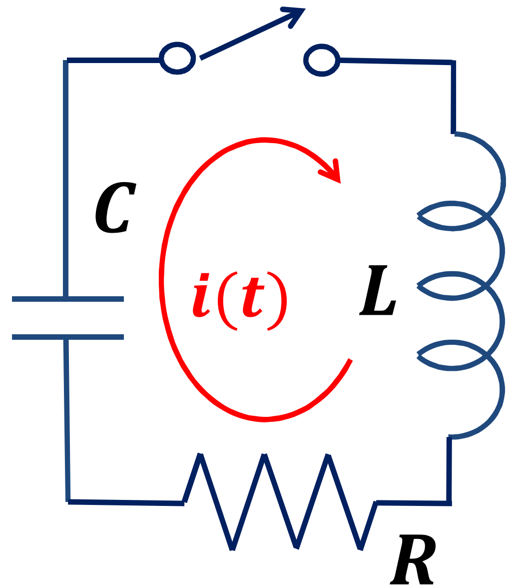
$$x(t)=Ae^{-\frac{\gamma}{2}t}e^{it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}}+Be^{-\frac{\gamma}{2}t}e^{-it\sqrt{(\omega_0)^2-\frac{\gamma^2}{4}}}$$

- Critically damped motion:

$$x(t)=(A+Bt)e^{-\frac{\gamma}{2}t}$$

- ***You must be able to use the initial conditions to solve for the constants of integration***

Example



Sum of potential differences:

$$-L\frac{di}{dt} - i(t)R - \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Initial charge, Q_0, defines the initial conditions.

Example

$$L\frac{di}{dt} + i(t)R + \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = 0$$

Differentiate once with respect to time:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

$$\frac{d^2i}{dt^2} + \gamma\frac{di}{dt} + \omega_0^2 i(t) = 0$$

Remember, the solution is $\boldsymbol{i(t)}$ but the initial conditions might be in terms of $\boldsymbol{Q(t) = Q_0 + \int i(t)dt}$

(See examples from Lecture 7)

Forced Harmonic Motion

- Now the differential equation is

$$m\ddot{x} + b\dot{x} + kx = F(\omega) = F_0 \cos \omega t$$

- Driving function is not always given in terms of a force… remember Question #2 on Assignment #3:

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = -\frac{d^2\eta}{dt^2} = C\omega^2 \cos \omega t$$

- General properties:
  - Steady state properties: $t \gg 1/\gamma$
  - Solution is $y(t) = A\cos(\omega t - \delta)$
  - Amplitude, A, and phase, δ, depend on ω

Forced Harmonic Motion

"Q" quantifies the amount of damping:

$$Q = \frac{\omega_0}{\gamma}$$

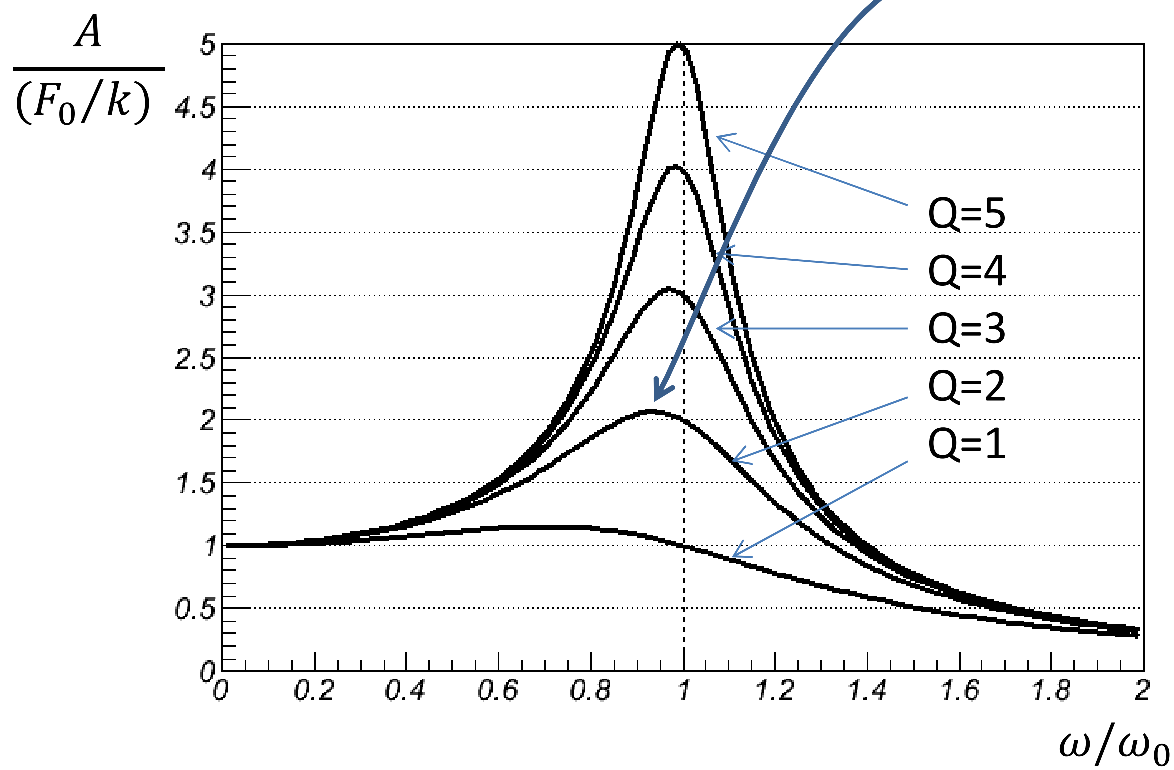
(large Q means small damping force)

$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

$$\delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right)$$

Resonance

- Qualitative features: amplitude



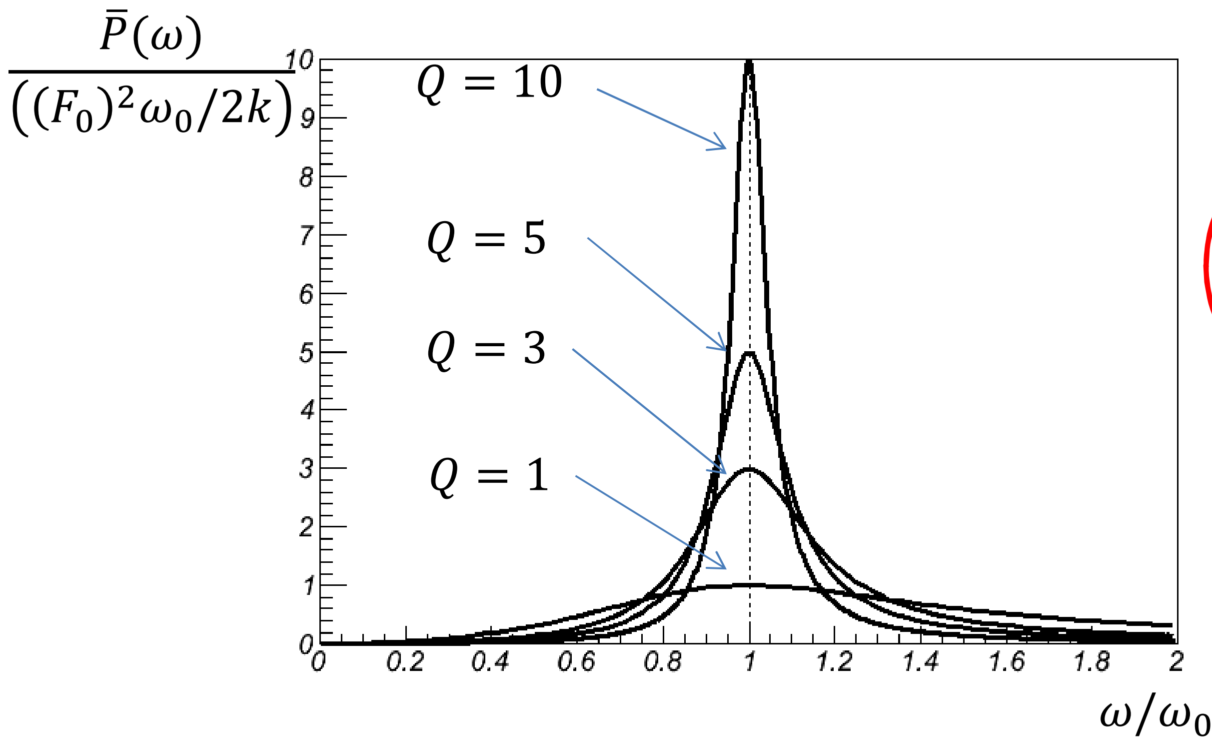
$$\frac{\omega_{free}}{\omega_0} = 1$$

$$\omega_{free} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

Average Power

- The rate at which the oscillator absorbs energy is:

$$\bar{P}(\omega) = \frac{(F_0)^2\omega_0}{2kQ}\frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$



Full-Width-at-Half-Max:

$$\boldsymbol{FWHM = \frac{\omega_0}{Q} = \gamma}$$

Resonance

- Qualitative features: phase shift

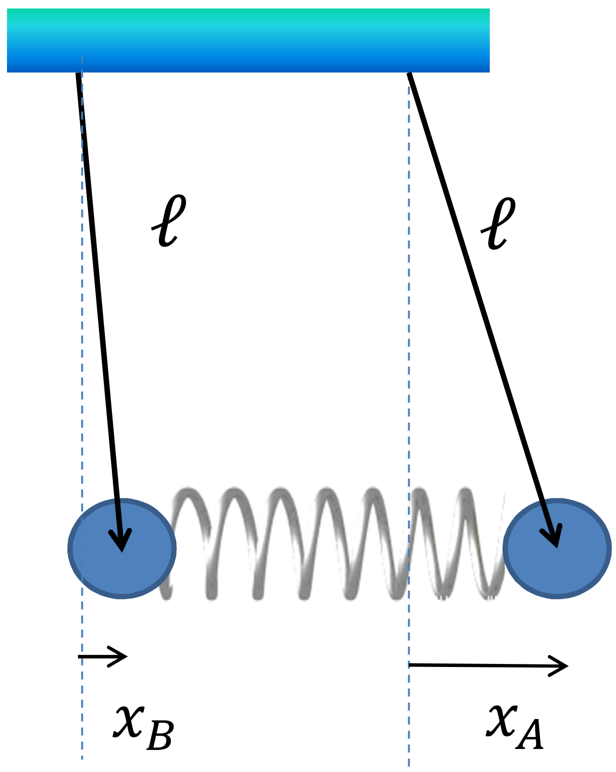
$$\delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right)$$

$\delta \to 0$ at low frequencies

$\delta \to \pi$ at high frequencies

$\boldsymbol{\delta = \frac{\pi}{2}}$ **when** $\boldsymbol{\omega = \omega_0}$

Coupled Oscillators



- Restoring force on pendulum A:
  $$F_A = -k(x_A - x_B)$$
- Restoring force on pendulum B:
  $$F_B = k(x_A - x_B)$$

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$

$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

Coupled Oscillators

- ***You must be able to draw the free-body diagram and set up the system of equations.***

$$m\ddot{x}_A + \frac{mg}{\ell}x_A + k(x_A - x_B) = 0$$
$$m\ddot{x}_B + \frac{mg}{\ell}x_B - k(x_A - x_B) = 0$$

- ***You must be able to write this system as a matrix equation.***

$$\begin{pmatrix} \ddot{x}_A \\ \ddot{x}_B \end{pmatrix} + \begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 \end{pmatrix} \begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = 0$$

Coupled Oscillators

- Assume solutions are of the form

$$\begin{pmatrix} x_A(t) \\ x_B(t) \end{pmatrix} = \begin{pmatrix} x_A \\ x_B \end{pmatrix} \cos(\omega t - \delta)$$

- Then,

$$\begin{pmatrix} (\omega_0)^2 + (\omega_c)^2 - \omega^2 & -(\omega_c)^2 \\ -(\omega_c)^2 & (\omega_0)^2 + (\omega_c)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = 0$$

- ***You must be able to calculate the eigenvalues of a 2x2 or 3x3 matrix.***
  - ***Calculate the determinant***
  - ***Calculate the roots by factoring the determinant or using the quadratic formula.***
- These are the frequencies of the normal modes of oscillation.

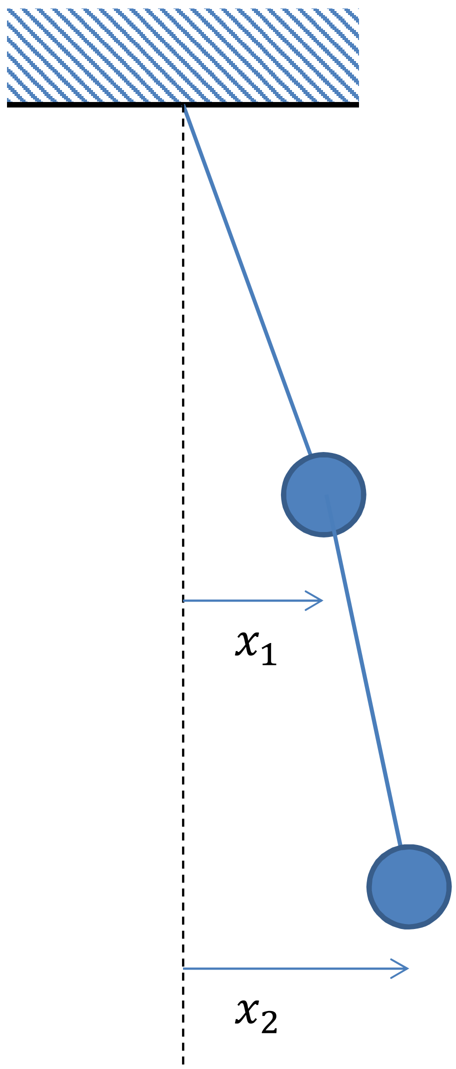
Coupled Oscillators

- ***You must be able to calculate the eigenvectors of a 2x2 or 3x3 matrix***
- General solution:

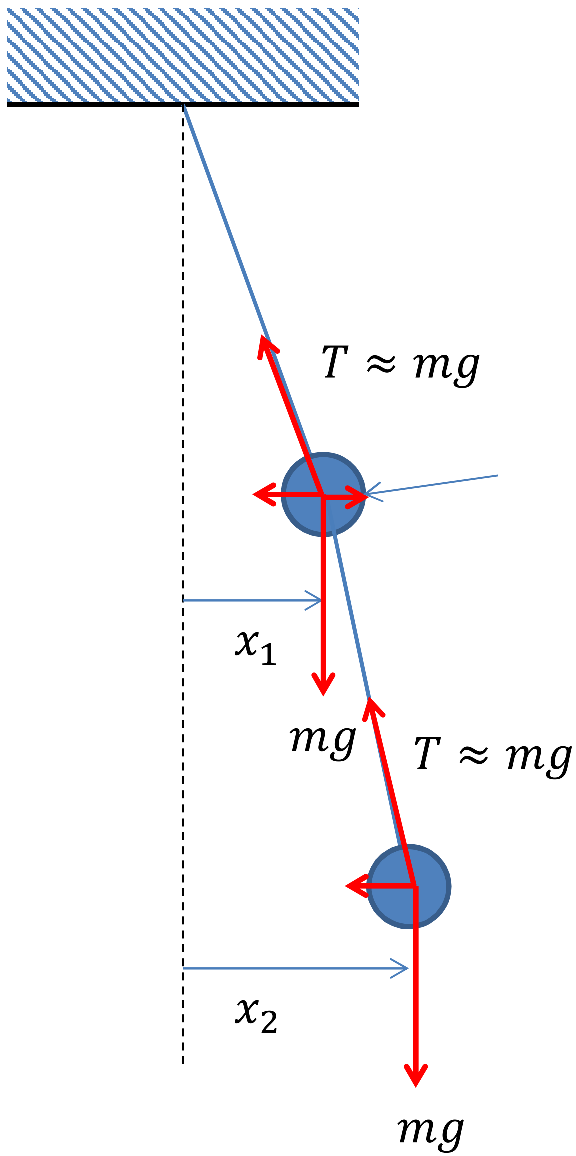
$$\vec{x}(t) = \boldsymbol{A}\vec{x}_1 \cos(\omega_1 t - \boldsymbol{\alpha}) + \boldsymbol{B}\vec{x}_2 \cos(\omega_2 t - \boldsymbol{\beta}) + \cdots$$

- ***You must be able to solve for the constants of integration using the initial conditions.***

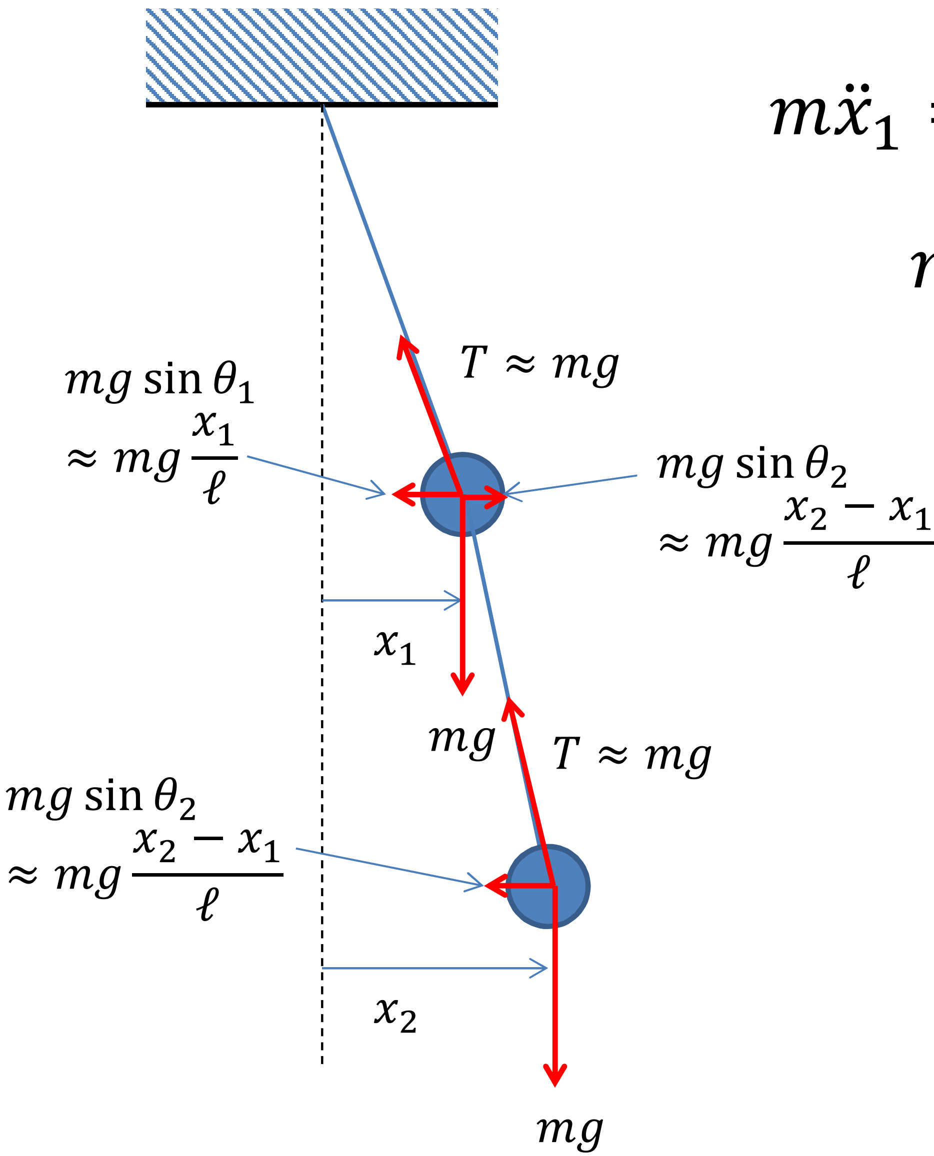
Example



Example



Example



$$m\ddot{x}_1 = -mg\frac{x_1}{\ell} + mg\frac{x_2 - x_1}{\ell}$$

$$m\ddot{x}_2 = -mg\frac{x_2 - x_1}{\ell}$$

Example

Equations of motion:

$$m\ddot{x}_1 = -mg\frac{x_1}{\ell} + mg\frac{x_2 - x_1}{\ell}$$

$$m\ddot{x}_2 = -mg\frac{x_2 - x_1}{\ell}$$

Write as a matrix:

$$\ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 = 0$$

$$\ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Example

Normal modes:

$$\vec{x}_i(t) = \vec{x}_i \cos(\omega t - \delta)$$

Eigenvalue problem:

$$\begin{pmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{vmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{vmatrix} = 0$$

$$(-\omega^2 + 2\omega_0^2)(-\omega^2 + \omega_0^2) - \omega_0^4 = 0$$

$$(\lambda - 2\omega_0^2)(\lambda - \omega_0^2) - \omega_0^4 = 0$$

$$\lambda = \omega^2 = \frac{3}{2}\omega_0^2 \pm \frac{1}{2}\sqrt{\omega_0^4 + 4\omega_0^4}$$

Example

- Eigenvector for $\omega_1^2 = \omega_0^2(\frac{3-\sqrt{5}}{2})$

$$\omega_0^2 \begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \binom{x_1}{x_2} = 0$$

$$x_2 = \frac{1+\sqrt{5}}{2} x_1$$

- Normal mode:

$$x_1(t) = A \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \cos(\omega_1 t - \alpha)$$

Example

- Eigenvector for $\omega_1^2 = \omega_0^2 (\frac{3+\sqrt{5}}{2})$

$$\omega_0^2 \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \binom{x_1}{x_2} = 0$$

$$x_2 = \frac{1-\sqrt{5}}{2} x_1$$

- Normal mode:

$$x_2(t) = B \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \cos(\omega_2 t - \beta)$$