PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 20 – French, Chapter 8

Spring 2013 Semester

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Midterm Exam:

Date:	Wednesday, March 6^{th}
Time:	8:00 – 10:00 pm
Room:	PHYS 203
Material:	French, chapters 1-8

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- The solution, $y(x,t)$, describes the shape of a string as a function of x and t.
- This is a transverse wave: the displacement is perpendicular to the direction of propagation.
- This would confuse the following discussion…
- Instead, let’s now consider longitudinal waves, like the pressure waves due to the propagation of sound in a gas.

Waves in Three Dimensions

- Wave equation in one dimension:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- The solution, $p(x, t)$, describes the excess pressure in the gas as a function of x and t.
- What if the wave was propagating in the y-direction?

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- What if the wave was propagating in the z-direction?

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

Waves in Three Dimensions

- The excess pressure is now a function of $\vec{x}$ and t.
- Wave equation in three dimensions:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- But we like to write it this way:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- Where ∇^2 is called the "Laplacian operator", but you just need to think of it as a bunch of derivatives:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Waves in Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- How do we solve this? Here's how…

$$p(\vec{x},t) = p_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

- One partial derivatives:

$$\frac{\partial p}{\partial x} = i p_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\frac{\partial}{\partial x}\left(\vec{k}\cdot\vec{x}-\omega t\right)$$

$$= i p_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\frac{\partial}{\partial x}\left(k_x x + k_y y + k_z z - \omega t\right)$$

$$= i k_x p(\vec{x},t)$$

- Second derivative:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2\, p(\vec{x},t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

- Second derivatives:

$$\frac{\partial^2 p}{\partial x^2} = -k_x^2\, p(\vec{x}, t)$$
$$\frac{\partial^2 p}{\partial y^2} = -k_y^2\, p(\vec{x}, t)$$
$$\frac{\partial^2 p}{\partial z^2} = -k_z^2\, p(\vec{x}, t)$$
$$\frac{\partial^2 p}{\partial t^2} = -\omega^2\, p(\vec{x}, t)$$

Waves in Two and Three Dimensions

- Wave equation in three dimensions:

$$\nabla^2 p = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

$$-\left(k_x^2 + k_y^2 + k_z^2\right)p(\vec{x},t) = -\frac{\omega^2}{v^2}p(\vec{x},t)$$

- Any values of k_x, k_y, k_z satisfy the equation, provided that

$$\omega = v\sqrt{k_x^2 + k_y^2 + k_z^2} = v\left|\vec{k}\right|$$

- If $k_y = k_z = 0$ then $p(\vec{x},t) = p_0 e^{i(k_x x - \omega t)}$ but this described a wave propagating in the $+x$ direction.

Waves in Three Dimensions

$$p(\vec{x}, t) = p_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

- The vector, $\vec{k}$, points in the direction of propagation
- The wavelength is $\lambda = 2\pi/|\vec{k}|$
- How do we visualize this solution?
  - Pressure is equal at all points $\vec{x}$ such that $\vec{k} \cdot \vec{x} - \omega t = \phi$ where ϕ is some constant phase.
  - Let $\vec{x}'$ be some other point such that $\vec{k} \cdot \vec{x}' - \omega t = \phi$
  - We can write $\vec{x}' = \vec{x} + \vec{u}$ and this tells us that $\vec{k} \cdot \vec{u} = 0$.
  - $\vec{k}$ and $\vec{u}$ are perpendicular.
  - *All points in the plane perpendicular to $\vec{k}$ have the same phase.*

Waves in Three Dimensions

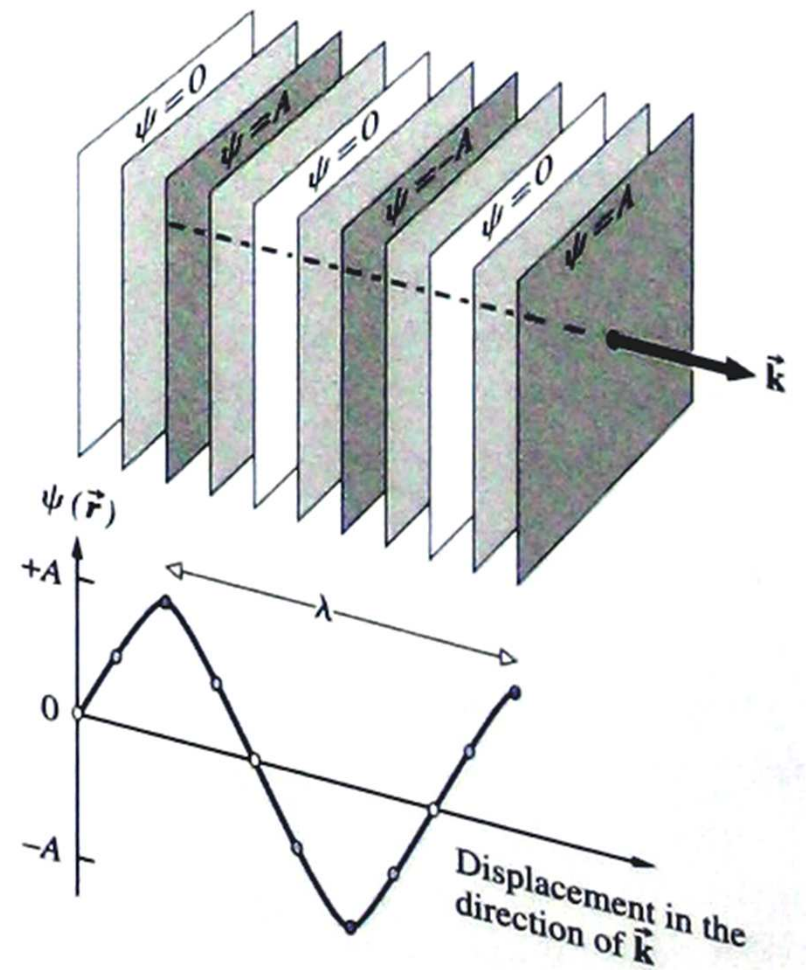
- As usual, we are mainly interested in the real component:

$$\psi(\vec{r},t) = A\cos\left(\vec{k}\cdot\vec{x} - \omega t\right)$$

- A wave propagating in the opposite direction would be described by

$$\psi'(\vec{r},t) = A'\cos\left(\vec{k}\cdot\vec{x} + \omega t\right)$$

- The points in a plane with a common phase is called the "wavefront".



Waves in Three Dimensions

$$\psi(\vec{r}, t) = A\cos\left(\vec{k} \cdot \vec{x} \mp \omega t\right)$$

- Sometimes we are free to pick a coordinate system in which to describe the wave motion.
- If we choose the x-axis to be in the direction of propagation, we get back the one-dimensional solution we are familiar with:

$$\psi(\vec{r}, t) = A\cos(kx \mp \omega t)$$

- But in one-dimension we saw that any function that satisfied $f(x \pm vt)$ was a solution to the wave equation.
- What is the corresponding function in three dimensions?

Waves in Three Dimensions

$$\omega = v\sqrt{k_x^2 + k_y^2 + k_z^2} = v\left|\vec{k}\right|$$

- General solution to the wave equation are functions that are twice-differentiable of the form:

$$\psi(\vec{r}, t) = C_1 f\left(\hat{k} \cdot \vec{r} - vt\right) + C_2 g\left(\hat{k} \cdot \vec{r} + vt\right)$$

where $\hat{k} = \vec{k}/\left|\vec{k}\right|$

- Just like in the one-dimensional case, these do not have to be harmonic functions.

Example

- Is the function $\psi(\vec{x}, t) = (ax + bt + c)^2$ a solution to the wave equation?
- It should be because we can write it as

$$\psi(\vec{x}, t) = (a(\boldsymbol{x + vt}) + c)^2$$

where $v = b/a$ which is of the form $g(x + vt)$

- We can check explicitly:

$$\frac{\partial \psi}{\partial x} = 2a(ax + bt + c) \qquad \frac{\partial \psi}{\partial x} = 2b(ax + bt + c)$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2a^2 \qquad \frac{\partial^2 \psi}{\partial x^2} = 2b^2$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \quad \Rightarrow \quad 2a^2 = 2b^2/v^2 \quad \Rightarrow \quad v = b/a$$

Example

- Is the function $\psi(\vec{x}, t) = ax^{-2} + bt$, where $a > 0, b > 0$, a solution to the wave equation?
- It is twice differentiable…

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{6a}{x^4} \qquad \frac{\partial^2 \psi}{\partial t^2} = 0$$

- But it is not a solution:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \Rightarrow \frac{6a}{x^4} = 0$$

  – Only true if $a = 0$, which we already said was not the case.
- This is not a solution to the wave equation.

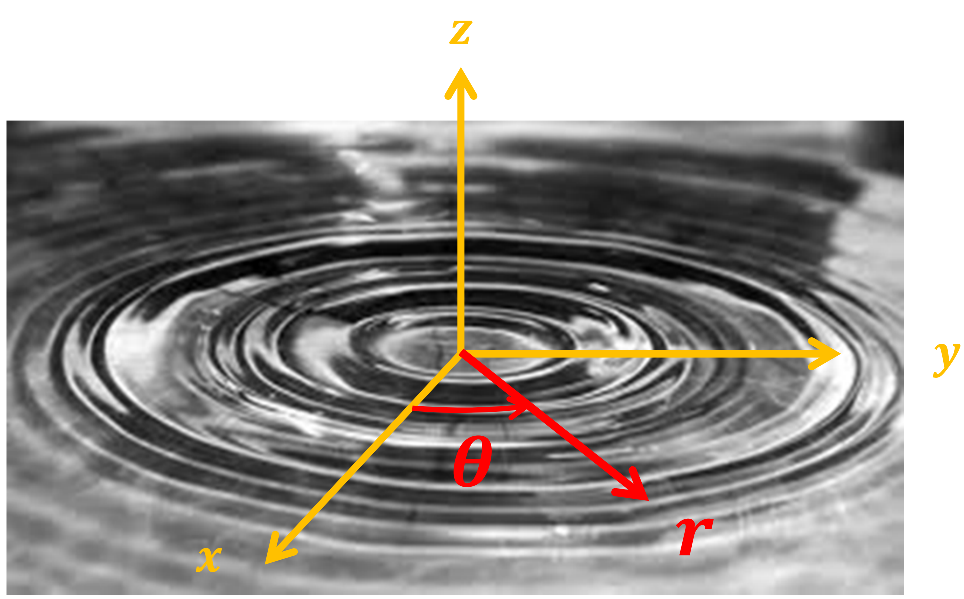
Waves in Two Dimensions

- Plane waves frequently provide a good description of physical phenomena, but this is usually an approximation:

- This looks like a wave… can the wave equation describe this?

Waves in Two Dimensions

- Rotational symmetry:
  - Cartesian coordinates are not well suited for describing this problem.
  - Use polar coordinates instead.
  - Motion should depend on r but should be independent of θ



Waves in Two Dimensions

- Wave equation: $\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$
- How do we write ∇^2 in polar coordinates?

$$r = \sqrt{x^2 + y^2}$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

- Derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

Waves in Two Dimensions

$$
\begin{aligned}
&= \cos\theta[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial r}] + \sin\theta[\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}] \\
&= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}
\end{aligned}
$$

Similarly,

$$
\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}.
$$

Taking one more derivative, we see

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= r\frac{\partial}{\partial \theta}[-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}] \\
&= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r[-\sin\theta(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial \theta}) + \cos\theta(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta})] \\
&= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r[-\sin\theta(-r\sin\theta\frac{\partial^2 u}{\partial x^2} + r\cos\theta\frac{\partial^2 u}{\partial x \partial y}) + \cos\theta(-r\sin\theta\frac{\partial^2 u}{\partial x \partial y} + r\cos\theta\frac{\partial^2 u}{\partial y^2})] \\
&= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r^2[\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}] \\
&= -r\frac{\partial u}{\partial r} + r^2[\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}]
\end{aligned}
$$

Now we're ready to put everything together:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} \\
&= (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial x^2} + (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}
\end{aligned}
$$

Waves in Two Dimensions

- Laplacian in polar coordinates:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

- When the geometry is does not depend on θ or z:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)$$

- Wave equation:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Waves in Two Dimensions

- Wave equation:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

- If we assume that $\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi$ then the equation is:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\omega^2}{v^2}\psi = 0$$

- Change of variables: Let $\rho = r\omega/v$

$$\frac{\omega^2}{v^2}\frac{\partial^2\psi}{\partial\rho^2} + \frac{\omega^2}{v^2}\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\omega^2}{v^2}\psi = 0$$

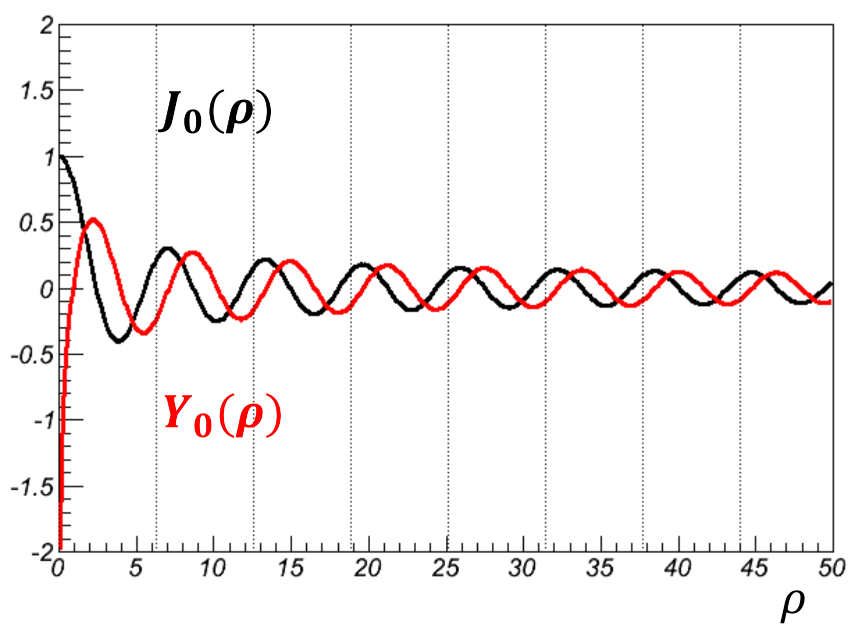
$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \psi(\rho) = 0$$

Waves in Two Dimensions

- Bessel's Equation:

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \psi(\rho) = 0$$

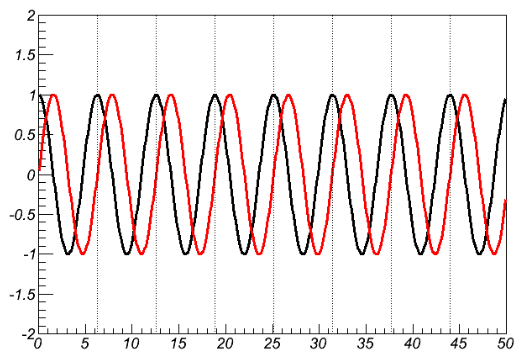
- Solutions are "Bessel functions": $J_0(\rho), Y_0(\rho)$



Bessel Functions?

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\omega^2}{v^2}\psi=0$$

- Solutions: $\sin kx, \cos kx$
- Graphs:

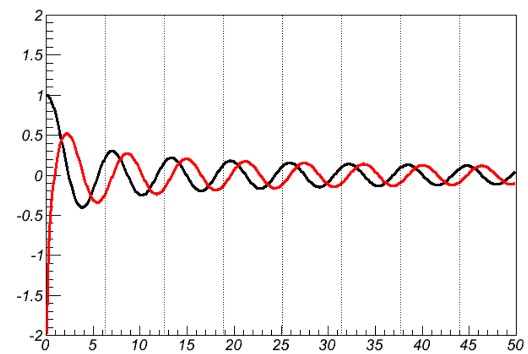


- Series representation:

$$\cos kx=\sum_{n=0}^{\infty}\frac{(-1)^n(kx)^{2n}}{(2n)!}$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\omega^2}{v^2}\psi=0$$

- Solutions: $J_0(kr), Y_0(kr)$
- Graphs:

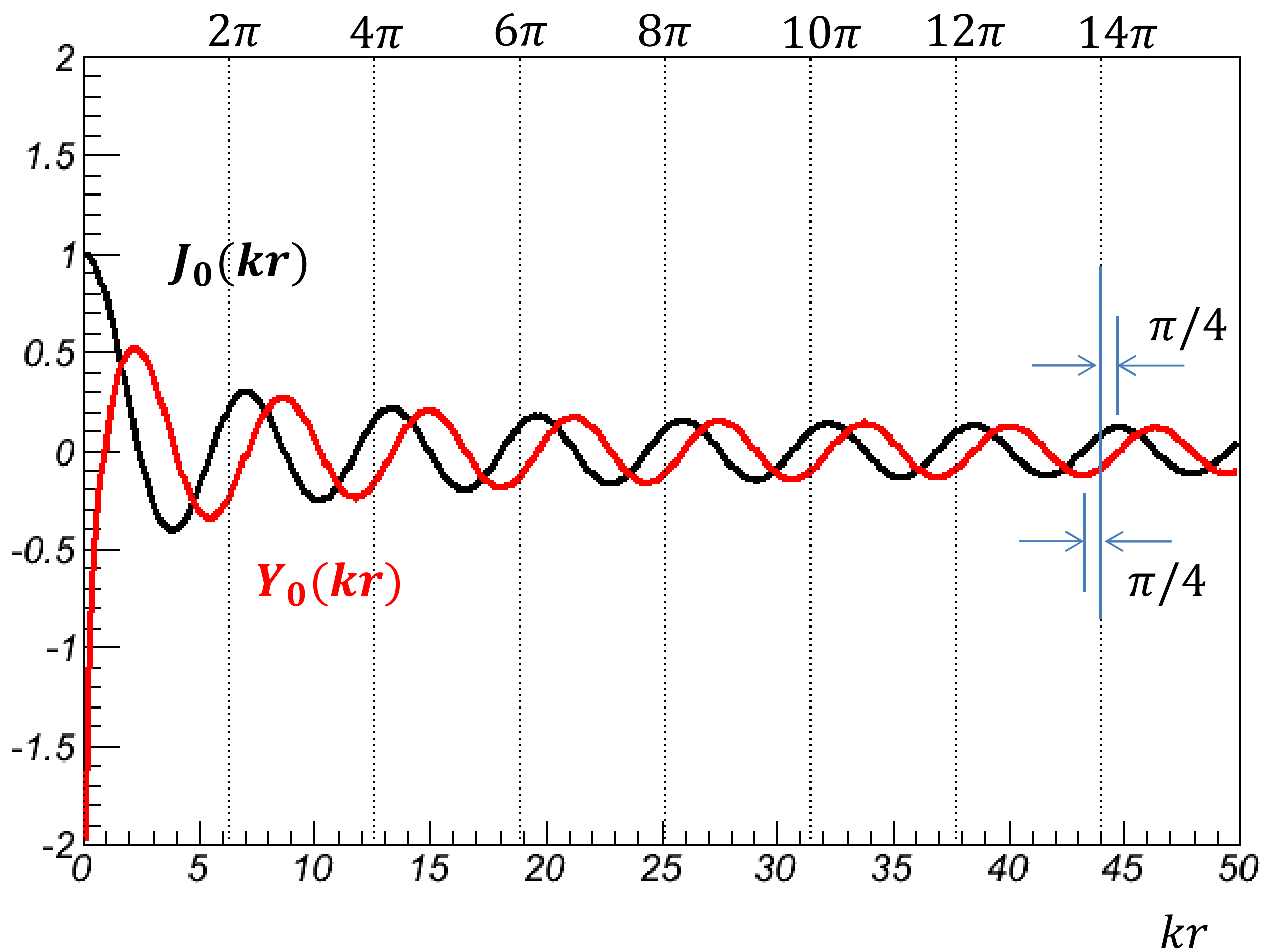


- Series representation:

$$J_0(kr)=\sum_{n=0}^{\infty}\frac{(-1)^n(kr)^{2n}}{2^{2n}(n!)^2}$$

Asymptotic Properties

- At large values of r…

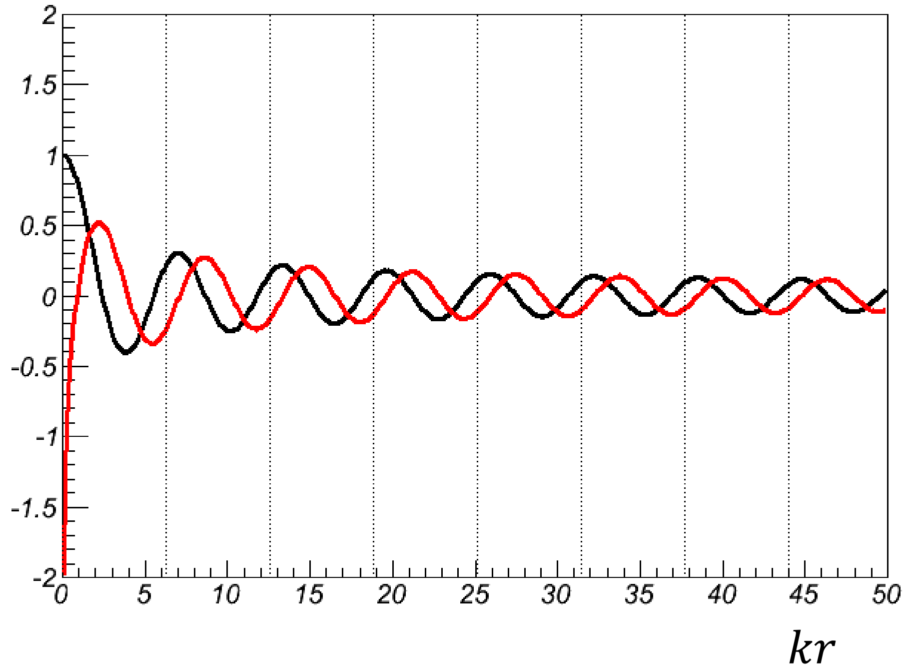


Asymptotic Properties

- When r is large, for example, $kr \gg 1$

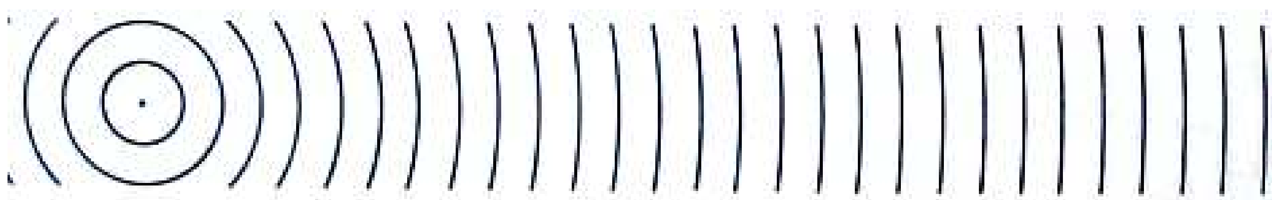
$$J_0(kr) \approx \sqrt{2/\pi}\frac{\cos(kr - \pi/4)}{\sqrt{kr}}$$

$$Y_0(kr) \approx \sqrt{2/\pi}\frac{\sin(kr - \pi/4)}{\sqrt{kr}}$$



Energy

- The energy carried by a wave is proportional to the square of the amplitude.
- When $\psi(r,t) \sim A\frac{\cos kr}{\sqrt{r}}$ the energy density decreases as $1/r$
- But the wave is spread out on a circle of circumference $2\pi r$
- The total energy is constant, independent of r
- At large r they look like plane waves:



Waves in Three Dimensions

- In spherical coordinates (r, θ, ϕ) the Laplacian is:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\psi) +$$

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2}$$

- When $\psi(\vec{r}, t)$ is independent of θ and ϕ then the second line is zero.
- This time, let $\psi(r, t) = \frac{f(r)}{r} \cos\omega t$
- Time derivative: $\frac{\partial^2\psi}{\partial t^2} = -\omega^2\, \psi$

Waves in Three Dimensions

- Let $\psi(r,t) = \frac{f(r)}{r}\cos\omega t$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}f(r)\cos\omega t = -\frac{\omega^2}{v^2}\frac{f(r)}{r}\cos\omega t$$

$$\frac{\partial^2 f}{\partial r^2} = -\frac{\omega^2}{v^2}f(r)$$

- We know the solution to this differential equation:

$$f(r) = Ae^{ikr}$$

- The solution to the wave equation is

$$\psi(r,t) = A\frac{e^{ikr}}{r}\cos\omega t$$

Waves in Three Dimensions

- Or we could write

$$\psi(r,t) = A\,\frac{\cos k(r \mp vt)}{r}$$

- Waves carry energy proportional to amplitude squared: $\propto 1/r^2$
- The energy is spread out over a surface with area $4\pi r^2$
- Energy is conserved
- Looks like a plane wave at large r