PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 17 – French, Chapter 7

Spring 2013 Semester

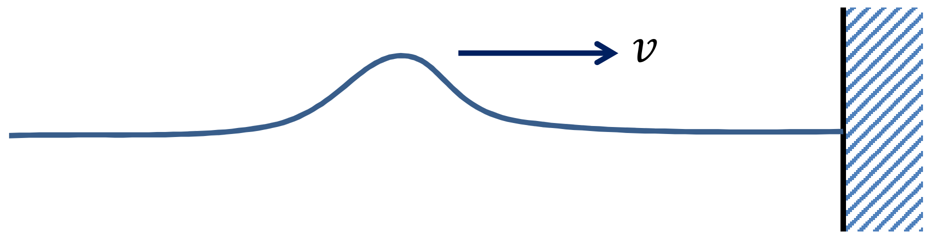
Matthew Jones

Midterm Exam:

Date:	Wednesday, March 6th
Time:	8:00 – 10:00 pm
Room:	PHYS 203
Material:	French, chapters 1-8

Reflection from Boundaries

- Consider a pulse propagating on a string, moving to the right, towards a fixed end:



- We know that this can be represented as a linear combination of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos \omega t$$

- Since the problem is linear, we just need to analyze one normal mode to see what happens next…

Reflection from a Boundary

- Considering just one normal mode:

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\omega t$$

- Trigonometric identity:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

- Re-write this as two travelling waves:

$$y_n(x,t) = \frac{A_n}{2}\left[\sin\left(\frac{n\pi x}{L} + \omega_n t\right) + \sin\left(\frac{n\pi x}{L} - \omega_n t\right)\right]$$

- At the end of the string, $x = L$, this is:

$$y_n(L,t) = \pm\frac{A_n}{2}[\sin(\omega_n t) - \sin(\omega_n t)] = 0$$

Reflection from a Boundary

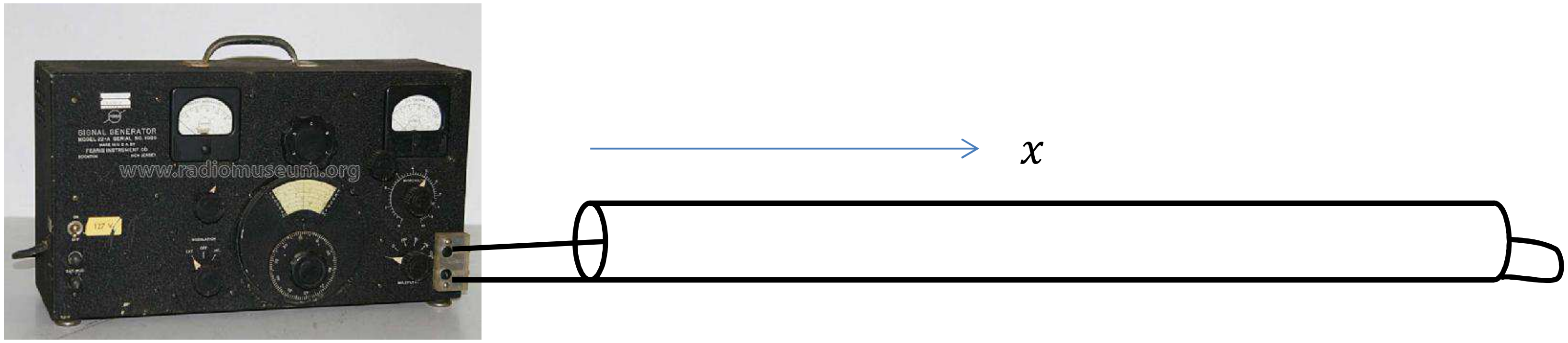
$$y_n(L,t) = \pm\frac{A_n}{2}[\sin(\omega_n t) - \sin(\omega_n t)] = 0$$

- The component of the wave that moves to the left has a displacement that is equal and opposite to the displacement of the incident wave.

Another way to see this:

- The function $y(x,t)$ that describes the shape of the string has two components that move in opposite directions: $y(x,t) = y_i(x,t) + y_r(x,t)$
- At the end of the string, the two components are equal and opposite, which ensures that the boundary condition $y(L,t) = 0$ is satisfied.

Reflection from a Boundary



- Wave equation for the potential difference between the conductors along a transmission line:

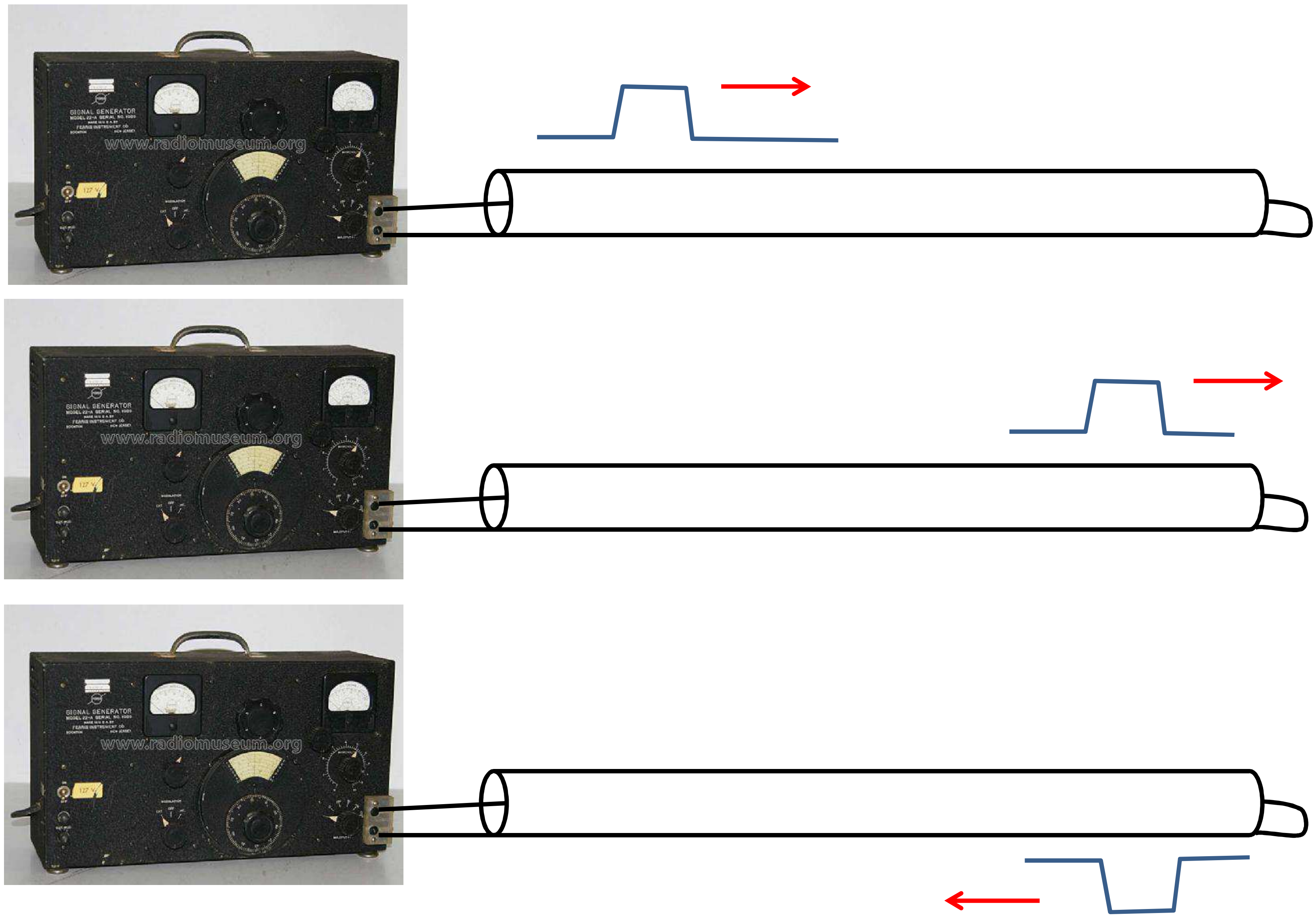
$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}$$

- If the inner and outer conductor are shorted at $x = L$, then the potential difference is zero.

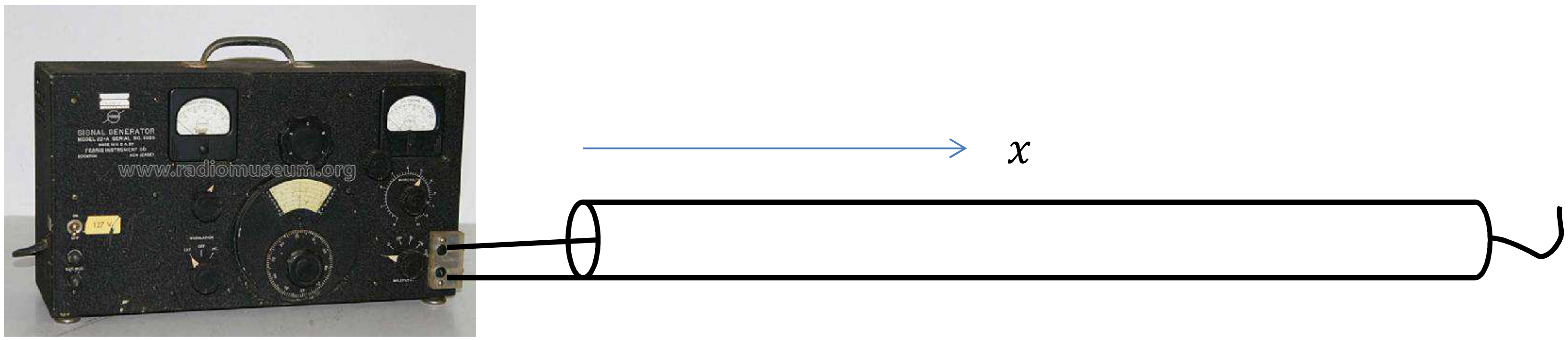
$$V(L,t) = 0$$

- A reflected pulse will propagate back towards the source with opposite amplitude.

Reflection from a Boundary

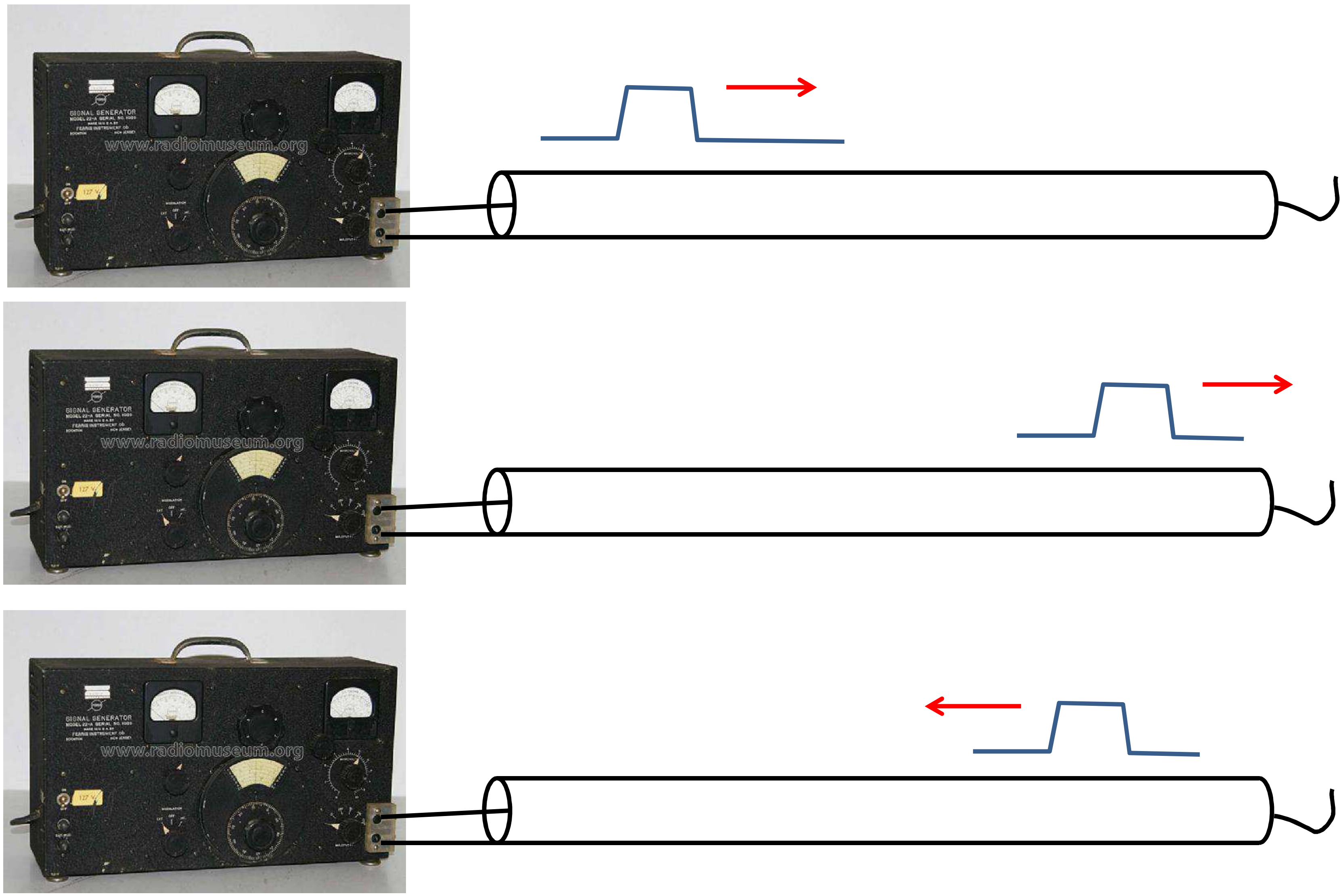


Reflection from a Boundary



- If the end of the transmission line is open, then the incident pulse produces a voltage across the end:
$$V(L,t) = V_L(t)$$
- This acts like a source for a wave propagating to the left.
- The reflected wave is not inverted

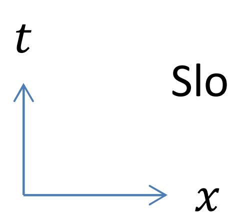
Reflection from a Boundary



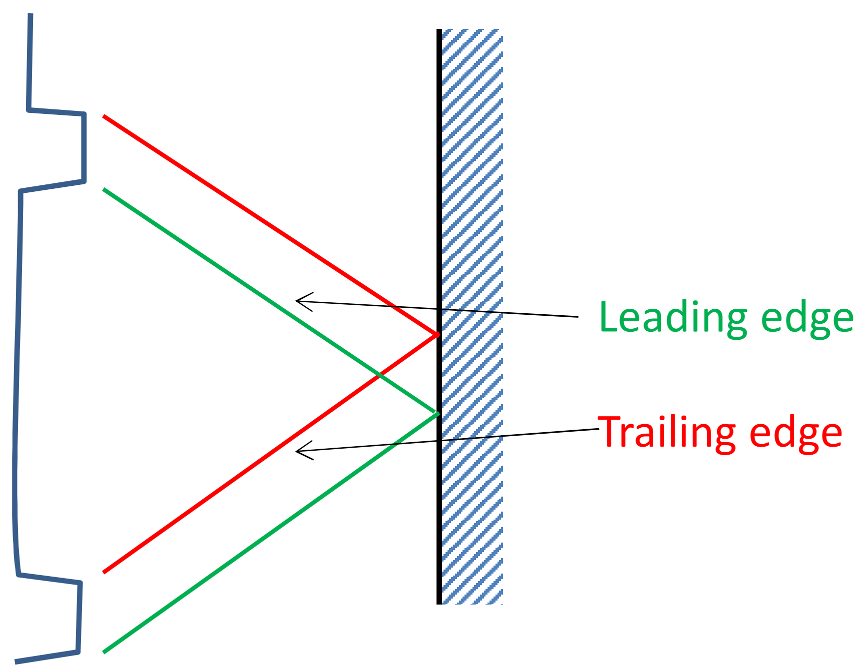
Reflection from a Boundary

- So far we have considered just two cases:
  - Reflection with inversion
  - Reflection without inversion
- We can draw a graphical description of the incident and reflected waves:

Incident and reflected pulses do not overlap.

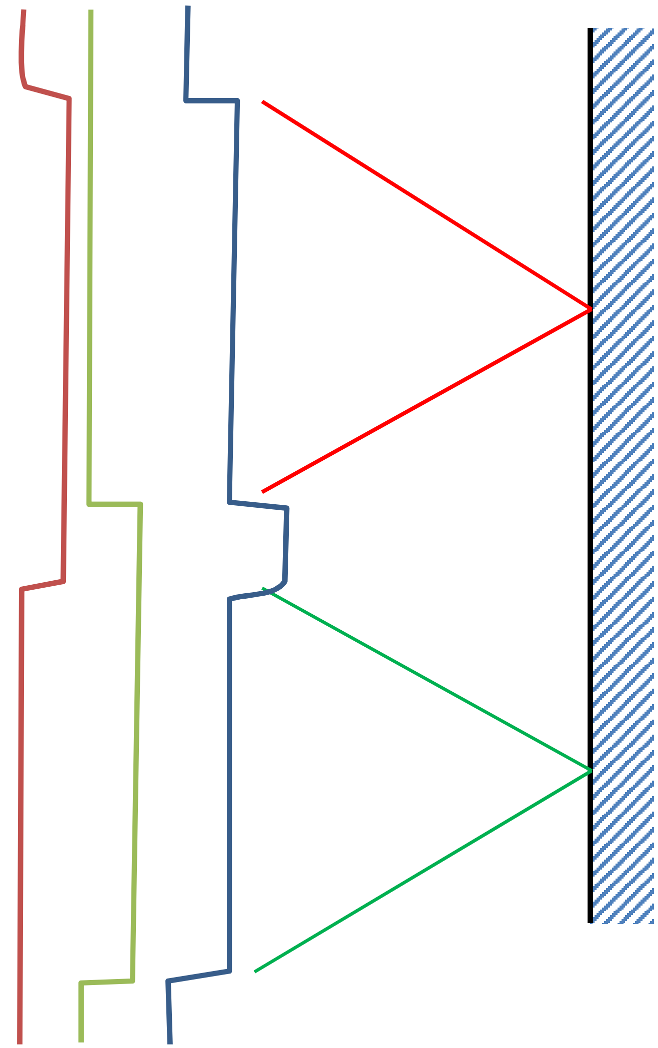


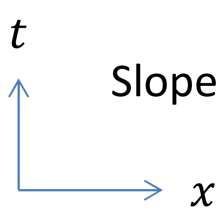
Slope $= 1/v$



Reflection from a Boundary

- We can draw a graphical description of the incident and reflected waves:





Reflection from a Discontinuity

- Suppose a pulse propagates on a string with an abrupt change in mass per unit length:

- Velocity in each section:

$$v_1 = \sqrt{T/\mu_1} \qquad\qquad v_2 = \sqrt{T/\mu_2}$$

- The function $y(x,t)$ needs three components:
  - Incident pulse: $y_i(x - v_1 t)$
  - Reflected pulse: $y_r(x + v_1 t)$
  - Transmitted pulse: $y_t(x - v_2 t)$
- Boundary conditions:
  - Continuity of the function $y(x,t)$ and its derivative.

Reflection from a Discontinuity

- Function for the wave in the string with μ_1:

$$y_1(x,t) = y_i(x - v_1 t) + y_r(x + v_1 t)$$

- Function for the wave in string with μ_2:

$$y_2(x,t) = y_t(x - v_2 t)$$

- Continuity at the boundary at $x = 0$:

$$y_i(0,t) + y_r(0,t) = y_t(0,t)$$

$$\frac{\partial y_i}{\partial x} + \frac{\partial y_r}{\partial x} = \frac{\partial y_t}{\partial x}$$

- But there is a relation between the derivatives:
  - Let $u = x - vt$
  - Then $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\,\frac{\partial u}{\partial t} = -v\,\frac{\partial y}{\partial u}$

Reflection from a Discontinuity

$$\frac{\partial y}{\partial x} = \pm \frac{1}{v}\frac{\partial y}{\partial t}$$

- Continuity at the boundary at $x = 0$:

$$y_i(0,t) + y_r(0,t) = y_t(0,t)$$

$$\frac{\partial y_i}{\partial x} + \frac{\partial y_r}{\partial x} = \frac{\partial y_t}{\partial x}$$

$$-\frac{1}{v_1}\frac{\partial y_i}{\partial t} + \frac{1}{v_1}\frac{\partial y_r}{\partial t} = -\frac{1}{v_2}\frac{\partial y_t}{\partial t}$$

- This can be written:

$$v_2 y_i'(0,t) - v_2 y_r'(0,t) = v_1 y_t'(t)$$

- Now we can integrate with respect to t:

$$v_2 y_i(0,t) - v_2 y_r(0,t) = v_1 y_t(0,t)$$

Reflection from a Discontinuity

$$y_i(0,t) + y_r(0,t) = y_t(0,t)$$
$$v_2 y_i(0,t) - v_2 y_r(0,t) = v_1 y_t(0,t)$$

- Since we specified the initial function $y_i(x,t)$ we just have two equations in two unknowns:

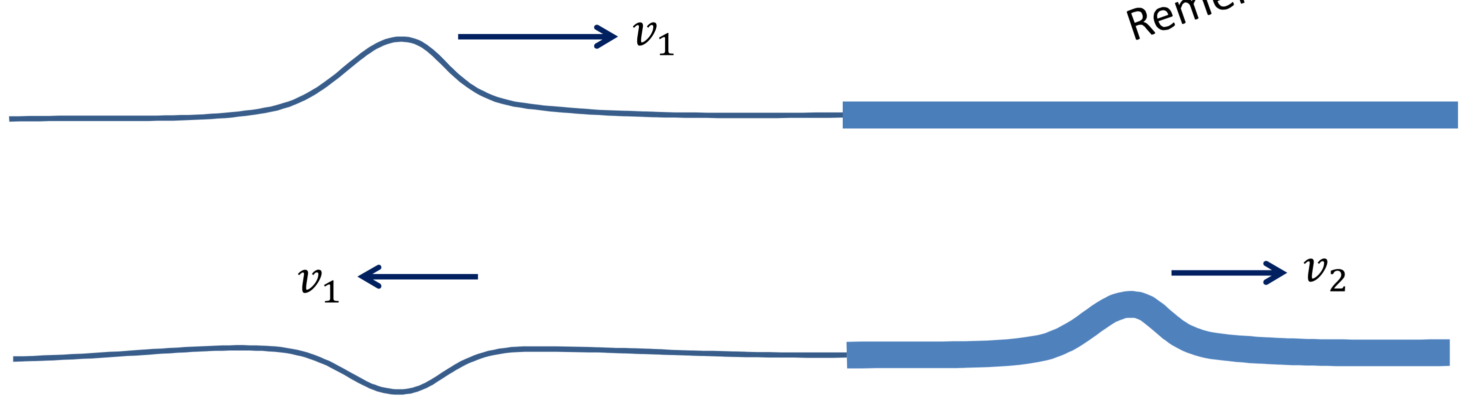
$$-y_r + y_t = y_i$$
$$v_2 y_r + v_1 y_t = v_2 y_i$$
$$\begin{pmatrix} -1 & 1 \\ v_2 & v_1 \end{pmatrix} \begin{pmatrix} y_r \\ y_t \end{pmatrix} = \begin{pmatrix} y_i \\ v_2 y_i \end{pmatrix}$$
$$y_r = y_i \left(\frac{v_2 - v_1}{v_2 + v_1} \right)$$
$$y_t = y_i \left(\frac{2v_2}{v_2 + v_1} \right)$$

Just use Kramer's rule…

Reflection from a Discontinuity

- Reflection coefficient: $\rho = (v_2 - v_1)/(v_2 + v_1)$
- Transmission coefficient: $\tau = 2v_2/(v_2 + v_1)$
- Example: $\mu_2 > \mu_1$ so $v_2 < v_1$

Remember $v = \sqrt{T/\mu}$



(This special case applies to the string, but not in general…)

Example from Optics

- The index of refraction is defined as the ratio:

$$n = \frac{c}{v}$$

- Speed of light in a dense medium: $v = c/n$
- Reflection coefficient:

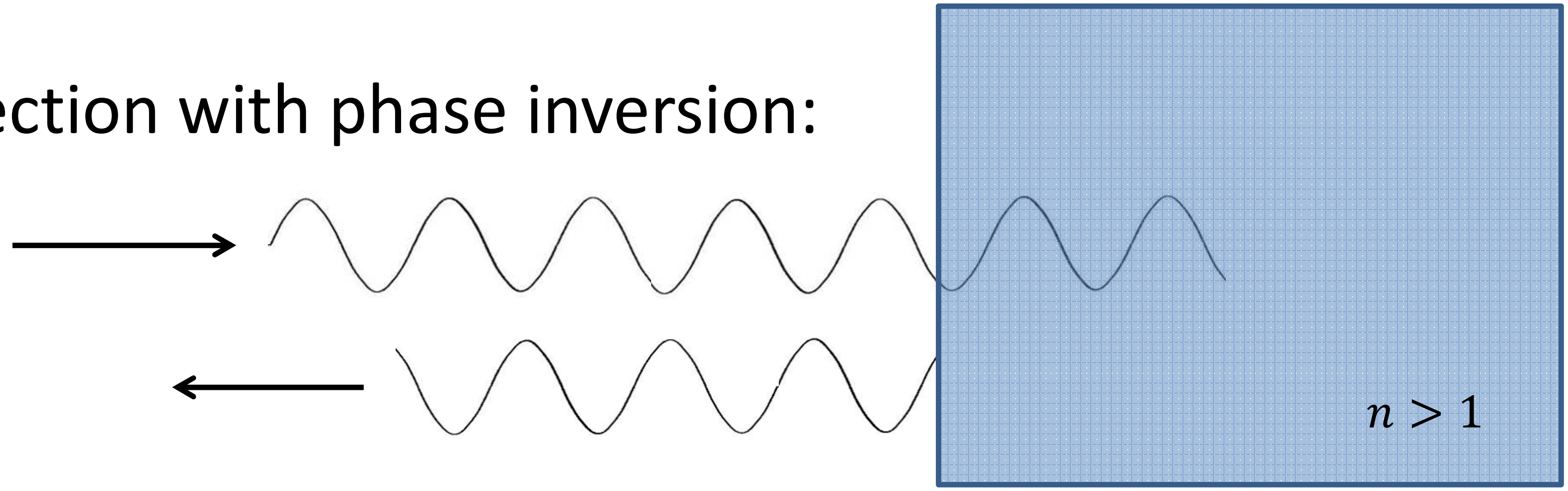
$$\rho = \frac{v_2 - v_1}{v_2 + v_1} = \frac{1/n_2 - 1/n_1}{1/n_2 + 1/n_1} = \frac{n_1 - n_2}{n_1 + n_2}$$

- Phase reversal when $n_2 > n_1$, but not when $n_1 > n_2$
- Transmission coefficient:

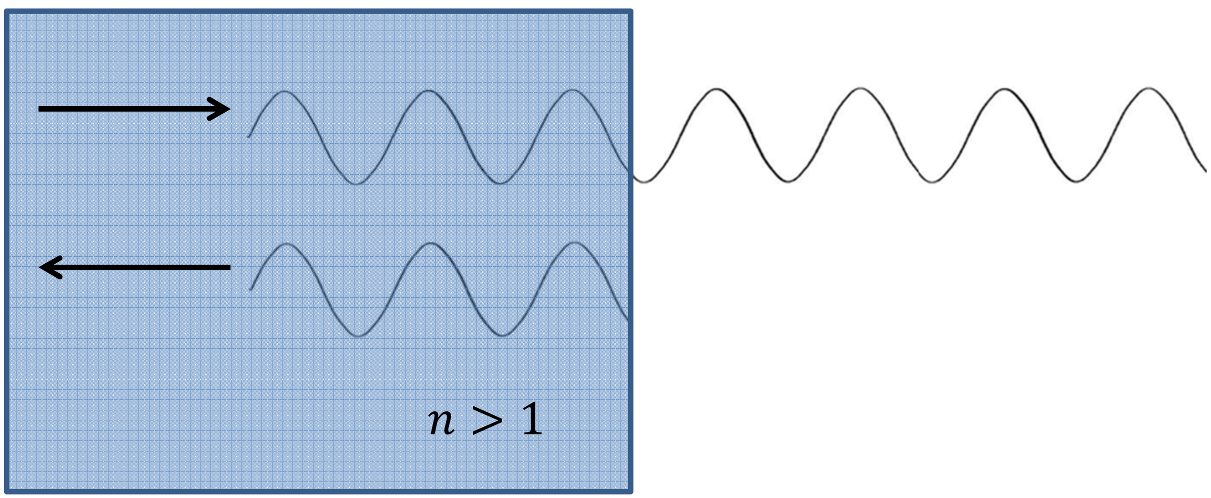
$$\tau = \frac{2v_2}{v_2 + v_1} = \frac{2/n_2}{n_1 + n_2}$$

Reflection

Reflection with phase inversion:



Reflection without phase inversion:



Reflection from a Discontinuity

- But be careful! So far we have assumed that the tension on both sides of the boundary are equal.

$$v_1 = \sqrt{T/\mu_1} \quad v_2 = \sqrt{T/\mu_2}$$

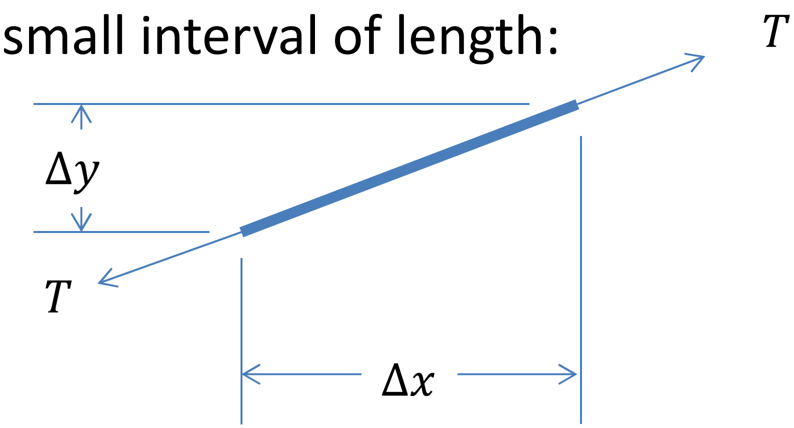
- In other situations this is not always the case:

Solid	Liquid	Gas
$v = \sqrt{Y/\rho}$ Y is Young's modulus	$v = \sqrt{K/\rho}$ K is the bulk modulus	$v = \sqrt{\gamma p/\rho}$ p is the gas pressure γ is a property of the gas.

- The restoring force can be produced by different physical effects.
- Next, let's look at how energy propagates in the medium…

Energy Carried by a Pulse

- Potential energy is stored in an elastic string when it is stretched into the shape of a pulse.
- Potential energy in one small interval of length:



- Work needed to stretch the string in the vertical direction:

$$\Delta W = T\int_0^{\Delta y} \frac{y}{\Delta x}\,dy = \frac{1}{2}\frac{T}{\Delta x}\,(\Delta y)^2$$

- Work per unit length:

$$\frac{\Delta W}{\Delta x} = \frac{1}{2}\,T\left(\frac{\Delta y}{\Delta x}\right)^2$$

Energy Carried by a Pulse

- Work per unit length:

$$\frac{\Delta W}{\Delta x} = \frac{1}{2}\, T \left(\frac{\Delta y}{\Delta x}\right)^2$$

- If the pulse maintains this shape but moves with velocity v then this is the potential energy per unit length.
- Total potential energy is

$$U = \frac{1}{2} T \int \left(\frac{\partial y}{\partial x}\right)^2 \, dx$$

- Written in terms of linear density and velocity, $T = \mu v^2$,

$$U = \frac{1}{2} \mu v^2 \int \left(\frac{\partial y}{\partial x}\right)^2 \, dx$$

Power Carried by a Wave

- A pulse has a finite amount of energy that moves with speed v
- It is also convenient to describe harmonic waves

$$y(x,t) = A\cos\left(\frac{2\pi x}{\lambda} - \omega t\right)$$

which extend in space over many wavelengths

- First derivative:

$$y'(x) = -\frac{2\pi A}{\lambda}\sin\frac{2\pi x}{\lambda}$$

- Energy in one wavelength (cycle):

$$U' = \frac{1}{2}\mu v^2 \int_0^\lambda \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}\mu v^2 \frac{\lambda}{2}\left(\frac{2\pi A}{\lambda}\right)^2$$
$$= \frac{1}{2}\lambda\mu\omega^2 A^2$$

Power Carried by a Wave

- Energy per cycle:

$$U' = \frac{1}{2}\lambda\mu\omega^2 A^2$$

- Cycles passing a point in space per unit time: v/λ
- Average power carried by the wave:

$$P = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}Z\omega^2 A^2$$

- Depends on the characteristic impedance of the medium
  - In this case, $Z = \mu v = T/v$
- Also depends on the properties of the wave
  - Amplitude and frequency

Transmission and Reflection

- For the pulse propagating on the string we had:

$$\rho = (v_2 - v_1)/(v_2 + v_1)$$
$$\tau = 2v_2/(v_2 + v_1)$$

- We want to write this in terms of the properties of the medium, not just the velocity.
- In this case,

$$Z_1 = T/v_1 = \sqrt{T\mu_1} \qquad Z_2 = T/v_2 = \sqrt{T\mu_2}$$

- General expressions:

$$\boldsymbol{\rho} = \frac{\boldsymbol{Z_1 - Z_2}}{\boldsymbol{Z_1 + Z_2}}$$
$$\boldsymbol{\tau} = \frac{\boldsymbol{2Z_1}}{\boldsymbol{Z_1 + Z_2}}$$

Reflections in Elastic Media

Solid	Liquid	Gas
$v = \sqrt{Y/\rho}$ Y is Young's modulus $Y \sim 10^{10}\ N/m^2$	$v = \sqrt{K/\rho}$ K is the bulk modulus $K \sim 10^9 N/m^2$	$v = \sqrt{\gamma p/\rho}$ p is the gas pressure γ is a property of the gas. For air, $\gamma p \sim 1.42 \times 10^5 N/m^2$

- Consider reflection coefficients for a typical interface:
  - Air, $v_{air} = 340\, m/s, Z_{air} = 417\ \sqrt{kg/s}$
  - Water, $v_{water} = 1500\, m/s, Z_{water} = 1.47 \times 10^6\ \sqrt{kg/s}$
- Reflection coefficient:

$$\rho = \frac{Z_1 - Z_2}{Z_1 + Z_2} = -0.9994 \approx -1$$

- Transmission coefficient:

$$\tau = \frac{2Z_1}{Z_1 + Z_2} = 0.0006$$

- How much power is transferred across the interface?

Transmitted Power

- Transmitted amplitude: $A_t = \tau A$
- Power carried by a wave:

$$P = \frac{1}{2} Z \omega^2 A^2$$

- Incident power:

$$P_i = \frac{1}{2} Z_1 \omega^2 A^2$$

- Transmitted power:

$$P_t = \frac{1}{2} Z_2 \omega^2 (\tau A)^2 = \frac{2 Z_2 Z_1^2}{(Z_1 + Z_2)^2} \omega^2 A^2$$

$$\frac{P_t}{P_i} = \frac{4 Z_1^2}{(Z_1 + Z_2)^2}$$

- Reflected power: $P_r = P_i - P_t$ (conserves energy).