PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 17 – French, Chapter 7

Spring 2013 Semester

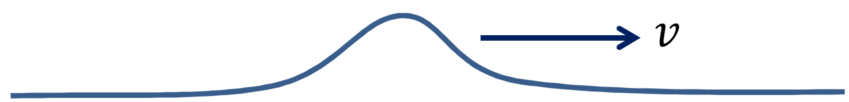
Matthew Jones

Midterm Exam:

Date:	Wednesday, March 6th
Time:	8:00 – 10:00 pm
Room:	PHYS 203
Material:	French, chapters 1-8

Wave Propagation

- We are considering the propagation of a disturbance on a continuous string, far from the ends:



- For convenience, we can define $t = 0$ to be the point where the pulse passes $x = 0$.
- Suppose that at this time the pulse has a Gaussian profile:

$$g(x) \sim \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- Let's fix this up so that the dimensions are more physical…

Wave Propagation

- Gaussian pulse:

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

  - Normalized to unit area:

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx = 1$$

  - Characteristic width:

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2}\, dx = \sigma^2$$

  - Change to dimensionless variables:

Let $u = x/\sigma$, then $dx = \sigma du$

and $g(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$

Fourier Transform

- The amplitudes of the different frequency components of the pulse are given by the Fourier transform:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx)\, dx$$
$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} \cos(kx)\, dx$$

- From your table of integrals:

$$\int_{-\infty}^{\infty} e^{-ax^2} \cos bx\, dx = \sqrt{\frac{\pi}{a}}\, e^{-b^2/4a}$$

- Substituting $a = 1/2\sigma^2$ and $b = k$ gives:

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma^2/2}$$

- The frequency components are also Gaussian

Notes about Fourier Transforms

- Fourier cosine transform:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx)\, dx$$

- Fourier sine transform:

$$B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx)\, dx$$

- The factor $1/\sqrt{2\pi}$ is there for convenience.
  - What matters is how to relate $A(k)$ and $B(k)$ to the original function
  - With this definition,

$$g(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} A(k) \cos(kx)\, dk + \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} B(k) \sin(kx)\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx)\, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx)\, dk$$

Notes about Fourier Transforms

- For the Gaussian pulse,

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

- The amplitudes of the frequency components are:

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2\sigma^2/2},\ B(k) = 0$$

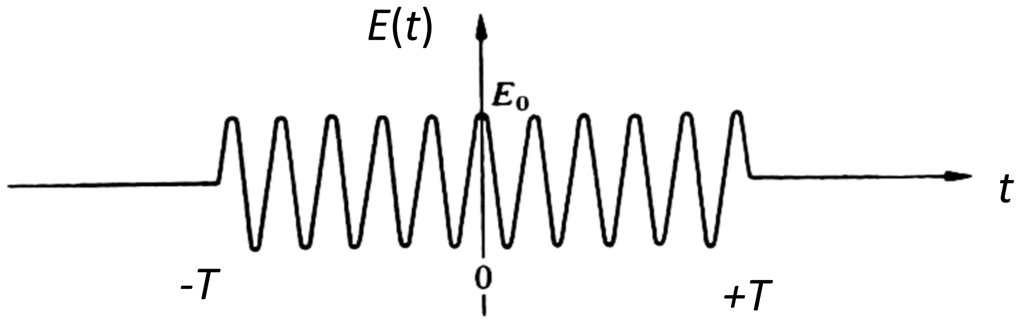
- When the pulse is narrow, $\sigma \ll 1$, then the exponent in $A(k)$ is large for a large range of k
  - Since $\omega = v/k$, a narrow pulse has a wide range of frequency components.
- Conversely, a wide pulse has a narrow range of frequencies.

Another Example

- Light is characterized by an oscillating electric and magnetic field.
- When we treat photons as particles, we think of them as being localized at a point in space that moves at the speed of light.
  - Localized at $x = 0$ implies a wide range of frequencies
  - A narrow range of frequencies implies that the photon is not localized in space…
- How, then, should we think about photons?

Another Example

- A photon can be described as a localized oscillation:



At $x = 0$, $E(t) = \begin{cases} E_0 \cos(\omega t) & \text{when } |t| < T \\ 0 & \text{otherwise} \end{cases}$

At $t = 0$,$E(x) = \begin{cases} E_0 \cos(kx) & \text{when } |x| < cT \\ 0 & \text{otherwise} \end{cases}$

$$A(k') = \frac{E_0}{\sqrt{2\pi}} \int_{-cT}^{cT} \cos(kx) \cos(k'x) \, dx$$

Another Example

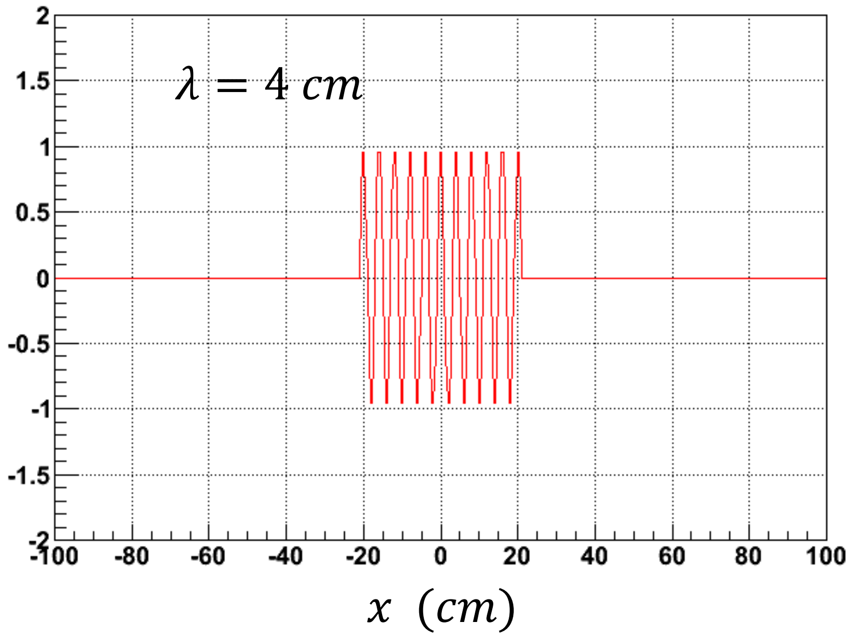
$$A(k') = \frac{E_0}{\sqrt{2\pi}} \int_{-cT}^{cT} \cos(kx) \cos(k'x)\, dx$$

- Trigonometric identity:

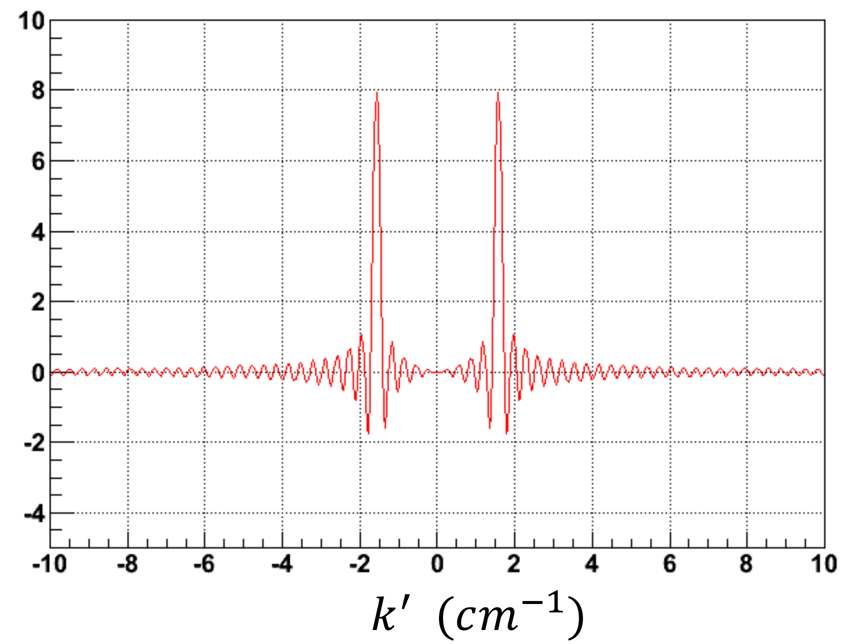
$$\cos\alpha \cos\beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$A(k') = \frac{E_0}{\sqrt{2\pi}} \left[\frac{\sin\big((k - k')cT\big)}{k - k'} + \frac{\sin\big((k + k')cT\big)}{k + k'} \right]$$

Another Example



$$k = \frac{2\pi}{\lambda} = 1.571\ cm^{-1}$$



Frequency Representation

- Why would we want to represent a function in terms of its frequency components?
  - Both representations contain the same information
- Physical properties can depend on the frequency
- Examples:
  - Maximum frequency for discrete masses

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

$$\omega_{max} = 2\omega_0$$

  - Speed of light depends on wavelength: $v = c/n(\lambda)$

Time Dependence

- The time-dependent description of the propagating pulse is, in the case where $B(k) = 0$,

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx - \omega t)\, dk$$

$$\omega = v(k)/k$$

- Phase velocity is $v(k) = \omega k$ but this is the speed at which the individual harmonic waves move.
- How fast does the pulse move?

Group Velocity

- Consider two harmonic waves with wavenumbers k_1 and k_2 which have frequencies ω_1 and ω_2
- The phase velocity of each wave is

$$v_1 = \omega_1/k_1$$
$$v_2 = \omega_2/k_2 \neq v_1$$

- Superposition:

$$y(x,t) = A[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)]$$

- Trigonometric identity:

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$y(x,t) = 2A\cos(kx - \omega t)\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$

Group Velocity

$$y(x,t) = 2A\underbrace{\cos(kx - \omega t)}\underbrace{\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)}$$

Fast component with velocity $v = \omega/k$

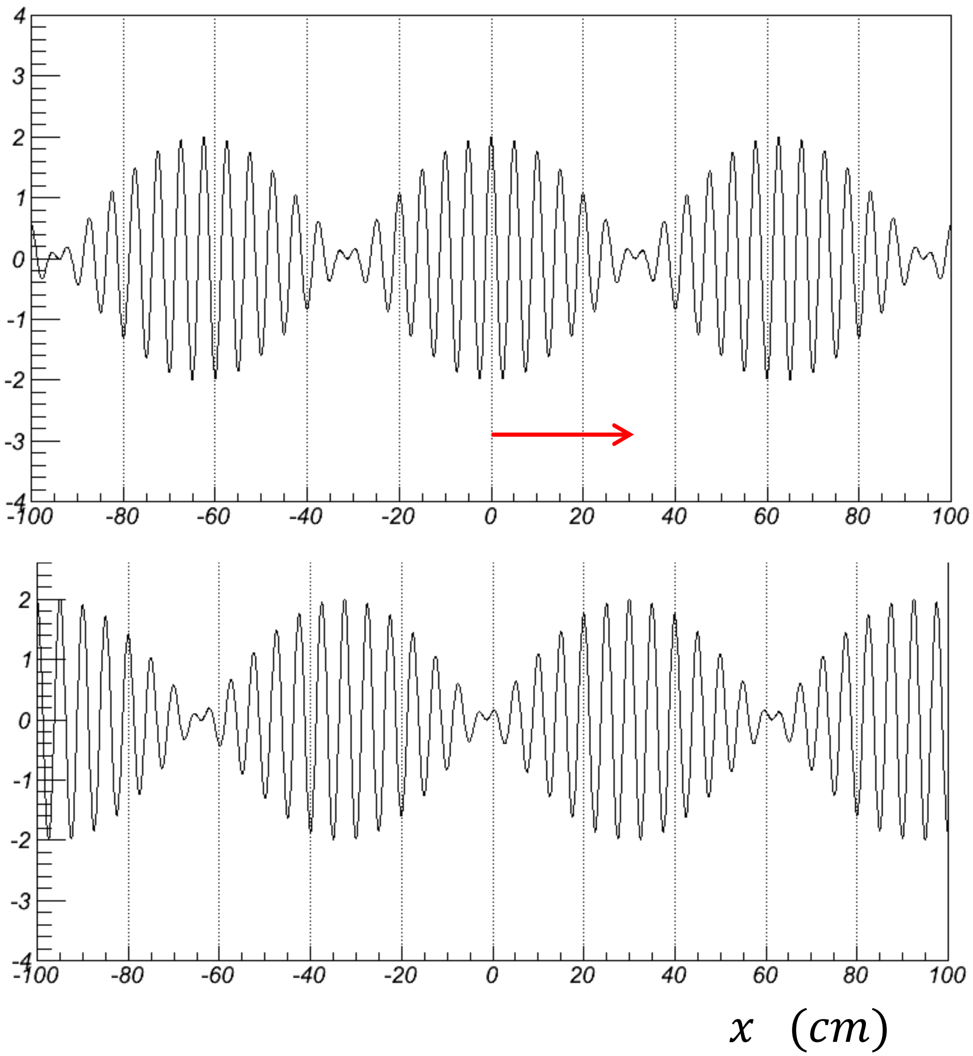
Slow component with velocity $v' = \Delta\omega/\Delta k$

We generalize this to define the group velocity as

$$v_g = \frac{d\omega}{dk}$$

which we evaluate at the average frequency, k.

Example

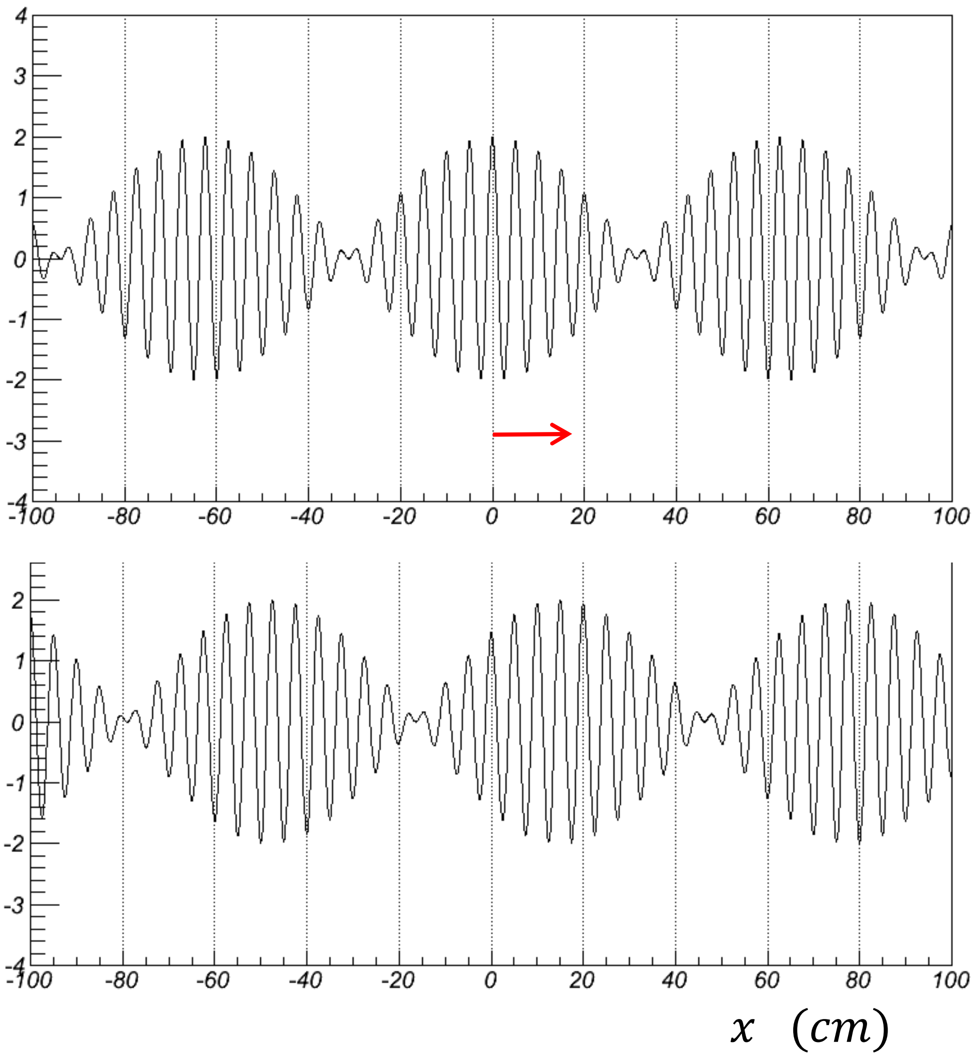


Group velocity same as phase velocity.

$$\lambda = 5\ cm$$
$$\Delta\lambda = 0.1\ cm$$
$$t = 0$$

$$t = 1\ ns$$

Example



$$v = \frac{\omega}{k} = 30\, cm/ns$$

$$\frac{\Delta\omega}{\Delta k} = 15\, cm/ns$$

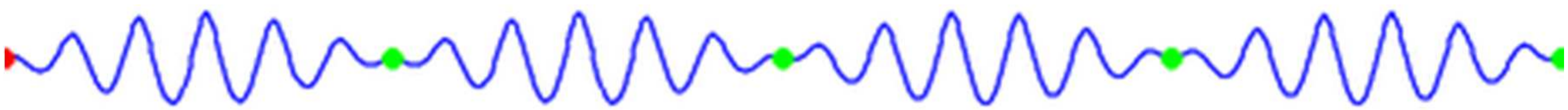
$$\lambda = 5\, cm$$

$$\Delta\lambda = 0.1\, cm$$

$$t = 0$$

$$t = 1\, ns$$

Example from Wikipedia



- Green points are at the minima between the pulses and move with the group velocity.
- The red point is constant in phase and moves with the phase velocity.
- The group velocity determines the rate of energy transfer.

Dispersion

- Different frequencies propagate with different speeds.
- For light, the speed depends on the index of refraction:

$$v = \frac{\omega}{k} = \frac{c}{n}$$

- This is the phase velocity.
- The index of refraction can be expressed as a function of wavelength:

$$\omega = vk = \frac{c}{n(k)}k = \frac{c}{n(\lambda)}\frac{2\pi}{\lambda}$$

- Let's calculate the group velocity…

Dispersion

- Frequency, expressed as a function of wavelength:

$$\omega(k) = kv(k)$$

- Group velocity:

$$v_g = \frac{d\omega}{dk} = v(k) + k\frac{dv}{dk}$$

- Phase velocity:

$$v(k) = \frac{c}{n(k)} \qquad \frac{dv}{dk} = -\frac{c}{n^2}\frac{dn}{dk}$$

$$v_g = \frac{c}{n(k)}\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

$$= c\left(n - \lambda\frac{dn}{d\lambda}\right)^{-1}$$

Optical Dispersion