PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 13 – French, Chapter 5

Spring 2013 Semester

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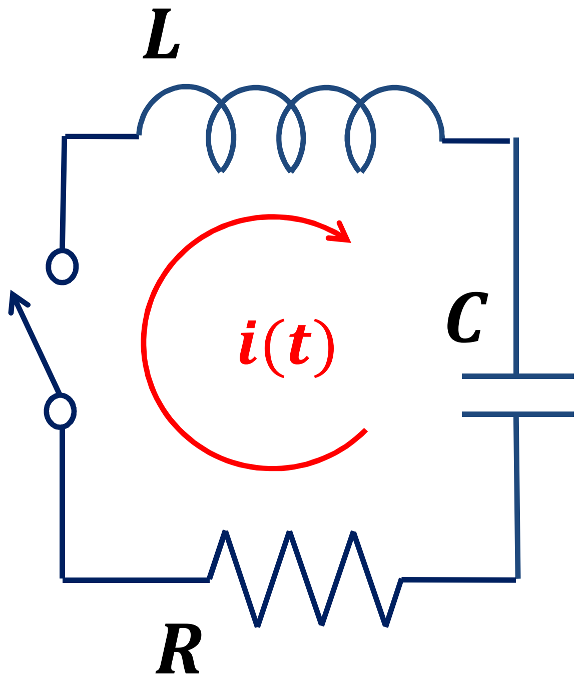
The Eigenvalue Problem

- If $\boldsymbol{A}$ is an $n \times n$ matrix and $\vec{\boldsymbol{u}}$ is a vector, find the numbers λ that satisfy
$$\boldsymbol{A}\,\vec{\boldsymbol{u}} = \lambda\,\vec{\boldsymbol{u}}$$
- Re-write the equation this way:
$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\,\vec{\boldsymbol{u}} = 0$$
- This is true only if
$$\det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = 0$$
- For a 2×2 matrix, this is:
$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$
- This is a second order polynomial in λ. Use the quadratic formula to find the roots.

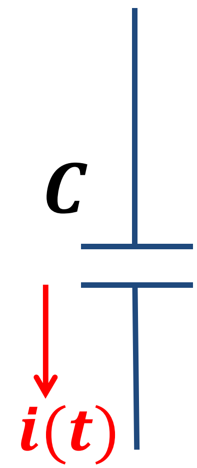
The Eigenvalue Problem

- The eigenvectors are vectors $\vec{\boldsymbol{u}}_i$ such that
$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\vec{\boldsymbol{u}}_i = 0$$
- There are n eigenvalues and n eigenvectors
- If $\vec{\boldsymbol{u}}_i$ is an eigenvector, then $\alpha\vec{\boldsymbol{u}}_i$ is also an eigenvector.
- Sometimes it is convenient to choose the eigenvectors so that they have unit length:
$$\hat{\boldsymbol{u}}_i \cdot \hat{\boldsymbol{u}}_i = 1$$
- Eigenvectors are orthogonal:
$$\vec{\boldsymbol{u}}_i \cdot \vec{\boldsymbol{u}}_j = 0 \text{ when } i \neq j$$
- An arbitrary vector $\vec{\boldsymbol{v}}$ can be written as a linear combination of the eigenvectors:
$$\vec{\boldsymbol{v}} = a_1\hat{\boldsymbol{u}}_1 + a_2\hat{\boldsymbol{u}}_2 + \cdots$$

A Circuit with One Loop



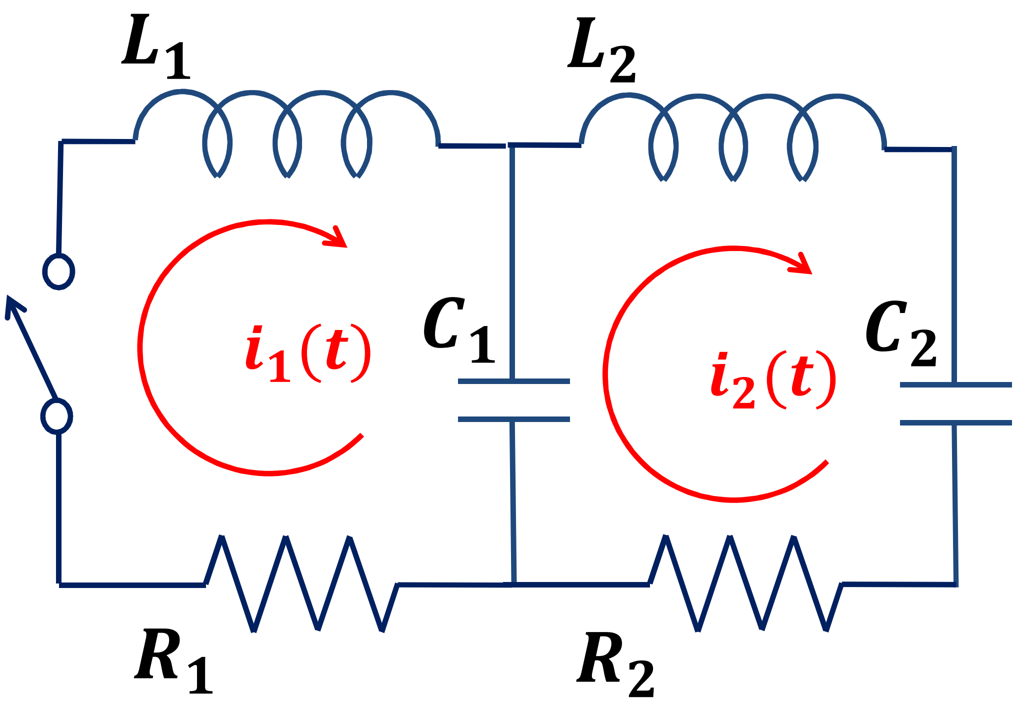
$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0$$

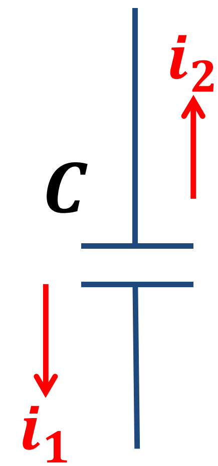


$$\frac{d^2 i}{dt^2} + \gamma\frac{di}{dt} + (\omega_0)^2 i(t) = 0$$

$$-\frac{1}{C}\int i(t)\,dt$$

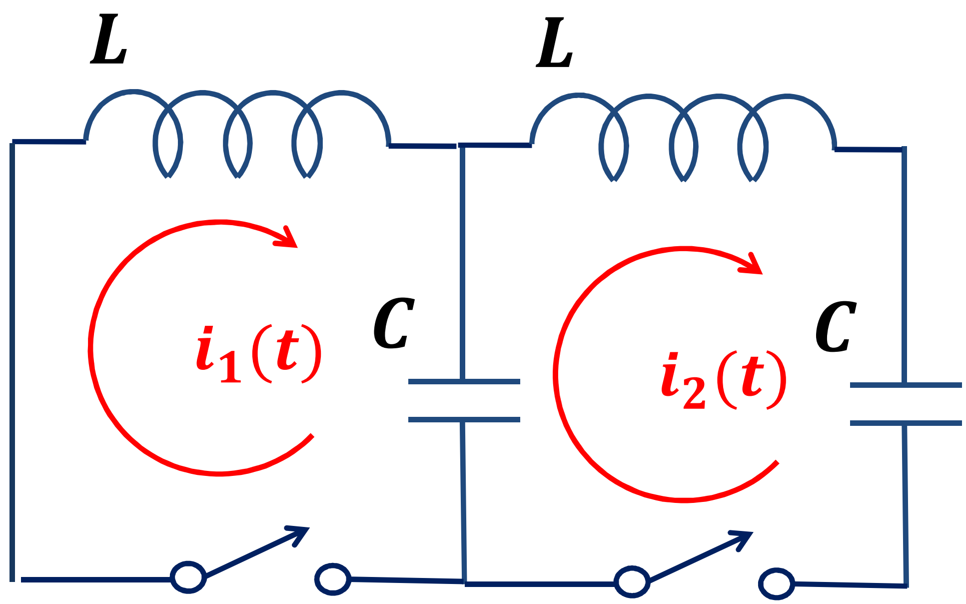
A Circuit with Two Loops





$$-\frac{1}{C}\int (i_1 - i_2)\, dt$$

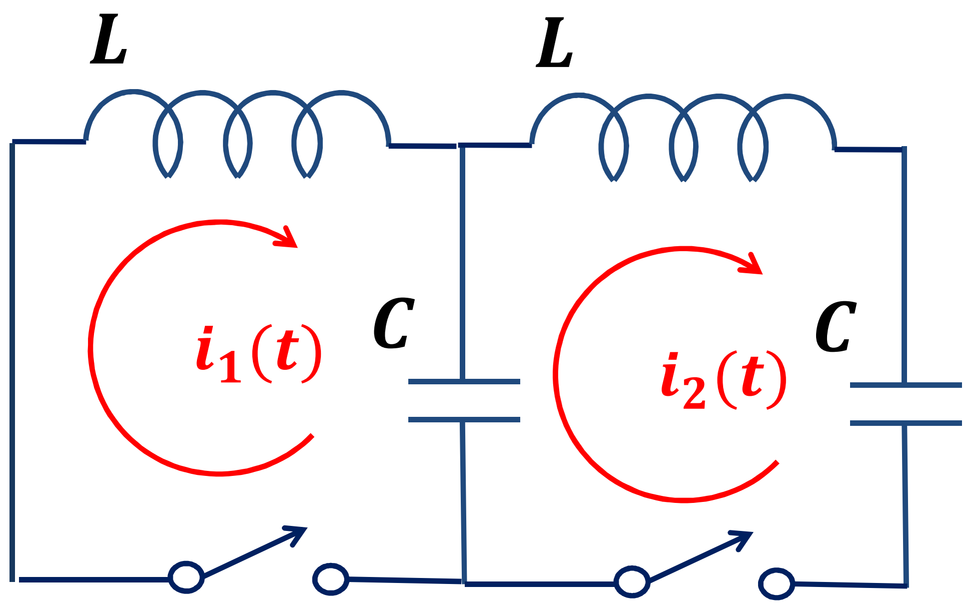
Example



$$-L\frac{di_1}{dt}-\frac{1}{C}\int(i_1-i_2)dt=0$$

$$-L\frac{di_2}{dt}-\frac{1}{C}\int i_2dt-\frac{1}{C}\int(i_2-i_1)dt=0$$

Example



$$\frac{d^2 i_1}{dt^2} + (\omega_0)^2 (i_1 - i_2) = 0$$

$$\frac{d^2 i_2}{dt^2} + (\omega_0)^2 (2i_2 - i_1) = 0$$

Normal Modes of Oscillation

- What are the frequencies of the normal modes of oscillation?
  - Let $\vec{\imath}(t) = \vec{\imath}\cos\omega t$
  - Then $\frac{d^2\vec{\imath}}{dt^2} = -\omega^2\vec{\imath}(t)$
- Substitute into the pair of differential equations:

$$(-\omega^2 + (\omega_0)^2)i_1 - (\omega_0)^2 i_2 = 0$$
$$(-\omega^2 + 2(\omega_0)^2)i_2 - (\omega_0)^2 i_1 = 0$$

- Write it as a matrix:

$$\begin{pmatrix} (\omega_0)^2 - \omega^2 & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

Eigenvalue Problem

$$\begin{pmatrix} (\omega_0)^2 - \omega^2 & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \omega^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

- For simplicity, let $\lambda = \omega^2$ and calculate the determinant:

$$\begin{vmatrix} (\omega_0)^2 - \lambda & -(\omega_0)^2 \\ -(\omega_0)^2 & 2(\omega_0)^2 - \lambda \end{vmatrix} = (\lambda - (\omega_0)^2)(\lambda - 2(\omega_0)^2) - (\omega_0)^4$$

$$= \lambda^2 - 3\lambda(\omega_0)^2 + (\omega_0)^4 = 0$$

- Roots of the polynomial:

$$\lambda = \frac{3}{2}(\omega_0)^2 \pm \frac{1}{2}\sqrt{9(\omega_0)^4 - 4(\omega_0)^4}$$

$$\omega^2 = (\omega_0)^2 \left(\frac{3 \pm \sqrt{5}}{2}\right)$$

Eigenvalue Problem

- The eigenvectors are obtained by substituting in each eigenvalue.
  - When $\omega^2 = (\omega_0)^2\left(\frac{3+\sqrt{5}}{2}\right)$

$$\frac{(\omega_0)^2}{2}\begin{pmatrix} -1-\sqrt{5} & -2 \\ -2 & 1-\sqrt{5} \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

$$i_1 = \left(\frac{1-\sqrt{5}}{2}\right) i_2$$

  - First normal mode of oscillation:

$$\vec{q}_1 = \boldsymbol{A}\begin{pmatrix} 1-\sqrt{5} \\ 2 \end{pmatrix}\cos(\omega_1 t + \boldsymbol{\alpha})$$

Eigenvalue Problem

- The eigenvectors are obtained by substituting in each eigenvalue.
  - When $\omega^2 = (\omega_0)^2 \left(\frac{3-\sqrt{5}}{2}\right)$

$$\frac{(\omega_0)^2}{2}\begin{pmatrix} -1+\sqrt{5} & -2 \\ -2 & 1+\sqrt{5} \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = 0$$

$$i_1 = \left(\frac{1+\sqrt{5}}{2}\right) i_2$$

  - Second normal mode of oscillation:

$$\vec{q}_2 = \boldsymbol{B}\begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix}\cos(\omega_2 t + \boldsymbol{\beta})$$

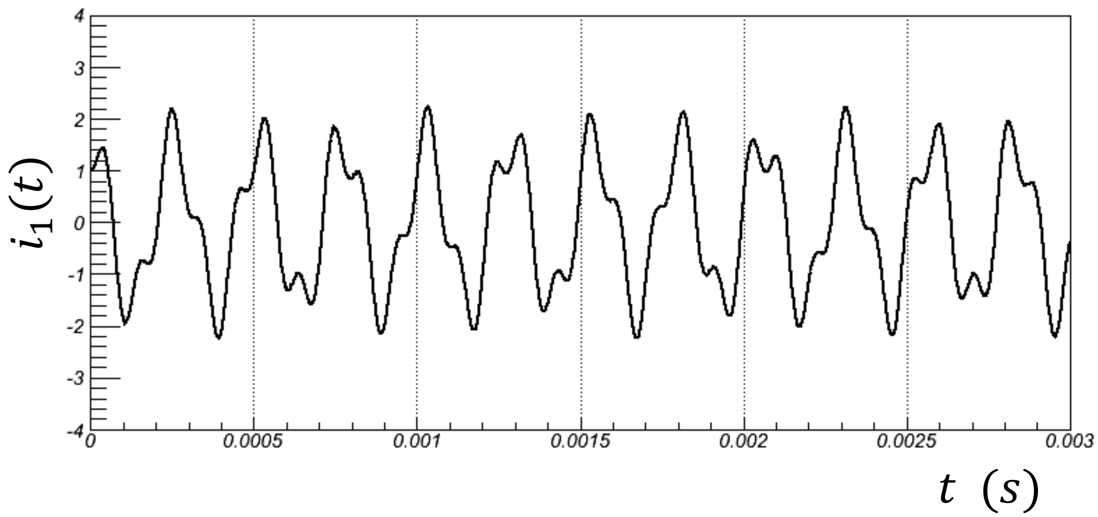
Eigenvalue Problem

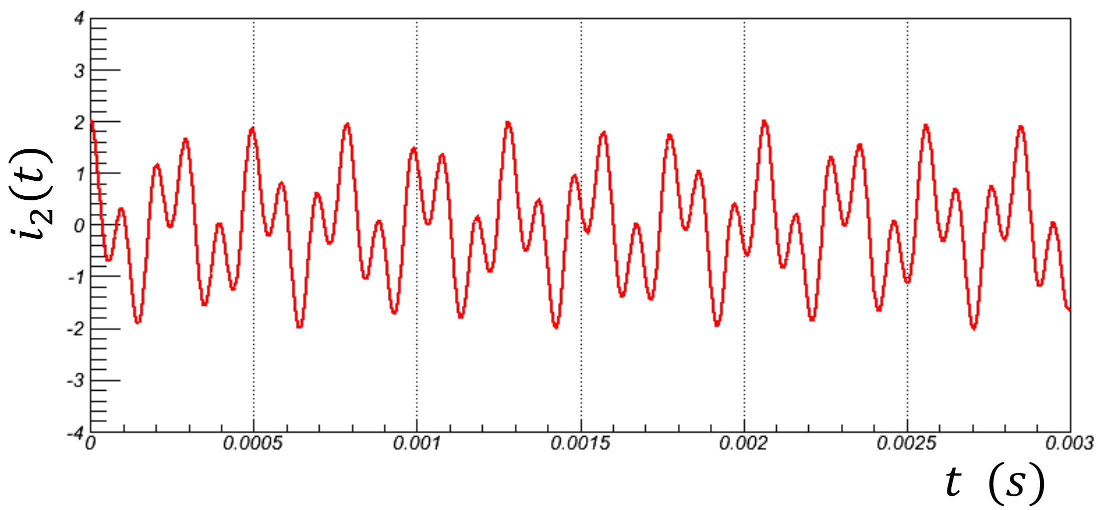
- The original "coordinates" are the sum of the normal modes of oscillation:

$$i_1(t) = A\left(1-\sqrt{5}\right)\cos(\omega_1 t + \alpha) + B(1+\sqrt{5})\cos(\omega_2 t + \beta)$$
$$i_2(t) = 2A\cos(\omega_1 t + \alpha) + 2B\cos(\omega_2 t + \beta)$$

- The constants of integration can be chosen to satisfy the initial conditions
  - For example, suppose that $i_1(0) = i_o$ and $i_2(0) = 0$
  - Then $A = -B, 2A = i_o$ ➔ $A = \frac{i_0}{2}, B = -\frac{i_0}{2}$

Two Loop Circuit



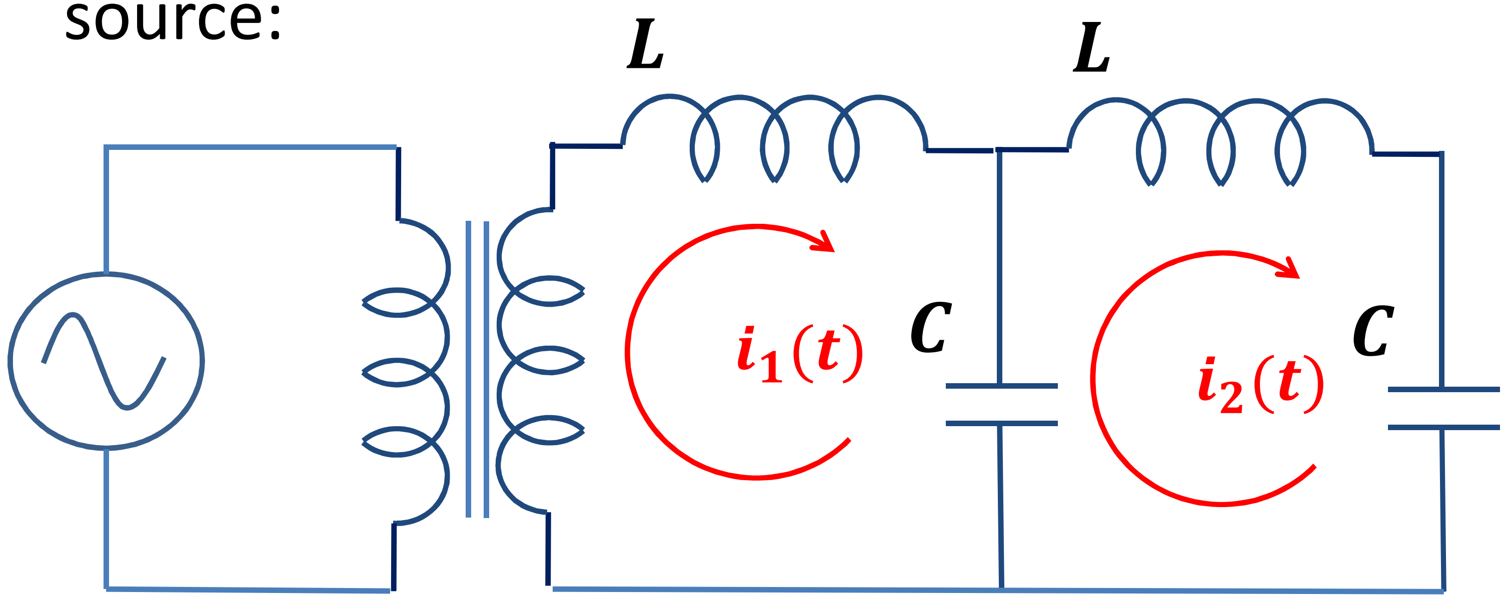


$$f_0 = \frac{\omega_0}{2\pi} = 1\ kHz$$

$$i_0 = 1\ A$$

Forced Coupled Circuit

- If the two loops were driven with a sinusoidal voltage source:

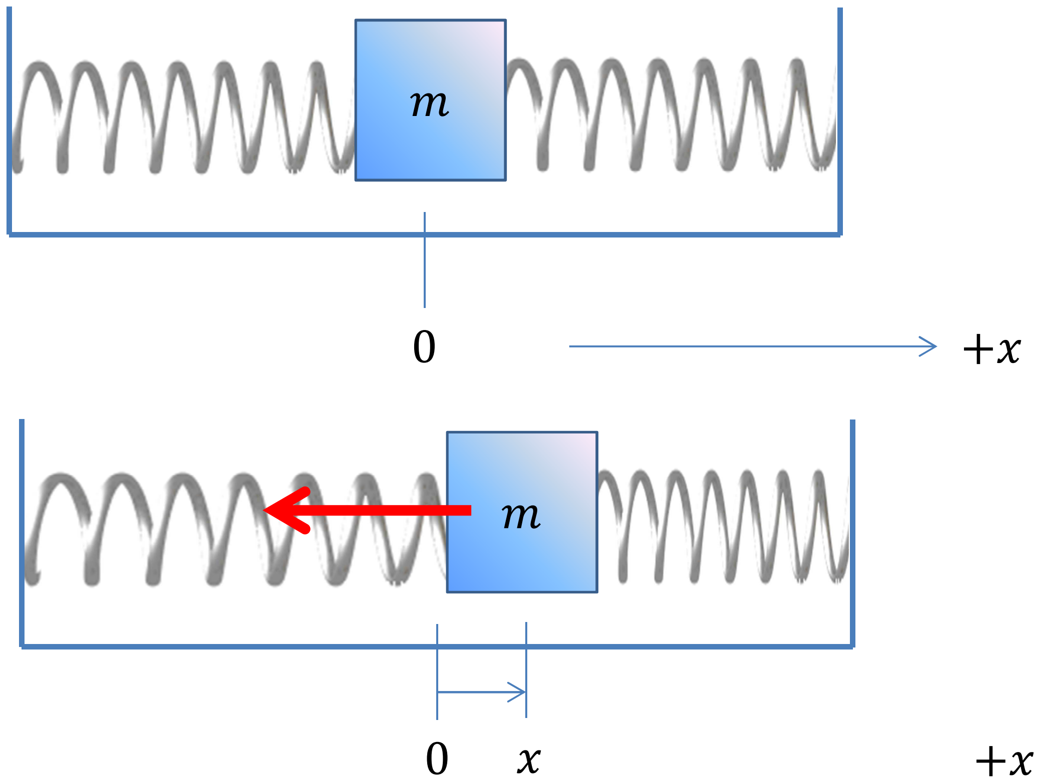


- Resonance would occur at the frequency of each normal mode:

$$\omega^2 = (\omega_0)^2 \left(\frac{3 \pm \sqrt{5}}{2}\right)$$

One Mass

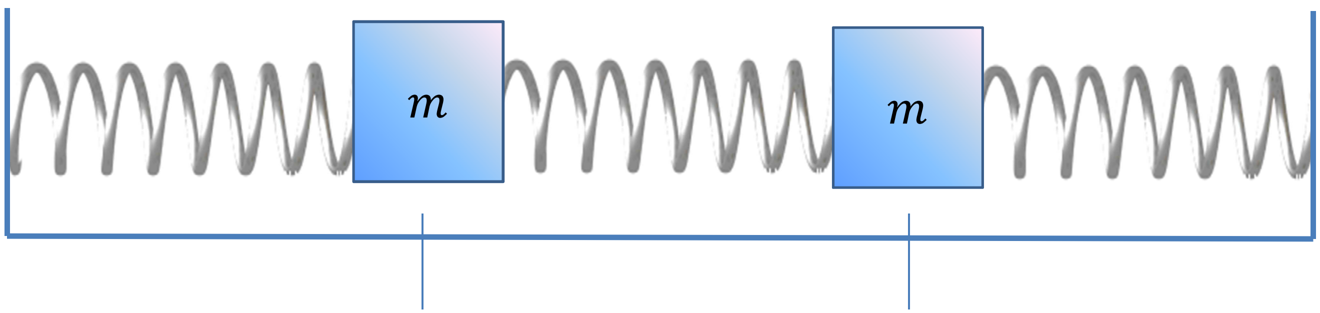
Consider one mass with two springs:



$$F = -2kx$$

Two Masses

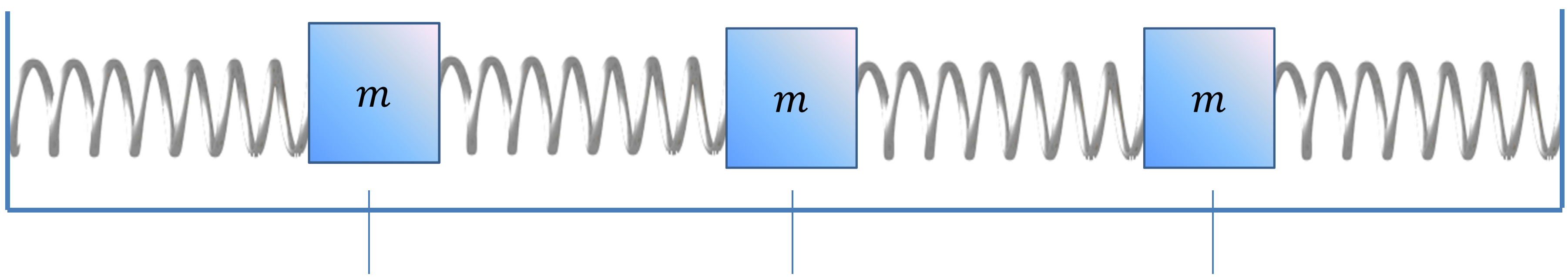
Consider two masses with three springs:



$$F_1 = -kx_1 - kx_1 + kx_2 = k(x_2 - 2x_1)$$
$$F_2 = kx_1 - kx_2 - kx_2 = k(x_1 - 2x_2)$$

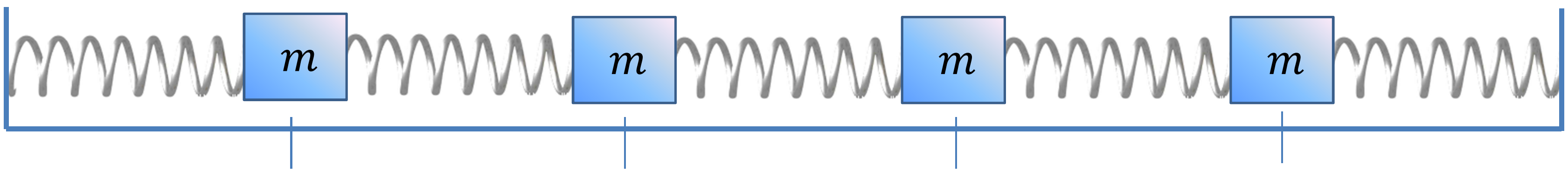
Three Masses

Consider three masses with four springs:



$$F_1 = -kx_1 - kx_1 + kx_2 = k(x_2 - 2x_1)$$
$$F_2 = -k(x_2 - x_1) - k(x_2 - x_3) = k(x_1 - 2x_2 + x_3)$$
$$F_3 = -kx_3 - kx_3 + kx_2 = k(x_2 - 2x_3)$$

Four Masses



$$F_1 = -kx_1 - kx_1 + kx_2 = k(x_2 - 2x_1)$$
$$F_2 = -k(x_2 - x_1) - k(x_2 - x_3) = k(x_1 - 2x_2 + x_3)$$
$$F_3 = -k(x_3 - x_2) - k(x_3 - x_4) = k(x_2 - 2x_3 + x_4)$$
$$F_4 = -kx_4 - kx_4 + kx_3 = k(x_3 - 2x_4)$$

- This pattern repeats for more and more masses.
- Except at the ends,

$$F_i = -k(x_i - x_{i-1}) - k(x_i - x_{i+1}) = k(x_{i-1} - 2x_i + x_{i+1})$$

- Equations of motion:

$$m\,\ddot{x}_i - k(x_{i-1} - 2x_i + x_{i+1}) = 0$$

Many Coupled Oscillators

$$m\,\ddot{x}_i - k(x_{i-1} - 2x_i + x_{i+1}) = 0$$
$$\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2(x_{i-1} + x_{i+1}) = 0$$

- Apply the same techniques we used before:
  - Suppose $x_i(t) = A_i \cos \omega t$
  - Then $\ddot{x}_i(t) = -\omega^2 A_i \cos \omega t$

$$(-\omega^2 + 2(\omega_0)^2)A_i - (\omega_0)^2(A_{i-1} + A_{i+1}) = 0$$
$$\frac{A_{i-1} + A_{i+1}}{A_i} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- Guess at a solution:

$$A_n = C \sin(n\Delta\theta)$$

- Will this work?

Many Coupled Oscillators

$$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- Proposed solution:

$$A_n = C\sin(n\Delta\theta)$$

- Boundary conditions: $A_0 = A_{N+1} = 0$
- This implies that $(N+1)\Delta\theta = k\pi$

$$A_n = C\sin\left(\frac{nk\pi}{N+1}\right)$$

$$A_{n-1} + A_{n+1} = C\sin\left(\frac{(n-1)k\pi}{N+1}\right) + C\sin\left(\frac{(n+1)k\pi}{N+1}\right)$$

$$= 2C\sin\left(\frac{nk\pi}{N+1}\right)\cos\left(\frac{k\pi}{N+1}\right)$$

$$\frac{A_{n-1} + A_{n+1}}{A_n} = 2\cos\left(\frac{k\pi}{N+1}\right) = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

Many Coupled Oscillators

$$\frac{A_{n-1} + A_{n+1}}{A_n} = 2\cos\left(\frac{k\pi}{N+1}\right) = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2}$$

- Solve for ω:

$$\omega^2 = 2(\omega_0)^2\left(1 - \cos\left(\frac{k\pi}{N+1}\right)\right)$$

$$= 4(\omega_0)^2 \sin^2\left(\frac{k\pi}{2(N+1)}\right)$$

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- There are N possible frequencies of oscillation.

Many Coupled Oscillators

- The motion of the masses depends on both the position of the mass (n) and the mode number (k):

$$A_{n,k} = C_n \sin\left(\frac{nk\pi}{N+1}\right)$$

$$\omega_k = 2\omega_0 \sin\left(\frac{k\pi}{2(N+1)}\right)$$

- When all the particles oscillate in the k^{th} normal mode, the n^{th} particle's position is:

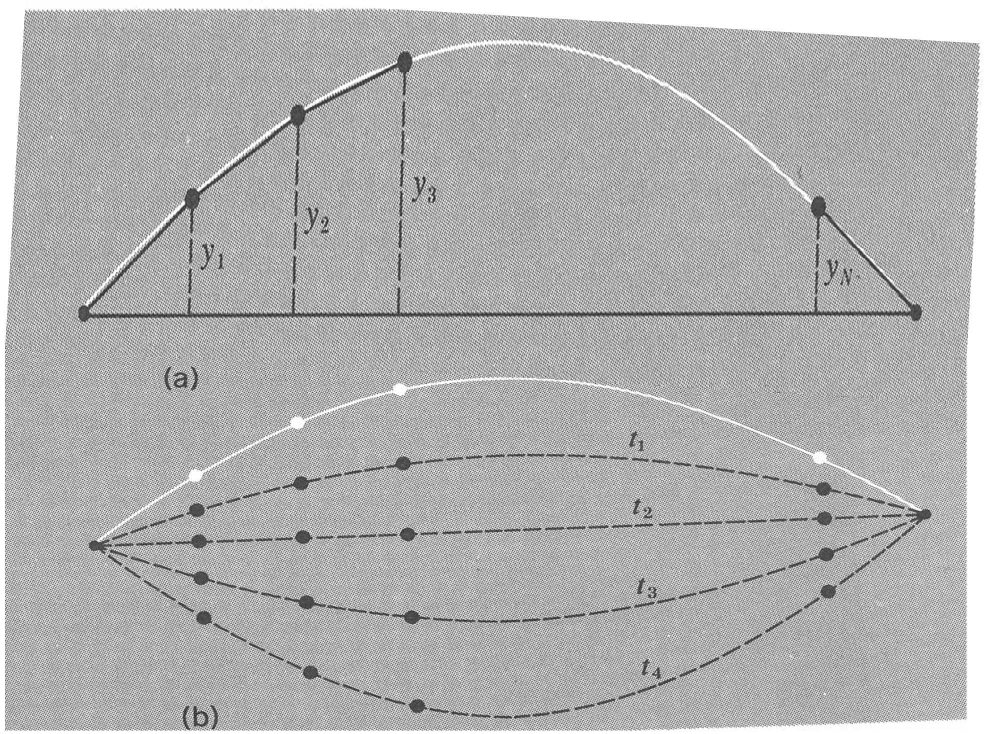
$$x_{n,k}(t) = A_{n,k} \cos(\omega_k t + \delta_k)$$

Many Coupled Oscillators

What do these modes look like?

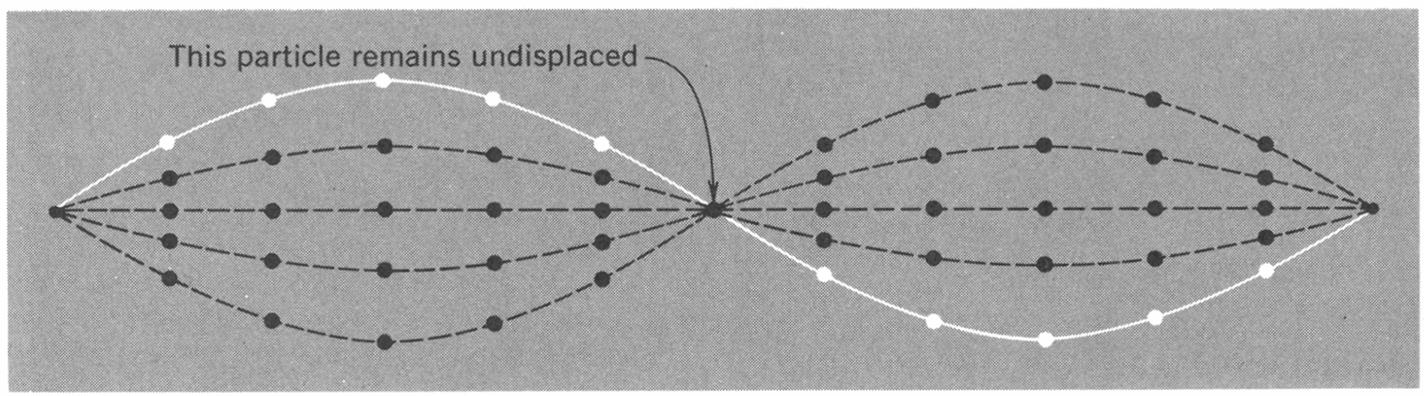
- Lowest order mode has $k = 1$…

$$x_{n,1}(t) = C_1 \sin\left(\frac{n\pi}{N+1}\right) \cos \omega_1 t$$



Many Coupled Oscillators

- Positions of masses in the second mode:



- Positions for 4 particles in modes $k = 1,2,3,4$: